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CIRCULANT AND NEGACYCLIC MATRICES VIA TETRANACCI NUMBERS

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Abstract. In this paper, the explicit determinants of the circulant and negacyclic matrix involving Tetranacci sequence M_n and Companion-Tetranacci sequence K_n are expressed by using only Tetranacci sequence M_n and Companion-Tetranacci sequence K_n . Also euclidean norms and spectral norms of circulant and negacyclic matrices have been obtained.

1. Introduction

Fibonacci, Lucas and Pell numbers and their generalizations arise in the examination of various areas of science and art. In fact these numbers are special case of a sequence which is defined as a linear combination as follows:

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n, \tag{1}$$

where $c_1, c_2, \cdots, c_k$ are real constants. The applications and identities related with these numbers can be seen in [5]. Fibonacci numbers form a sequence defined by the following recurrence relation: $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ (sequence A000045 in OEIS). The characteristic equation of F_n is $x^2 - x - 1 = 0$ and hence the roots of it are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Moreover its Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for $n \geq 0$. Lucas numbers [5, 6] L_n are defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ (sequence A000032 in OEIS). There

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are a lot of algebraic identities between Fibonacci and Lucas numbers. Some of them can be given as $L_n = F_{n-1} + F_{n+1}$, $F_{m+n} = \frac{F_m L_n + L_m F_n}{2}$, $F_{m-n} = \frac{(-1)^n (F_m L_n - L_m F_n)}{2}$, $L_n^2 - 5F_n^2 = 4(-1)^n$ and $F_{2n} = F_n L_n$.

Tetranacci sequence [9] M_n and Companion-Tetranacci sequence K_n are defined by a fourth-order recurrence

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} \tag{2}$$

$$K_n = K_{n-1} + K_{n-2} + K_{n-3} + K_{n-4}, \tag{3}$$

with initial values $M_0 = 1,\ M_1 = 2,\ M_2 = 2,\ M_3 = 4$ and $K_0 = 4,\ K_1 = 1,\ K_2 = 3,\ K_3 = 7$ for $n \geq 4$. The characteristic equation of them is $x^4 - x^3 - x^2 - x - 1 = 0$ and if its roots are denoted by α, β, γ and δ then the following equalities holds

$$\begin{aligned}
\alpha\beta + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta + \alpha\gamma &= -1 \\
\left\{ \begin{array}{c} (\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2 \\ +(\alpha\beta\gamma\delta)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \end{array} \right\} &= -4 \\
\left\{ \begin{array}{c} \alpha^2(\alpha\gamma + \gamma\delta + \beta\delta) + \beta^2(\alpha\delta + \gamma\delta + \alpha\gamma) \\ +\gamma^2(\alpha\delta + \alpha\beta + \beta\delta) + \delta^2(\beta\gamma + \alpha\beta + \beta\delta) \end{array} \right\} &= 5.
\end{aligned}$$

Furthermore, by utilizing the method in [8], the Binet formulas for the Tetranacci sequence is

$$M_n = X\alpha^n + Y\beta^n + Z\gamma^n + W\delta^n \tag{4}$$

for

$$X = \frac{\alpha^5 - \alpha^4}{2\alpha^4 - 5}, Y = \frac{\beta^5 - \beta^4}{2\beta^4 - 5}, Z = \frac{\gamma^5 - \gamma^4}{2\gamma^4 - 5}, W = \frac{\delta^5 - \delta^4}{2\delta^4 - 5}$$

and the Binet formulas for the Companion-Tetranacci sequence is

$$K_n = \alpha^n + \beta^n + \gamma^n + \delta^n.$$

Note that Tetranacci numbers (sequence A000078 in OEIS), Companion-Tetranacci numbers (sequence A073817 in OEIS).

There are many interests in properties and generalization of some special matrices with Fibonacci and Lucas numbers and also third order-recurrence, e.g., Tribonacci and Tribonacci-Lucas sequences. For example, some authors have give various algorithms for the determinants and inverses of circulant matrices. The circulant matrices have been extended in recent years in many directions. It has important applications including image processing, communications, signal processing, encoding, solving Toeplitz matrix problems and others [1, 2, 10, 11, 12].

The circulant and negacyclic matrices formed a square matrix for T_n and K_n are defined to be

$$C(M_0, M_1, \cdots, M_{n-1}) = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-1} \\ M_{n-1} & M_0 & M_1 & \cdots & M_{n-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ M_1 & M_2 & M_3 & \cdots & M_0 \end{bmatrix},$$

$$C(K_0, K_1, \cdots, K_{n-1}) = \begin{bmatrix} K_0 & K_1 & K_2 & \cdots & K_{n-1} \\ K_{n-1} & K_0 & K_1 & \cdots & K_{n-2} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ K_1 & K_2 & K_3 & \cdots & K_0 \end{bmatrix}$$

and

$$N(M_0, M_1, \cdots, M_{n-1}) = \begin{bmatrix} M_0 & M_1 & \cdots & M_{n-1} \\ -M_{n-1} & M_0 & \cdots & M_{n-2} \\ . & . & \ddots & . \\ . & . & \ddots & . \\ -M_1 & -M_2 & \cdots & M_0 \end{bmatrix},$$

