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SOME PROPERTIES OF BIVATIATE FIBONACCI AND LUCAS QUATERNION POLYNOMIALS

Arzu Özkoç Öztürk and Faruk Kaplan

Abstract. In this work, we introduce bivariate Fibonacci quaternion polynomials and bivariate Lucas quaternion polynomials. We present generating function, Binet formula, matrix representation, binomial formulas and some basic identities for the bivariate Fibonacci and Lucas quaternion polynomial sequences. Moreover we give various kinds of sums for these quaternion polynomials.
Keywords: Bivariate Fibonacci quaternion polynomials, Bivariate Lucas quaternion polynomials, Generating function, Binet formula.

1. Introduction

In mathematics, Fibonacci and Lucas or other special numbers are investigation topic of great interest. Classical Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$ is defined by a recurrence identity;

$$F_n = \begin{cases} 0 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ F_{n-1} + F_{n-2} & \text{if} \quad n \geq 2. \end{cases}$$

The Lucas sequence $\{L_n\}_{n\in\mathbb{N}}$ is defined by some recurrence identity with different starting values;

$$L_n = \begin{cases} 2 & \text{if} \quad n = 0 \\ 1 & \text{if} \quad n = 1 \\ L_{n-1} + L_{n-2} & \text{if} \quad n \geq 2. \end{cases}$$

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Let $p(x)$ and $q(x)$ be polynomials with real coefficients of the $(p,q)-$Fibonacci polynomials are defined by the recurrence relation

$$F_{p,q,n+1} = p(x)F_{p,q,n} + q(x)F_{p,q,n-1}$$

with the initial conditions $F_{p,q,0} = 0$, $F_{p,q,1} = 1$. Also for the $p(x)$ and $q(x)$ polynomials with real coefficients the $(p,q)-$Lucas polynomials are defined by the recurrence relation

$$L_{p,q,n+1} = p(x)L_{p,q,n} + q(x)L_{p,q,n-1}$$

with initial conditions $L_{p,q,0} = 2$, $L_{p,q,1} = p(x)$.

Definition 1.1. [1] For $n \geq 2$, bivariate Fibonacci polynomials are defined as recurrence relation

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y). \tag{1.1}$$

We can compute the first few bivariate Fibonacci polynomials as follow $F_0(x,y) = 0$, $F_1(x,y) = 1$, $F_2(x,y) = x$, $F_3(x,y) = x^2 + y$, $F_4(x,y) = x^3 + 2xy$. Characteristic equation of relation (1.1) is

$$h^2 - xh - y = 0 \tag{1.2}$$

and so the roots of (1.2) are $\alpha = \alpha(x,y) = \frac{x+\sqrt{x^2+4y}}{2}$ and $\beta = \beta(x,y) = \frac{x-\sqrt{x^2+4y}}{2}$. Also it has Binet's formula $F_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for $n \geq 0$.

Definition 1.2. [1] For $n \geq 2$, bivariate Lucas polynomials are defined as recurrence relation

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y)$$

with initial conditionals $L_0(x,y) = 2$ and $L_1(x,y) = 1$.

Likely, let compute the first few terms of Lucas polynomials $L_0(x,y) = 2$, $L_1(x,y) = 1$, $L_2(x,y) = x + 2y$, $L_3(x,y) = x^2 + 2xy + y$, $L_4(x,y) = x^3 + 2x^2y + 2xy + 2y^2$. Also it has Binet's formula $L_n(x,y) = \alpha^n + \beta^n$ for $n \geq 0$.

Some authors considered special sequence polynomials for example generalized Fibonacci and Lucas polynomials in [7] and also bivariate Fibonacci and Lucas like polynomials in [6].

Normed division algebra, nowadays which is so important topic consists of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$ and octonions $\mathbf{O}$. Prime facie, directly we can not extend sundry results on real and complex numbers to quaternions due to quaternions are noncommutative normed division algebra over the real numbers, even it looks like things are going to be done with quaternions **H** [3]. For $a_0, a_1, a_2, a_3 \in \mathbb{R}$, a quaternion is defined by

$$e = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$$

where $e_0 = 1, e_1, e_2,$ and e_3 are unit vectors which verifies the following rules

$$(e_1)^2 = (e_2)^2 = (e_3)^2 = e_1e_2e_3 = -1. \tag{1.3}$$

From equation (1.3), we get

$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_1e_3 = -e_3e_1 = e_2.$$

Some new quaternion and octonion polynomials are studied in [2, 4, 5, 8, 9].

2. Bivariate Fibonacci and Lucas quaternion polynomials

Now, we define new quaternion polynomials which are called bivariate Fibonacci quaternion polynomials (QBF) and bivariate Lucas quaternion polynomials (QBL).

Definition 2.1. Bivariate Fibonacci quaternion polynomials (QBF) are defined as the recurrence relation

$$\begin{aligned} QBF_n(x,y) &= \sum_{k=0}^{3} F_{n+k}(x,y)e_k \\ &= F_n(x,y)e_0 + F_{n+1}(x,y)e_1 + F_{n+2}(x,y)e_2 + F_{n+3}(x,y)e_3 \end{aligned} \tag{2.1}$$

where $F_{n+k}(x,y)$ is the $n-th$ bivariate Fibonacci polynomial with the initial conditions $QBF_0(x,y) = e_1 + xe_2 + (x^2+y)e_3$ and $QBF_1(x,y) = e_0 + xe_1 + (x^2+y)e_2 + (x^3+2xy)e_3$.

Furthermore,

$$\begin{aligned} QBF_{n+1}(x,y) &= \sum_{k=0}^{3} F_{n+1+k}(x,y)e_k \\ &= x\sum_{k=0}^{3} F_{n+k}(x,y)e_k + y\sum_{k=0}^{3} F_{n+k-1}(x,y)e_k. \end{aligned}$$

So we get recurrence relation as follow

$$QBF_{n+1}(x,y) = xQBF_n(x,y) + yQBF_{n-1}(x,y). \tag{2.2}$$

Similarly, bivariate Lucas quaternion polynomials QBL are defined as the recurrence relation

$$\begin{aligned} QBL_n(x,y) &= \sum_{k=0}^{3} L_{n+k}(x,y)e_k \\ &= L_n(x,y)e_0 + L_{n+1}(x,y)e_1 + L_{n+2}(x,y)e_2 + L_{n+3}(x,y)e_3 \end{aligned} \tag{2.3}$$

where $L_{n+k}(x,y)$ is the $n-th$ bivariate Lucas polynomial and with the initial conditions $QBL_0(x,y) = 2e_0 + e_1 + (x+2y)e_2 + (x^2+2xy+y)e_3$ and $QBL_1(x,y) = e_0 + (x+y)e_1 + (x^2+2xy+y)e_2 + (x^3+2x^2y+2xy+2y^2)e_3$. Moreover, recurrence relation is

$$QBL_{n+1}(x,y) = xQBL_n(x,y) + yQBL_{n-1}(x,y). \tag{2.4}$$

Let $\alpha(x,y) = \frac{x+\sqrt{x^2+4y}}{2}$ and $\beta(x,y) = \frac{x-\sqrt{x^2+4y}}{2}$ denote the roots of the characteristic equation such that $\sqrt{x^2+4y} = \Delta$,

$$t^2 - xt - yt = 0$$

on the recurrence relation of (2.2) and (2.4).

From now on, for convenience of representation, we adopt the following notation

$$\alpha(x,y) = \alpha,\ \beta(x,y) = \beta,\ \Delta = \sqrt{x^2+4y}.$$

Equations that can be obtained with these roots are as follow

$$\begin{aligned} \alpha + \beta &= x \\ \alpha - \beta &= \Delta \\ \alpha\beta &= -y \\ \frac{\alpha}{\beta} &= -\frac{\alpha^2}{y} \\ \frac{\beta}{\alpha} &= -\frac{\beta^2}{y}. \end{aligned} \tag{2.5}$$

We continue with the generating function results.