$$N(K_0, K_1, \cdots, K_{n-1}) = \begin{bmatrix} K_0 & K_1 & \cdots & K_{n-1} \\ -K_{n-1} & K_0 & \cdots & K_{n-2} \\ . & . & \ddots & . \\ . & . & \ddots & . \\ -K_1 & -K_2 & \cdots & K_0 \end{bmatrix}$$

respectively, which we will use shortly $C(M) = C(M_0, M_1, \cdots, M_{n-1})$, $N(M) = N(M_0, M_1, \cdots, M_{n-1})$ and $C(K) = C(K_0, K_1, \cdots, K_{n-1})$, $N(K) = N(K_0, K_1, \cdots, K_{n-1})$.

The eigenvalues of a $n \times n$ circulant matrix M are

$$\lambda_j(x) = \sum_{k=0}^{n-1} x_k w^{-jk}, \tag{5}$$

where $w = e^{\frac{2\pi i}{n}}, i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$.

Theorem 1.1. [3] *Let $N(x)$ be an $n \times n$ negacyclic matrix. Then*

$$N(x) = G\ diag(\lambda_0(x), \lambda_1(x), \cdots, \lambda_{n-1}(x))G^*,$$

where $\lambda_j(x) = \sum_{k=0}^{n-1} x_k w^{(2j+1)k/2}$, $j = 0,\ 1, \cdots, n-1$.

There are several papers on the norms of some special matrices [3, 4, 7]. Let $A = (a_{ij})$ be an $n \times n$ matrix. The Euclidean norm, spectral norm, the maximum column sum norm and maximum row sum norm of the matrix A are denoted as following respectively,

$$\begin{aligned} \|A\|_E &= \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}, \ \|A\|_2 = \left(\max_{1\le i\le n} \lambda_i(A^*A)\right)^{1/2}, \\ \|A\|_1 &= \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \text{ and } \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|. \end{aligned}$$

where A^*denotes the conjugate transpose of A.

It is well known that $\frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2 \le \|A\|_E$.

Lemma 1.2. *Let $w = e^{\frac{2\pi i}{n}}$satisfy the $n-th$ primitive root of unity where $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$ and a, b, c, d and g are complex numbers, then*

$$\begin{aligned} &\prod_{k=1}^{n} \left(a - bw^{-k} + cw^{-2k} - dw^{-3k}\right) \\ = \ & a^n - d^n + (2^{-n} - 2^{1-2n})b^n + 2^{1-n}\left(\frac{c-2ad}{b}\right)^n + 2^n\left(\frac{ad}{b}\right)^n \end{aligned}$$

and

$$\begin{aligned} &\prod_{k=1}^{n} \left(a - bw^{-k} + cw^{-2k} - dw^{-3k} + gw^{-4k}\right) \\ = \ & a^n + g^n + 2^{2-2n}(b^n + d^n) + 2^{1-3n}\left(\frac{4ac+b}{a}\right)^n + 2^{2-4n}\left(\frac{b}{a}\right)^n. \end{aligned}$$

2. Main Results

We present the exact formulae of determinants by some terms of Tetranacci and Companion-Tetranacci sequences, on the basis of the fourth-order recurrence, binet formulas, and other properties of these two sequences. Also we deduce euclidean norm, spectral norm and eigenvalues for the circulant and negacyclic matrices which terms are Tetranacci and Companion-Tetranacci numbers.

In this section, we consider some algebraic properties of M_n and K_n. We first consider the sums of them.

Theorem 2.1. *The sums of first n terms of M_n and K_n are*

$$\begin{aligned}\sum_{i=4}^{n} M_i &= \frac{1}{3}(4M_n + 3M_{n-1} + 2M_{n-2} + M_{n-3} - 25)\\ \sum_{i=4}^{n} K_i &= \frac{1}{3}(4K_n + 3K_{n-1} + 2K_{n-2} + K_{n-3} - 43).\end{aligned}$$

Proof. Notice that $M_n - M_{n-1} = M_{n-2} + M_{n-3} + M_{n-4}$. We deduce that

$$\begin{aligned}M_4 - M_3 &= M_2 + M_1\\ M_5 - M_4 &= M_3 + M_2 + M_1\\ &\ldots\\ M_{n-1} - M_{n-2} &= M_{n-3} + M_{n-4} + M_{n-5}\\ M_n - M_{n-1} &= M_{n-2} + M_{n-3} + M_{n-4}\end{aligned} \tag{6}$$

If we sum both side of (6), then we have

$$3(M_4 + M_5 + \cdots + M_n) = 4M_n + 3M_{n-1} + 2M_{n-2} + M_{n-3} - 25.$$

So we get

$$\sum_{i=4}^{n} M_i = \frac{1}{3}(4M_n + 3M_{n-1} + 2M_{n-2} + M_{n-3} - 25)$$

the desired result. The other assertion can be proved similarly. ∎

Theorem 2.2. *Let M_n and K_n denote the n^{th} Tetranacci and Companion-Tetranacci sequences. Then the difference between the terms of two sequences are*

$$M_n - K_n = M_{n+4} - M_{n+3} - M_{n+2} - M_{n+1} - K_{n+3} + K_{n+2} + K_{n+1} + K_{n-1}.$$