Theorem 2.1. *The generating functions for QBF and QBL polynomials are respectively*

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{QBF_0(x,y) + [QBF_1(x,y) - xQBF_0(x,y)]\,t}{1 - xt - yt^2}$$

and

$$\sum_{n=0}^{\infty} QBL_n(x,y)t^n = \frac{QBL_0(x,y) + [QBL_1(x,y) - xQBL_0(x,y)]\,t}{1 - xt - yt^2}.$$

Proof. To compute the generating function of QBF polynomials

$$\begin{aligned} &\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = \;& QBF_0(x,y) + QBF_1(x,y)t + QBF_2(x,y)t^2 + \cdots + QBF_n(x,y)t^n + \cdots \end{aligned}$$

then using the equations of $-xt\left(\sum_{n=0}^{\infty} QBF_n(x,y)t^n\right)$ and $-yt^2\left(\sum_{n=0}^{\infty} QBF_n(x,y)t^n\right)$

$$\begin{aligned} &\sum_{n=0}^{\infty} QBF_n(x,y)t^n + (-xt)\sum_{n=0}^{\infty} QBF_n(x,y)t^n + (-yt^2)\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = \;& QBF_0(x,y) + [QBF_1(x,y) - xQBF_0(x,y)]\,t \\ & + [QBF_2(x,y) - xQBF_1(x,y) - yQBF_0(x,y)]\,t^2 \\ & + \cdots + [QBF_n(x,y) - xQBF_{n-1}(x,y) - yQBF_{n-2}(x,y)]\,t^n + \cdots \end{aligned}$$

Consequently,

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n(1 - xt - yt^2) = QBF_0(x,y) + (QBF_1(x,y) - xQBF_0(x,y))t$$

is valid. Similar proof can be done for QBL polynomials. □

Now we can give the following theorems.

Lemma 2.1. *If we rearrange the Theorem 2.1, we have the generating functions as follows*

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{\frac{QBF_1(x,y) - \beta QBF_0(x,y)}{1-\alpha t} - \frac{QBF_1(x,y) - \alpha(x,y) QBF_0(x,y)}{1-\beta t}}{\alpha - \beta}$$

and

$$\sum_{n=0}^{\infty} QBL_n(x,y)t^n = \frac{\frac{QBL_1(x,y) - \beta QBL_0(x,y)}{1-\alpha t} - \frac{QBL_1(x,y) - \alpha(x,y) QBL_0(x,y)}{1-\beta t}}{\alpha - \beta}.$$

Proof. If we use Theorem 2.1 and recurrence relation (2.2), then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\
=\;& \left(\frac{QBF_0(x,y)+(QBF_1(x,y)-(\alpha+\beta)QBF_0(x,y))t}{(1-\alpha t)(1-\beta t)}\right) \\
&\times\left(\frac{\alpha-\beta}{\alpha-\beta}\right) \\
=\;& \frac{\left\{\begin{array}{c} QBF_1(x,y)(1-\beta t)+\beta QBF_0(x,y)(-1+\beta t) \\ +QBF_1(x,y)(-1+\alpha t)+\alpha QBF_0(x,y)(1-\alpha t)\end{array}\right\}}{(1-\alpha t)(1-\beta t)(\alpha-\beta)} \\
=\;& \frac{\frac{QBF_1(x,y)-\beta QBF_0(x,y)}{1-\alpha t}-\frac{QBF_1(x,y)-\alpha(x,y)QBF_0(x,y)}{1-\beta t}}{\alpha-\beta}.
\end{aligned}$$

Hence the proof is completed. The other QBL polynomials can be proved similarly. □

Lemma 2.2. *For $k\geq 0$, let bivariate Fibonacci and Lucas polynomials are $F_n(x,y)$ and $L_n(x,y)$. We have*

$$\begin{aligned}
\textbf{(i)}\; F_{k+1}(x,y)-\alpha F_k(x,y) &= \beta^k \\
\textbf{(ii)}\; F_{k+1}(x,y)-\beta F_k(x,y) &= \alpha^k \\
\textbf{(iii)}\; \frac{\alpha L_k(x,y)-L_{k+1}(x,y)}{\alpha-\beta} &= \beta^k \\
\textbf{(iv)}\; \frac{L_{k+1}(x,y)-\beta L_k(x,y)}{\alpha-\beta} &= \alpha^k.
\end{aligned}$$

Proof. (i) We can prove it by induction method. Let $k=1$, then $F_2(x,y)-\alpha F_1(x,y)=\beta$.

Now let us assume that the equation is $F_n(x,y)-\alpha F_{n-1}(x,y)=\beta^{n-1}$, for $k=n-1$. For $k=n$ it becomes,

$$\begin{aligned}
\beta^n &= \beta^{n-1}\beta \\
&= ((F_n(x,y)-\alpha F_{n-1}(x,y))\beta \\
&= \beta F_n(x,y)-\alpha\beta F_{n-1}(x,y) \\
&= (\alpha+\beta-\alpha)F_n(x,y)-\alpha F_n(x,y)-\alpha\beta F_n(x,y) \\
&= xF_n(x,y)+yF_n(x,y)-\alpha F_n(x,y) \\
&= F_{n-1}(x,y)-\alpha F_n(x,y).
\end{aligned}$$

so this completes the proof. (ii), (iii) and (iv) can be done similarly. □

Now we want to derive the Binet formulas for QBF and QBL polynomials. To get this we can give the following theorem.

Theorem 2.2. *The Binet formulas of QBF and QBL polynomials are given as*

$$\begin{aligned} QBF_n(x,y) &= \frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \\ QBL_n(x,y) &= \alpha^*\alpha^n + \beta^*\beta^n \end{aligned}$$

for $n \geq 0$, where $\alpha^* = \sum_{k=0}^{3} \alpha^k e_k$ and $\beta^* = \sum_{k=0}^{3} \beta^k e_k$.

Proof. Recall that generating function is

$$\sum_{n=0}^{\infty} QBF_n(x,y)t^n = \frac{QBF_0(x,y) + (QBF_1(x,y) - xQBF_0(x,y))t}{1 - xt - yt^2}.$$

So using the Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} QBF_n(x,y)t^n \\ = &\sum_{k=0}^{\infty}(F_{k+1} - \beta F_{k+1})e_k \sum_{n=0}^{\infty} \alpha^n t^n - \sum_{k=0}^{\infty}(F_{k+1} - \alpha F_{k+1})e_k \sum_{n=0}^{\infty} \beta^n t^n. \end{aligned}$$

So we get,

$$\sum_{n=0}^{\infty} \left(\frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta} \right) t^n$$

this is valid. Binet formula for the other QBL polynomial can be done similarly. □

We derive generating functions for the $(mk+s)-th$ order of QBF and QBL polynomials.

Theorem 2.3. *For all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{\infty} QBF_{mk+s}(x,y)x^k = \frac{QBF_s(x,y) - (-y)^m QBF_{s-m}(x,y)x}{(-y)^m - L_m(x,y) + 1}$$

and

$$\sum_{k=0}^{\infty} QBL_{mk+s}(x,y)x^k = \frac{QBL_s(x,y) - (-y)^m QBL_{s-m}(x,y)x}{(-y)^m - L_m(x,y) + 1}.$$

Proof. Using Binet formula and equation (2.5), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} QBF_{mk+s}(x,y)x^k \\
=& \sum_{k=0}^{\infty} \frac{\alpha^*\alpha^{mk+s}-\beta^*\beta^{mk+s}}{\alpha-\beta}x^k \\
=& \frac{\alpha^*\alpha^s}{\alpha-\beta}\sum_{k=0}^{\infty}\alpha^{mk}x^k - \frac{\beta^*\beta^s}{\alpha-\beta}\sum_{k=0}^{\infty}\beta^{mk}x^k \\
=& \frac{\alpha^*\alpha^s}{\alpha-\beta}\left(\frac{1}{1-\alpha^m x}\right) - \frac{\beta^*\beta^s}{\alpha-\beta}\left(\frac{1}{1-\beta^m x}\right) \\
=& \frac{\frac{\alpha^*\alpha^s-\beta^*\beta^s}{\alpha-\beta} - (\alpha\beta)^m\left(\frac{\alpha^*\alpha^{s-m}-\beta^*\beta^{s-m}}{\alpha-\beta}\right)x}{1-(\alpha^m+\beta^m)x+(\alpha\beta)^m x^2}
\end{aligned}$$

this is valid. The other result can be done similarly. □

We formulate the sum of the first n terms of these sequences of QBF and QBL polynomials.