Proof. From the defination of M_n and K_n, we know

$$\begin{aligned}M_n &= M_n + (M_{n+2} + M_{n+1} + M_{n-1}) - (M_{n-1} + M_{n+2} + M_{n+1})\\ &= M_{n+3} - M_{n-1} - M_{n+2} - M_{n+1}\\ &= (M_{n+3} + M_{n+2} + M_{n+1} + M_n) - (M_{n+2} + M_{n+1} + M_n + M_{n-1})\\ &\quad - M_{n+2} - M_{n+1}\\ &= M_{n+4} - M_{n+3} - M_{n+2} - M_{n+1}\end{aligned}$$

and

$$\begin{aligned}K_n &= K_n-(K_{n+1}+K_{n-1}+K_{n-2})+(K_{n+1}+K_{n-2}+K_{n-1})\\ &= K_{n+2}-K_{n-2}-K_{n+1}-K_{n-1}\\ &= (K_{n+2}+K_{n+1}+K_n+K_{n-1})-(K_{n+1}+K_n+K_{n-1}+K_{n-2})\\ &\quad -K_{n+1}-K_{n-1}\\ &= K_{n+3}-K_{n+2}-K_{n+1}-K_{n-1}.\end{aligned}$$

Hence we conclude that

$$M_n-K_n=M_{n+4}-M_{n+3}-M_{n+2}-M_{n+1}-K_{n+3}+K_{n+2}+K_{n+1}+K_{n-1}.$$

∎

Now we can give the following results for circulant matrices.

Theorem 2.3. *Let $C(M)$ and $C(K)$ denote the circulant matrices of M_n and K_n. Then*

1. *The Euclidean norms are*

$$\|C(M)\|_E = \sqrt{\left\{\begin{array}{c} n+nX^2\left(\frac{\alpha^2-\alpha^{2n}}{1-\alpha^2}\right)+nY^2\left(\frac{\beta^2-\beta^{2n}}{1-\beta^2}\right)+nZ^2\left(\frac{\gamma^2-\gamma^{2n}}{1-\gamma^2}\right)\\ +nW^2\left(\frac{\delta^2-\delta^{2n}}{1-\delta^2}\right)\\ +2n\left\{\begin{array}{c} XY\frac{\alpha\beta-(\alpha\beta)^n}{1-\alpha\beta}+ZW\frac{\gamma\delta-(\gamma\delta)^n}{1-\gamma\delta}+XZ\frac{\alpha\gamma-(\alpha\gamma)^n}{1-\alpha\gamma}\\ +XW\frac{\alpha\delta-(\alpha\delta)^n}{1-\alpha\delta}+YZ\frac{\beta\gamma-(\beta\gamma)^n}{1-\beta\gamma}+YW\frac{\beta\delta-(\beta\delta)^n}{1-\beta\delta}\end{array}\right\}\end{array}\right\}}$$

$$\|C(K)\|_E = \sqrt{\left\{\begin{array}{c} 16n+n\left(\frac{\alpha^4-\alpha^{2n+2}}{1-\alpha^2}\right)+n\left(\frac{\beta^4-\beta^{2n+2}}{1-\beta^2}\right)+n\left(\frac{\gamma^4-\gamma^{2n+2}}{1-\gamma^2}\right)\\ +n\left(\frac{\delta^4-\delta^{2n+2}}{1-\delta^2}\right)+2n\left\{\begin{array}{c} \frac{\alpha\beta-(\alpha\beta)^n}{1-\alpha\beta}+\frac{\gamma\delta-(\gamma\delta)^n}{1-\gamma\delta}+\frac{\alpha\gamma-(\alpha\gamma)^n}{1-\alpha\gamma}\\ +\frac{\alpha\delta-(\alpha\delta)^n}{1-\alpha\delta}+\frac{\beta\gamma-(\beta\gamma)^n}{1-\beta\gamma}+\frac{\beta\delta-(\beta\delta)^n}{1-\beta\delta}\end{array}\right\}\end{array}\right\}}$$

2. *The maximum column sum matrix and the maximum row sum matrix norms are*

$$\|C(M)\|_1=\|C(M)\|_\infty=\frac{1}{3}(4M_{n-1}+3M_{n-2}+2M_{n-3}+M_{n-4}-1)$$

$$\|C(K)\|_1=\|C(K)\|_\infty=\frac{1}{3}(4K_n+3K_{n-1}+2K_{n-2}+K_{n-3}+2).$$

3. *The spectral norms are*

$$\begin{aligned}\|C(M)\|_2 &= \frac{1}{3}(4M_{n-1}+3M_{n-2}+2M_{n-3}+M_{n-4}-1)\\ \|C(K)\|_2 &= \frac{1}{3}(4K_n+3K_{n-1}+2K_{n-2}+K_{n-3}+2).\end{aligned}$$