Theorem 2.4. *The sum of the first $n-$terms of the quaternion sequences $QBF_n(x,y)$ and $QBL_n(x,y)$ is given by*

$$\sum_{k=0}^{n} QBF_k(x,y) = \frac{\left\{\begin{array}{c} -yQBF_n(x,y)-QBF_{n+1}(x,y) \\ +QBF_0(x,y)-\frac{\alpha^*\beta-\beta^*\alpha}{\alpha-\beta} \end{array}\right\}}{(\alpha-1)(\beta-1)}$$

and

$$\sum_{k=0}^{n} QBL_k(x,y) = \frac{\left\{\begin{array}{c} -yQBL_n(x,y)-QBL_{n+1}(x,y) \\ +QBL_0(x,y)+\alpha^*\beta+\beta^*\alpha \end{array}\right\}}{(\alpha-1)(\beta-1)}.$$

Proof. Using Binet formula and equation (2.5), we get

$$\begin{aligned}
&\sum_{k=0}^{n} QBF_k(x,y) \\
=& \sum_{k=0}^{n} \frac{\alpha^*\alpha^k-\beta^*\beta^k}{\alpha-\beta} \\
=& \frac{1}{\alpha-\beta}\left\{\alpha^*\sum_{k=0}^{n}\alpha^k - \beta^*\sum_{k=0}^{n}\beta^k\right\}
\end{aligned}$$

$$= \frac{1}{\alpha-\beta}\left\{\alpha^*(\frac{\alpha^{n+1}-1}{\alpha-1})-\beta^*(\frac{\beta^{n+1}-1}{\beta-1})\right\}$$

$$= \frac{1}{(\alpha-1)(\beta-1)}\left\{\begin{array}{c}\frac{\alpha\beta(\alpha^*\alpha^n-\beta^*\beta^n)}{\alpha-\beta}-\frac{\alpha^*\alpha^{n+1}-\beta^*\beta^{n+1}}{\alpha-\beta}\\+\frac{\alpha^*-\beta^*}{\alpha-\beta}-\frac{\alpha^*\beta-\alpha\beta^*}{\alpha-\beta}\end{array}\right\}.$$

The other case can be done similarly. □

We derive summation formulas for the $(mk+s)-th$ order of QBF and QBL polynomials.

Theorem 2.5. *For all $n \in \mathbb{N}$ and $m, s \in \mathbb{Z}$,we have*

$$\sum_{k=0}^{n} QBF_{mk+s}(x,y) = \frac{\left\{\begin{array}{c}(-y)^m(QBF_{mn+s}(x,y)-QBF_{s-m}(x,y))\\-QBF_{mn+m+s}(x,y)+QBF_s(x,y)\end{array}\right\}}{(-y)^m-F_m(x,y)+1}$$

and

$$\sum_{k=0}^{n} QBL_{mk+s}(x,y) = \frac{\left\{\begin{array}{c}(-y)^m(QBL_{mn+s}(x,y)-QBL_{s-m}(x,y))\\-QBL_{mn+m+s}(x,y)+QBL_s(x,y)\end{array}\right\}}{(-y)^m-L_m(x,y)+1}.$$

Proof. Using Binet formula, equation (2.5), we have

$$\begin{aligned}
&\sum_{k=0}^{n} QBF_{mk+s}(x,y)\\
=\ & \sum_{k=0}^{n}\frac{\alpha^*\alpha^{mk+s}-\beta^*\beta^{mk+s}}{\alpha-\beta}\\
=\ & \frac{\alpha^*\alpha^s}{\alpha-\beta}\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right)-\frac{\alpha^*\alpha^s}{\alpha-\beta}\left(\frac{\alpha^{mn+m}-1}{\alpha^m-1}\right)\\
=\ & \frac{\left\{\begin{array}{c}\alpha^*(\alpha^{mn+s}\alpha^m\beta^m-\alpha^{mn+m+s}-\alpha^s\beta^m+\alpha^s)\\-\beta^*(\beta^{mn+s}\alpha^m\beta^m-\beta^{mn+m+s}-\alpha^m\beta^s+\beta^s)\end{array}\right\}}{(\alpha-\beta)(\alpha^m\beta^m-\alpha^m-\beta^m+1)}\\
=\ & \frac{(\alpha\beta)^m(\alpha^*\alpha^{mn+s}-\beta^*\alpha^{mn+s})-(\alpha^*\alpha^{mn+m+s}-\beta^*\beta^{mn+m+s})}{(\alpha-\beta)(\alpha^m\beta^m-\alpha^m-\beta^m+1)}\\
&+\frac{-(\alpha\beta)^m(\alpha^*\alpha^{s-m}-\beta^*\alpha^{s-m})-(\alpha^*\alpha^{mn+m+s}-\beta^*\beta^{mn+m+s})}{(\alpha-\beta)(\alpha^m\beta^m-\alpha^m-\beta^m+1)}.
\end{aligned}$$