Proof. 1. From the defination of the Euclidean norm, we get

$$\|C(M)\|_E^2 = n\sum_{i=0}^{n-1} M_i^2.$$

From (4), we obtain

$$\begin{aligned}\sum_{i=1}^{n-1} M_i^2 &= \sum_{i=1}^{n-1}\left(X\alpha^i + Y\beta^i + Z\gamma^i + W\delta^i\right)^2\\ &= X^2\sum_{i=1}^{n-1}\alpha^{2i} + Y^2\sum_{i=1}^{n-1}\beta^{2i} + Z^2\sum_{i=1}^{n-1}\gamma^{2i} + W^2\sum_{i=1}^{n-1}\delta^{2i}\\ &\quad +2XY\sum_{i=1}^{n-1}(\alpha\beta)^i + 2ZW\sum_{i=1}^{n-1}(\gamma\delta)^i + 2XZ\sum_{i=1}^{n-1}(\alpha\gamma)^i\\ &\quad +2XW\sum_{i=1}^{n-1}(\alpha\delta)^i + 2YZ\sum_{i=1}^{n-1}(\beta\gamma)^i + 2YW\sum_{i=1}^{n-1}(\beta\delta)^i\end{aligned}$$

Applying the fact that $\sum\limits_{k=1}^{j} t^k = \frac{t-t^{j+1}}{1-t}$, hence

$$\begin{aligned}\sum_{i=1}^{n-1} M_i^2 &= (X\alpha)^2\frac{1-\alpha^{2n-2}}{1-\alpha^2} + (Y\beta)^2\frac{1-\beta^{2n-2}}{1-\beta^2} + (Z\gamma)^2\frac{1-\gamma^{2n-2}}{1-\gamma^2}\\ &\quad +(W\delta)^2\frac{1-\delta^{2n-2}}{1-\delta^2} \qquad (7)\\ &\quad +2\left(\begin{array}{c} XY\frac{\alpha\beta-(\alpha\beta)^n}{1-\alpha\beta} + ZW\frac{\gamma\delta-(\gamma\delta)^n}{1-\gamma\delta} + XZ\frac{\alpha\gamma-(\alpha\gamma)^n}{1-\alpha\gamma}\\ +XW\frac{\alpha\delta-(\alpha\delta)^n}{1-\alpha\delta} + YZ\frac{\beta\gamma-(\beta\gamma)^n}{1-\beta\gamma} + YW\frac{\beta\delta-(\beta\delta)^n}{1-\beta\delta}\end{array}\right) \qquad (8)\end{aligned}$$

Therefore we get $\|C(M)\|_E^2 = n\left(\sum\limits_{i=1}^{n-1} M_i^2 + M_0^2\right) = n\left(\sum\limits_{i=1}^{n-1} M_i^2 + 1\right)$. $\|C(K)\|_E$ can be obtained similarly.

2. Since the circulant matrix $C(M)$ is normal, there exist a unitary matrix $P \in M_n$ such that $U^H C(M) U = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where λ_i is eigenvalue of $C(M)$. So

$$U^H (C(M))^H C(M) U = diag(|\lambda_1|^2, |\lambda_2|^2, \cdots, |\lambda_n|^2).$$

The circulant matrix $C(M)$ is given by its spectral radius. Since $C(M)$ is nonnegative, its spectral radius $\rho(C(M))$ satisfy

$$\min_{1\le i\le n}\sum_{j=1}^{n} a_{ij} \le \rho(C(M)) \le \max_{1\le i\le n}\sum_{j=1}^{n} a_{ij}$$

so

$$\sum_{j=1}^{n} a_{ij} = \sum_{l=4}^{n-1} M_l = \frac{1}{3}(4M_{n-1} + 3M_{n-2} + 2M_{n-3} + M_{n-4} - 1)$$

for any $i = 1, 2, \cdots, n$.

$$\|C(M)\|_2 = \frac{1}{3}(4M_{n-1} + 3M_{n-2} + 2M_{n-3} + M_{n-4} - 1).$$

The other assertion can be proved similarly.

∎

In the following theorem, we give the determinant and eigenvalues of circulant matrices with Tetranacci and Companion-Tetranacci numbers.

We can define the identities for the following theorems

$$\begin{aligned}
Q &= \alpha\beta\gamma\delta,\ P = \alpha+\beta+\gamma+\delta,\ N = X+Y+Z+W \\
R &= \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta,\ K = X\beta\gamma\delta+Y\alpha\gamma\delta+Z\alpha\beta\delta+W\alpha\beta\gamma \\
T &= X(\beta\gamma+\beta\delta+\gamma\delta)+Y(\beta\alpha+\alpha\delta+\gamma\delta)+Z(\beta\alpha+\alpha\delta+\beta\delta) \\
&\quad +W(\beta\alpha+\alpha\delta+\beta\gamma).
\end{aligned}$$

Theorem 2.4. *Let $C(M)$ and $C(K)$ denote the circulant matrices of M_n and K_n. Then*

1. *The eigenvalues are*

$$\lambda_j\left(C(M)\right) = \frac{\left\{\begin{array}{c} (QM_{n-1}-K)w^{-3j} + (M_n + PM_{n+1} - M_{n+2} + T)w^{-2j} \\ +(PM_n - M_{n+1} + M_1 - PN)w^{-j} - M_n + N \end{array}\right\}}{Qw^{-4j} - Rw^{-3j} - w^{-2j} - Pw^{-j} + 1}$$

$$\lambda_j\left(C(K)\right) = \frac{\left\{\begin{array}{c} (QK_{n-1}+R)w^{-3j} + (K_n + PK_{n+1} - K_{n+2} - 2)w^{-2j} \\ +((K_n-3)P - K_{n+1})w^{-j} - K_n + 4 \end{array}\right\}}{Qw^{-4j} - Rw^{-3j} + Sw^{-2j} + Pw^{-j} + 1}$$

2. *The determinants are*

$$\det(C(M)) = \frac{\left\{\begin{array}{c} -(K-QM_{n-1})^n+(-M_n+N)^n \\ +2^{1-n}\left(\frac{M_n+PM_{n+1}-M_{n+2}+T-(2QM_{n-1}-2K)(N-M_n)}{P(-M_n+N)+M_{n+1}-M_1}\right)^n \\ +\left(\frac{(2QM_{n-1}-2K)(N-M_n)}{P(-M_n+N)+M_{n+1}-M_1}\right)^n \\ -2^{1-2n}\left(P(-M_n+N)+M_{n+1}-M_1\right)^n \\ +2^{-n}\left(P(-M_n+N)+M_{n+1}-M_1\right)^n \end{array}\right\}}{1+Q^n+2^{1-3n}(P-4)^n-2^{2-2n}(P^n+R^n)+2^{2-4n}P^n}$$

$$\det(C(K)) = \frac{\left\{\begin{array}{c} -(-QK_{n-1}-R)^n \\ +(2^{-n}-2^{1-2n})(K_{n+1}-(K_n-3)P)^n \\ +2^{1-n}\left(\frac{K_n+PK_{n+1}-K_{n+2}-2-2(4-K_n)(-QK_{n-1}-R)}{K_{n+1}-(K_n-3)P}\right)^n \\ +2^n\left(\frac{(4-K_n)(-QK_{n-1}-R)}{K_{n+1}-(K_n-3)P}\right)^n+(4-K_n)^n \end{array}\right\}}{1+Q^n+2^{2-2n}(R^n+(-P)^n)+2^{1-3n}(4S-P)^n+2^{2-4n}(-P)^n}.$$

Proof. 1. For the sequence M_n, we have

$$\begin{aligned} \lambda_j(C(M)) &= \sum_{k=0}^{n-1} M_k w^{-jk} \\ &= \sum_{k=0}^{n-1}\left(X\alpha^k+Y\beta^k+Z\gamma^k+W\delta^k\right)w^{-jk} \\ &= X\left(\frac{(\alpha w^{-j})^n-1}{(\alpha w^{-j})-1}\right)+Y\left(\frac{(\beta w^{-j})^n-1}{(\beta w^{-j})-1}\right) \\ &\quad +Z\left(\frac{(\gamma w^{-j})^n-1}{(\gamma w^{-j})-1}\right)+W\left(\frac{(\delta w^{-j})^n-1}{(\delta w^{-j})-1}\right). \end{aligned}$$

Including $(\alpha w^{-j})^n=\alpha^n$, $(\beta w^{-j})^n=\beta^n$,$(\gamma w^{-j})^n=\gamma^n$,$(\delta w^{-j})^n=\delta^n$, the equation equals

$$\lambda_j(C(M)) = \frac{\left\{\begin{array}{c} X(\alpha^n-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1) \\ +Y(\beta^n-1)(\alpha w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1) \\ +Z(\gamma^n-1)(\alpha w^{-j}-1)(\beta w^{-j}-1)(\delta w^{-j}-1) \\ +W(\delta^n-1)(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1) \end{array}\right\}}{(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1)}.$$

For the given values we deduce the numarator

$$\left\{\begin{array}{c} X\alpha^n(\beta\gamma\delta)-X(\beta\gamma\delta)+Y\beta^n(\alpha\gamma\delta)-Y(\alpha\gamma\delta) \\ +Z\gamma^n(\alpha\beta\delta)-Z(\alpha\beta\delta)+W\delta^n(\alpha\beta\gamma)-W(\alpha\beta\gamma) \end{array}\right\}w^{-3j}$$
$$+\left\{\begin{array}{c} -X\alpha^n(\gamma\beta+\delta\beta+\gamma\delta)+X(\gamma\beta+\delta\beta+\gamma\delta) \\ -Y\beta^n(\gamma\alpha+\delta\alpha+\gamma\delta)+Y(\gamma\alpha+\delta\alpha+\gamma\delta) \\ -Z\gamma^n(\alpha\beta+\alpha\delta+\beta\delta)+Z(\alpha\beta+\alpha\delta+\beta\delta) \\ -W\delta^n(\alpha\beta+\alpha\gamma+\beta\gamma)+W(\alpha\beta+\alpha\gamma+\beta\gamma) \end{array}\right\}w^{-2j}$$
$$+\left\{\begin{array}{c} X\alpha^n(\beta+\gamma+\delta)-X(\beta+\gamma+\delta) \\ +Y\beta^n(\alpha+\gamma+\delta)-Y(\alpha+\gamma+\delta) \\ +Z\gamma^n(\alpha+\beta+\delta)-Z(\alpha+\beta+\delta) \\ +W\delta^n(\alpha+\beta+\gamma)-W(\alpha+\beta+\gamma) \end{array}\right\}w^{-j}$$
$$-X\alpha^n+X-Y\beta^n+Y-Z\gamma^n+Z-W\delta^n+W$$