Other case can be done similarly. □

Now, some new results for binomial summation of these sequences are derived by using their Binet forms.

Theorem 2.6. *Let n be a non-negative integer. Then we have the following binomial sum formulas for odd and even terms,*

$$\begin{aligned} \textbf{\textit{(i)}} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k QBF_k(x,y) &= QBF_{2n}(x,y) \\ \textbf{\textit{(ii)}} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k QBF_k(x,y) &= QBF_{2n+1}(x,y) \\ \textbf{\textit{(iii)}} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k QBL_k(x,y) &= QBL_{2n}(x,y) \\ \textbf{\textit{(iv)}} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k QBL_k(x,y) &= QBL_{2n+1}(x,y). \end{aligned}$$

Proof. (i) Let $P = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k QBF_k(x,y)$. From Binet formula, we change the right-hand side of P into:

$$P = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k \left(\frac{\alpha^* \alpha^k - \beta^* \beta^k}{\alpha - \beta} \right).$$

Elementary calculations implies that

$$P = \frac{\alpha^*(y + x\alpha)^n - \beta^*(y + x\beta)^n}{\alpha - \beta}.$$

From equation (2.5), we get

$$\frac{\alpha^* \alpha^{2n} - \beta^* \beta^{2n}}{\alpha - \beta} = QBF_{2n}(x,y).$$

The other cases (ii),(iii) and (iv) can be done similarly. □

Now we can also formulate the Catalan's identity, Cassini's identity and d'Ocagne's identity by using Binet formulas.

Theorem 2.7. *(Catalan's Identity) For n and k non-negative integer such that $k \leq n$, we have*

$$\begin{aligned} & QBF_{n+k}(x,y) QBF_{n-k}(x,y) - QBF_n^2(x,y) \\ = \quad & (-y)^{n-k} F_{n-k}(x,y) \left(\frac{\alpha^* \beta^* \beta^k - \beta^* \alpha^* \alpha^k}{(\alpha - \beta)} \right) \end{aligned}$$

and

$$\begin{aligned} & QBL_{n+k}(x,y) QBL_{n-k}(x,y) - QBL_n^2(x,y) \\ = \quad & (-y)^{n-k} F_{n-k}(x,y) \sqrt{\triangle} (\alpha^* \beta^* \beta^k - \beta^* \alpha^* \alpha^k). \end{aligned}$$

Proof. Using Binet formula, we obtain

$$\begin{aligned} & QBF_{n+k}(x,y)QBF_{n-k}(x,y) - QBF_n^2(x,y) \\ = & \left(\frac{\alpha^*\alpha^{n+k} - \beta^*\beta^{n+k}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n-k} - \beta^*\beta^{n-k}}{\alpha-\beta}\right) - \left(\frac{\alpha^*\alpha^{n} - \beta^*\beta^{n}}{\alpha-\beta}\right) \\ = & \frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left(\alpha^*\beta^*\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right) + \beta^*\alpha^*\left(\frac{\beta^k-\alpha^k}{\alpha-\beta}\right)\right) \\ = & (\alpha\beta)^{n-k}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^*\beta^*\beta^k - \beta^*\alpha^*\alpha^k}{\alpha-\beta}\right). \end{aligned}$$

□

Theorem 2.8. *For any natural number n, Cassini's identities for QBF and QBL polynomials are*

$$QBF_{n+1}(x,y)QBF_{n-1}(x,y) - QBF_n^2(x,y) = (-y)^{n-1}\left(\frac{\alpha^*\beta^*\beta - \beta^*\alpha^*\alpha}{\alpha-\beta}\right)$$

and

$$QBL_{n+1}(x,y)QBL_{n-1}(x,y) - QBL_n^2(x,y) = (-y)^{n-1}\sqrt{\Delta}(\alpha^*\beta^*\beta - \beta^*\alpha^*\alpha).$$