and denominator

$$\begin{array}{l} \alpha\beta\gamma\delta w^{-4j}-\alpha\beta\gamma w^{-3j}-\alpha\beta\delta w^{-3j}+\alpha\beta w^{-2j}-\alpha\gamma\delta w^{-3j} \\ +\alpha\gamma w^{-2j}+\alpha\delta w^{-2j}-\alpha w^{-j}-\beta\gamma\delta w^{-3j}+\beta\gamma w^{-2j} \\ +\beta\delta w^{-2j}-\beta w^{-j}+\gamma\delta w^{-2j}-\gamma w^{-j}-\delta w^{-j}+1 \end{array}$$

for $\lambda_j\left(C(M)\right)$. For the given values

$$\lambda_j\left(C(M)\right)=\frac{\left\{\begin{array}{c} (QM_{n-1}-K)w^{-3j}+(M_n+PM_{n+1}-M_{n+2}+T)w^{-2j} \\ +(PM_n-M_{n+1}+M_1-PN)w^{-j}-M_n+N \end{array}\right\}}{Qw^{-4j}-Rw^{-3j}-w^{-2j}-Pw^{-j}+1}.$$

The other assertion can be proved similarly.

2. By considering the eigenvalue $\lambda_j C(M)$ and Lemma 1.2, we have

$$\begin{aligned} & \det(C(M)) \\ = & \prod_{j=0}^{n-1}\lambda_j(C(M)) \\ = & \prod_{j=0}^{n-1}\frac{\left\{\begin{array}{c} (QM_{n-1}-K)w^{-3j}+(M_n+PM_{n+1}-M_{n+2}+T)w^{-2j} \\ +(PM_n-M_{n+1}+M_1-PN)w^{-j}-M_n+N \end{array}\right\}}{Qw^{-4j}-Rw^{-3j}-w^{-2j}-Pw^{-j}+1} \\ = & \frac{\left\{\begin{array}{c} -(K-QM_{n-1})^n+2^{1-n}\left(\frac{M_n+PM_{n+1}-M_{n+2}+T-(2QM_{n-1}-2K)(N-M_n)}{P(-M_n+N)+M_{n+1}-M_1}\right)^n \\ +\left(\frac{(2QM_{n-1}-2K)(N-M_n)}{P(-M_n+N)+M_{n+1}-M_1}\right)^n-2^{1-2n}\left(P(-M_n+N)+M_{n+1}-M_1\right)^n \\ +2^{-n}\left(P(-M_n+N)+M_{n+1}-M_1\right)^n+(-M_n+N)^n \end{array}\right\}}{1+Q^n+2^{1-3n}(P-4)^n-2^{2-2n}(P^n+R^n)+2^{2-4n}P^n}. \end{aligned}$$

The other assertion can be proved similarly.

∎

Now we can give the following theorem for negacyclic matrices.

Theorem 2.5. *Let $N(M)$ and $N(K)$ denote the negacyclic matrices of M_n and K_n. Then*

1. *The eigenvalues are*

$$\lambda_j(N(M)) = \frac{\left\{ \begin{array}{c} (K-QM_{n-1})w^{\frac{6j+3}{2}} + (M_{n+1}-PM_n+M_1-PN)w^{\frac{2j+1}{2}} \\ +(M_{n+2}-M_n-PM_{n+1}-T)w^{2j+1}+M_n+N \end{array} \right\}}{Qw^{4j+2}-Rw^{\frac{6j+3}{2}}-w^{2j+1}-Pw^{\frac{2j+1}{2}}+1}$$

$$\lambda_j(N(K)) = \frac{\left\{ \begin{array}{c} (-R-QK_{n-1})w^{\frac{6j+3}{2}} + K_n w^{2j+1}+K_n+4 \\ (K_{n+1}-PK_n+K_1-4P)w^{\frac{2j+1}{2}} \end{array} \right\}}{Qw^{4j+2}-Rw^{\frac{6j+3}{2}}-w^{2j+1}-Pw^{\frac{2j+1}{2}}+1}$$

2. *The determinants are*

$$\det(N(M)) = \frac{(-1)^n \left\{ \begin{array}{c} (-M_n-N)^n - \left(2^{-n}-2^{1-2n}\right) \\ \times\left(-w(M_{n+2}-M_n-PM_{n+1}-T)\right)^n \\ -2^{1-n}\left(\frac{\left\{ \begin{array}{c} (\sqrt{w})^{-1}(M_{n+1}-PM_n+M_1-PN) \\ -2\sqrt{w}(K-QM_{n-1})(-M_n-N) \end{array}\right\}}{M_n+PM_{n-1}+T-M_{n+2}} \right)^n \\ -2^n\left(\frac{\sqrt{w}(K-QM_{n-1})(-M_n-N)}{M_n+PM_{n-1}+T-M_{n+2}}\right)^n - \sqrt{w^{3n}}(K-QM_{n-1})^n \end{array} \right\}}{\left\{ \begin{array}{c} 1^n+Q^n w^{2n}+2^{2-2n}\left((R\sqrt{w^3})^n+(P\sqrt{w})^n\right) \\ -2^{1-3n}\left(R(\sqrt{w})^{-1}Q^{-1}-4w\right)+2^{2-4n}\left(RQ^{-1}\sqrt{w})^{-1}\right)^n \end{array} \right\}}$$

$$\det(N(K)) = \frac{(-1)^n \left\{ \begin{array}{c} (-K_n-4)^n + \left(2^{-n}-2^{1-2n}\right)(-wK_n)^n \\ -2^{1-n}\left(\frac{\sqrt{w}(K_{n+1}-PK_n+K_1-4P)-2\sqrt{w^3}(-R-QK_{n-1})(-K_n-4)}{-wK_n}\right)^n \\ -2^n\left(\frac{\sqrt{w^3}(-R-QK_{n-1})(-K_n-4)}{-wK_n}\right)^n - \sqrt{w^{3n}}(-R-QK_{n-1})^n \end{array} \right\}}{\left\{ \begin{array}{c} 1^n+Q^n w^{2n}+2^{2-2n}\left((R\sqrt{w^3})^n+(P\sqrt{w})^n\right) \\ -2^{1-3n}\left(R(\sqrt{w})^{-1}Q^{-1}-4w\right)+2^{2-4n}\left(RQ^{-1}\sqrt{w})^{-1}\right)^n \end{array} \right\}}.$$

Proof. 1. Using binet formulas for the sequence M_n, we get

$$\begin{aligned}
\lambda_j(N(M)) &= \sum_{k=0}^{n-1} M_k w^{\frac{(2j+1)k}{2}} \\
&= \sum_{k=0}^{n-1} \left[X\alpha^k + Y\beta^k + Z\gamma^k + W\delta^k\right] w^{\frac{(2j+1)k}{2}} \\
&= X\left(\frac{(\alpha w^{\frac{2j+1}{2}})^n - 1}{\alpha w^{\frac{2j+1}{2}} - 1}\right) + Y\left(\frac{(\beta w^{\frac{2j+1}{2}})^n - 1}{\beta w^{\frac{2j+1}{2}} - 1}\right) \\
&\quad + Z\left(\frac{(\gamma w^{\frac{2j+1}{2}})^n - 1}{\gamma w^{\frac{2j+1}{2}} - 1}\right) + W\left(\frac{(\delta w^{\frac{2j+1}{2}})^n - 1}{\delta w^{\frac{2j+1}{2}} - 1}\right).
\end{aligned}$$

Note that $(\alpha w^{\frac{2j+1}{2}})^n = -\alpha^n$, $(\beta w^{\frac{2j+1}{2}})^n = -\beta^n$, $(\gamma w^{\frac{2j+1}{2}})^n = -\gamma^n$, $(\delta w^{\frac{2j+1}{2}})^n = -\delta^n$, so

$$\begin{aligned}
\lambda_j(N(M)) &= \frac{\left\{\begin{array}{c} (-X\alpha^n - X)(\beta w^{\frac{2j+1}{2}} - 1)(\gamma w^{\frac{2j+1}{2}} - 1)(\delta w^{\frac{2j+1}{2}} - 1) \\ +(-Y\beta^n - Y)(\alpha w^{\frac{2j+1}{2}} - 1)(\gamma w^{\frac{2j+1}{2}} - 1)(\delta w^{\frac{2j+1}{2}} - 1) \\ +(-Z\gamma^n - Z)(\alpha w^{\frac{2j+1}{2}} - 1)(\beta w^{\frac{2j+1}{2}} - 1)(\delta w^{\frac{2j+1}{2}} - 1) \\ +(-W\delta^n - W)(\alpha w^{\frac{2j+1}{2}} - 1)(\beta w^{\frac{2j+1}{2}} - 1)(\gamma w^{\frac{2j+1}{2}} - 1) \end{array}\right\}}{(\alpha w^{\frac{2j+1}{2}} - 1)(\beta w^{\frac{2j+1}{2}} - 1)(\gamma w^{\frac{2j+1}{2}} - 1)(\delta w^{\frac{2j+1}{2}} - 1)} \\
&= \frac{\left\{\begin{array}{c} M_n + N \\ + \left(-X\alpha^{n-1}Q - Y\beta^{n-1}Q - Z\gamma^{n-1}Q - W\delta^{n-1}Q - K\right) w^{\frac{6j+3}{2}} \\ + \left(\begin{array}{c} X\alpha^n(P-\alpha) + Y\beta^n(P-\beta) \\ +Z\gamma^n(P-\gamma) + W\delta^n(P-\delta) \end{array}\right) w^{\frac{2j+1}{2}} \\ + \left\{\begin{array}{c} -M_n - PM_{n+1} + M_{n+2} - X(\beta\gamma + \beta\delta + \gamma\delta) \\ -Y(\alpha\gamma + \alpha\delta + \gamma\delta) - Z(\alpha\beta + \alpha\delta + \beta\delta) \\ -W(\alpha\beta + \alpha\gamma + \beta\gamma) \end{array}\right\} w^{2j+1} \end{array}\right\}}{Qw^{4j+2} - Rw^{\frac{6j+3}{2}} - w^{2j+1} - Pw^{\frac{2j+1}{2}} + 1} \\
&= \frac{\left\{\begin{array}{c} (K - QM_{n-1})w^{6j+3} + (M_{n+1} - PM_n + M_1 - PN)w^{\frac{2j+1}{2}} \\ +(M_{n+2} - M_n - PM_{n+1} - T)w^{2j+1} + M_n + N \end{array}\right\}}{Qw^{4j+2} - Rw^{\frac{6j+3}{2}} - w^{2j+1} - Pw^{\frac{2j+1}{2}} + 1}.
\end{aligned}$$