Proof. Let $k=1$ in Catalan's identity so the proof is completed for both of QBF and QBL polynomials. □

Theorem 2.9. *(d'Ocagne's Identity) Let QBF_n and QBL_n be n-th QBF and QBL polynomials. The d'Ocagne's identities are*

$$\begin{aligned} & QBF_k(x,y)QBF_{n+1}(x,y) - QBF_{k+1}(x,y)QBF_n(x,y) \\ = & \frac{(-1)^n y^n}{\alpha-\beta}(\alpha^*\beta^*\alpha^{k-n} - \beta^*\alpha^*\beta^{k-n}) \end{aligned}$$

and

$$\begin{aligned} & QBL_k(x,y)QBL_{n+1}(x,y) - QBL_{k+1}(x,y)QBL_n(x,y) \\ = & (\alpha-\beta)(\beta^*\alpha^*\beta^k\alpha^n - \alpha^*\beta^*\alpha^k\beta^n). \end{aligned}$$

Proof. From Binet formula to left -hand side, we get

$$\begin{aligned} & QBF_k(x,y)QBF_{n+1}(x,y) - QBF_{k+1}(x,y)QBF_n(x,y) \\ = & \left(\frac{\alpha^*\alpha^k - \beta^*\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right) - \left(\frac{\alpha^*\alpha^{k+1} - \beta^*\beta^{k+1}}{\alpha-\beta}\right)\left(\frac{\alpha^*\alpha^{n} - \beta^*\beta^{n}}{\alpha-\beta}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} (\alpha^*)^2\alpha^{n+k+1} - \beta^*\alpha^*\beta^k\alpha^{n+1} - \alpha^*\beta^*\alpha^k\beta^{n+1} + (\beta^*)^2\beta^{n+k+1} \\ -(\alpha^*)^2\alpha^{n+k+1} + \beta^*\alpha^*\beta^{k+1}\alpha^n + \alpha^*\beta^*\alpha^{k+1}\beta^n - (\beta^*)^2\beta^{n+k+1} \end{array}\right\} \\
&= \frac{\beta^*\alpha^*\beta^k\alpha^n(\beta-\alpha) + \alpha^*\beta^*\alpha^k\beta^n(\alpha-\beta)}{(\alpha-\beta)^2} \\
&= \frac{(\alpha\beta)^n}{(\alpha-\beta)}(\alpha^*\beta^*\alpha^{k-n} - \beta^*\alpha^*\beta^{k-n}).
\end{aligned}$$

The other case can be done similarly. □

The corresponding identities for QBF and QBL polynomials are contained in the next theorem.

Theorem 2.10. *For $n \geq 0$, the following statements hold:*

$$yQBF_n^2(x,y) + QBF_{n+1}^2(x,y) = \frac{(\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}}{\alpha-\beta}$$

and

$$yQBL_n^2(x,y) + QBL_{n+1}^2(x,y) = (\alpha-\beta)((\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}).$$

Proof. Using Binet formula and equation (2.5), we obtain

$$\begin{aligned}
& yQBF_n^2(x,y) + QBF_{n+1}^2(x,y) \\
&= y\left(\frac{\alpha^*\alpha^n - \beta^*\beta^n}{\alpha-\beta}\right)^2 + \left(\frac{\alpha^*\alpha^{n+1} - \beta^*\beta^{n+1}}{\alpha-\beta}\right)^2 \\
&= \frac{1}{(\alpha-\beta)^2}\left\{\begin{array}{c} y(\alpha^*)^2\alpha^{2n} - y\beta^*\alpha^* - y\alpha^*\beta^*(\alpha\beta)^n + y(\beta^*)^2\beta^{2n} + (\alpha^*)^2\alpha^{2n+2} \\ -\beta^*\alpha^*(\alpha\beta)^{n+1} - \alpha^*\beta^*(\alpha\beta)^{n+1} + (\beta^*)^2\beta^{2n+2} \end{array}\right\} \\
&= \frac{1}{(\alpha-\beta)^2}\left\{ y(\alpha^*)^2\alpha^{2n} + (\alpha^*)^2\alpha^{2n+2} + y(\beta^*)^2\beta^{2n} + (\beta^*)^2\beta^{2n+2} \right\} \\
&= \frac{(\alpha^*)^2\alpha^{2n+1} - (\beta^*)^2\beta^{2n+1}}{(\alpha-\beta)}.
\end{aligned}$$

Other case can be done similarly. □

Matrix method can use to get results for not only different identities but also algebraic representations in the study of recurrence relations.

In[10], the Pell quaternion matrix is defined by

$$R(n) = \begin{pmatrix} R_n & R_{n-1} \\ R_{n-1} & R_{n-2} \end{pmatrix}$$

and also was obtain equality as follow

$$\begin{pmatrix} R_n & R_{n-1} \\ R_{n-1} & R_{n-2} \end{pmatrix} = \begin{pmatrix} R_2 & R_1 \\ R_1 & R_0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{n-2}$$

where $n \geq 2$ is an integer.

Now, we define the matrix for $QBF_n(x, y)$ and $QBL_n(x, y)$. The matrix $QBF_n(x, y)(n)$ and $QBL_n(x, y)(n)$ that play role of $R(n)$. These are

$$QBF_n(x, y)(n) = \begin{pmatrix} QBF_{n+1}(x, y) & yQBF_n(x, y) \\ QBF_n(x, y) & yQBF_n(x, y) \end{pmatrix}$$

and

$$QBL_n(x, y)(n) = \begin{pmatrix} QBL_{n+1}(x, y) & yQBL_n(x, y) \\ QBL_n(x, y) & yQBL_n(x, y) \end{pmatrix}$$

for $n \geq 1$.

Theorem 2.11. *For an integer $n \geq 1$, we have*

$$QBF_n(x, y)(n) = \begin{pmatrix} QBF_2(x, y) & yQBF_1(x, y) \\ QBF_1(x, y) & yQBF_0(x, y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{n-1}$$

and

$$QBL_n(x, y)(n) = \begin{pmatrix} QBL_2(x, y) & yQBL_1(x, y) \\ QBL_1(x, y) & yQBL_0(x, y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{n-1}.$$

Proof. Induction method can be used to prove it. Let $n = 1$, then basis step is clear. Now let us assume that the equation is valid for $n = k - 1$. For $n = k$, it becomes

$$\begin{aligned} & \begin{pmatrix} QBF_2(x, y) & yQBF_1(x, y) \\ QBF_1(x, y) & yQBF_0(x, y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{k-1} \\ = & \begin{pmatrix} QBF_2(x, y) & yQBF_1(x, y) \\ QBF_1(x, y) & yQBF_0(x, y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}^{k-2} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} QBF_k(x, y) & yQBF_{k-1}(x, y) \\ QBF_{k-1}(x, y) & yQBF_{k-2}(x, y) \end{pmatrix} \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} QBF_{k+1}(x, y) & yQBF_k(x, y) \\ QBF_k(x, y) & yQBF_{k-1}(x, y) \end{pmatrix}. \end{aligned}$$

which completes the proof. The other case can be done similarly. □

3. Conclusion

This work studied bivariate Fibonacci and Lucas quaternion polynomials. Since bivariate Fibonacci and Lucas quaternion polynomials were not intensive studied until now, we expect to find in the future more and surprising new properties. For this purpose, Fibonacci and Lucas quaternion polynomials was used and investigated in detail particularly in the first part. Also in the other part, Binet formulas, generating functions, matrix representation and some identities of bivariate Fibonacci and Lucas quaternion polynomials were computed. Quaternions have great importance as they are used in quantum physics, applied mathematics, graph theory and differential equations. Thus, in our future studies we plan to examine bivariate Fibonacci and Lucas octonion polynomials and their key features.

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Arzu Özkoç Öztürk
Duzce University
Science and Art Faculty
Department of Mathematics
Duzce, Turkey

arzuozkoc@duzce.edu.tr

Faruk Kaplan
Duzce University
Science and Art Faculty
Department of Mathematics
Duzce, Turkey
farkaplan@gmail.com