The other assertion can be proved similarly.

2. From above case, we get

$$\det(N(M)) = \prod_{j=0}^{n-1} \lambda_j(N(M))$$

$$= \prod_{j=0}^{n-1} \frac{\left\{ \begin{array}{c} (K - QM_{n-1})w^{\frac{6j+3}{2}} + (M_{n+1} - PM_n + M_1 - PN)w^{\frac{2j+1}{2}} \\ +(M_{n+2} - M_n - PM_{n+1} - T)w^{2j+1} + M_n + N \end{array} \right\}}{Qw^{4j+2} - Rw^{\frac{6j+3}{2}} - w^{2j+1} - Pw^{\frac{2j+1}{2}} + 1}.$$

Then multiplying numerator and denominator with $\prod_{j=0}^{n-1} w^{-4j}$,we get

$$\det(N(M)) = \frac{\prod_{j=0}^{n-1} w^{-j} \prod_{j=0}^{n-1} \left\{ \begin{array}{c} w^{-3j}(M_n + N) \\ +((M_{n+1} - PM_n + M_1 - PN)\sqrt{w})w^{-2j} \\ +((M_{n+2} - M_n - PM_{n+1} - T)w)w^{-j} \\ +(K - QM_{n-1})\sqrt{w^3} \end{array} \right\}}{\prod_{j=0}^{n-1} w^{-4j} - (P\sqrt{w})w^{-3j} - ww^{-2j} - R\sqrt{w^3}w^{-j} + Qw^2}.$$

By applying Lemma 1.2, we get the desire result.
The other assertion can be proved similarly.

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References

[1] P. J. Davis,Circulant Matrices(John Wiley &Sons, New York, 1979).

[2] R. A. Horn. C.R. Matrix Analysis (Cambridge University Press, 1985).

[3] H. Karner, J.C. Schneid and W. Ueberhuber. Spectral decomposition of real circulant matrices, *Linear Algebra and Its Applications* **367**, (2003), 301–311.

[4] E.G. Kocer, Circulant, Negacyclic and Semicirculant Matrices with the Modified Pell, Jacobsthal-Lucas numbers. *Hacettepe Journal of Mathematics and Statistics* **36**(2)(2007), 133-142.

[5] T. Koshy. *Fibonacci and Lucas Numbers with Applications.* John Wiley and Sons, 2001.P.J. Davis, Circulant Matrices, John Wiley and Sons, 1979.

[6] P. Ribenboim. *My Numbers, My Friends, Popular Lectures on Number Theory,* Springer-Verlag, New York, Inc. 2000.

[7] S.Q. Shen, J.M. Cen, On the determinants and inverses of circulant matrices with the Fibonacci and Lucas numbers, *Applied Mathematics and Computation*, 217, (2011), 9790-9797.

[8] W. R. Spickerman & R. N. Joyner. Binet's Formula for the Recursive Sequence of Order k. Fibonacci Quarterly 22.4 (1984), 327-31.

[9] M.E. Waddill. *The Tetranacci Sequence and Generalizations.* Fibonacci Quarterly 30.1(1992), 9-20.

[10] Z. Yanpeng, S. Shon, S. Lee and D. Oh. *Determinant of the Generalized Lucas RSFMLR Circulant Matrices in Communication.* International Conference on Information Computing and Applications. Springer-Verlag Berlin, Heidelberg, Part I, CCIS 391, pp. (2013), 72–81.

[11] Z. Yanpeng, S. Shon. *Exact Inverse Matrices of Fermat and Mersenne Circulant Matrix.* Abstract and Applied Analysis. Vol. (2015), Article ID 760823, 10 pages.

[12] Z. Yanpeng, S. Shon. *Exact Determinants and Inverses of Generalized Lucas Skew Circulant Type Matrices.* Applied Mathematics and Computation. 270(2015), 105–113.

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