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APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

Explicit bounds on certain integral inequalities via conformable fractional calculus

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Abstract: In this paper, we present some explicit upper bounds for integral inequalities with the help of Katugampola-type conformable fractional calculus. The results have been obtained to cover the previous published studies for Gronwall–Bellman and Bihari like integral inequalities.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Mathematical Analysis

Keywords: integral inequality; conformable fractional differential equation; global existence

AMS subject classifications: 26D15; 26A51; 26A33; 26A42

1. Introduction and preliminaries

In the history of development calculus, integral inequalities have been thought of as a key factor in the theory of differential and integral equations. For instance, Gronwall, Bellman and Bihari have great contribution in the literature (Bellman, 1943; Bihari, 1965; Dragomir, 1987, 2002; Gronwall, 1919; Pachpatte, 1995). However, in non-integer order of situations, the bounds provided by the above authors are not feasible.

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Fuat Usta received his BSc (Mathematical Engineering) degree from Istanbul Technical University, Turkey in 2009 and MSc (Mathematical Finance) from the University of Birmingham, UK in 2011 and PhD (Applied Mathematics) from University of Leicester, UK in 2015. At present, he is working as an assistant professor in the Department of Mathematics Düzce University (Turkey). He is interested in the applications of RBFs in finance, especially practical high-dimensional approximation using sparse grid methods. His second research area is theory of inequalities.

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PUBLIC INTEREST STATEMENT

Differential and integral inequalities play a vital role in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential and integral equations. One can hardly imagine these theories without the well-known Gronwall inequality and its non-linear version Bihari inequality. In addition to this, fractional calculus has a number of fields of application such as control theory, computational analysis and engineering. Thus, a number of new definitions have been introduced in academia to provide the best method for fractional calculus. In this paper, we presented a retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities using the Katugampola conformable fractional calculus.

In addition to this, fractional calculus has a number of fields of application such as control theory, computational analysis and engineering (Kilbas, Srivastava, & Trujillo, 2006, see also Samko, Kilbas, & Marichev, 1993). Thus, a number of new definitions have been introduced in academia to provide the best method for fractional calculus. For instance, in more recent times, a new local, limit-based definition of a conformable derivative has been introduced in Abdeljawad (2015), Khalil, Al horani, Yousef, and Sababheh (2014), Katugampola (2014), with several follow-up papers (Anderson & Ulness, *in press*; Atangana, Baleanu, & Alsaedi, 2015; Hammad & Khalil, 2014a, 2014b; Iyiola & Nwaeze, 2016; Sarikaya, 2016; Usta & Sarikaya, 2016; Zheng, Feng, & Wang, 2015).

In this study, we presented a retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities using the Katugampola conformable fractional calculus. In detail, Katugampola conformable derivatives for $\alpha \in (0, 1]$ and $t \in [0, \infty)$ given by

$$D^\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \quad D^\alpha(f)(0) = \lim_{t \rightarrow 0} D^\alpha(f)(t), \tag{1.1}$$

provided the limits exist (for detail see, Katugampola, 2014). If f is fully differentiable at t , then

$$D^\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{1.2}$$

A function f is α -differentiable at a point $t \geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results.

THEOREM 1 Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $t > 0$. Then,

- (i) $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$, for all $a, b \in \mathbb{R}$,
- (ii) $D^\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,
- (iii) $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$,
- (iv) $D^\alpha\left(\frac{f}{g}\right) = \frac{fD^\alpha(g) - gD^\alpha(f)}{g^2}$ where $g(t) \neq 0$,
- (v) $D^\alpha(t^n) = nt^{n-\alpha}$ for all $n \in \mathbb{R}$,
- (vi) $D^\alpha(f \circ g)(t) = f'(g(t))D^\alpha(g)(t)$ for f is differentiable at $g(t)$.

Definition 1 (Conformable fractional integral) Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f: [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_\alpha x = \int_a^b f(x) x^{\alpha-1} dx$$

exists and is finite. All α -fractional integrable on $[a, b]$ is indicated by $L_\alpha^1([a, b])$

Remark 1

$$I_\alpha^a(f)(t) = I_1^\alpha(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

We will also use the following important results, which can be derived from the results above.

LEMMA 1 Let the conformable differential operator D^α be given as in (1.1), where $\alpha \in (0, 1]$ and $t \geq 0$, and assume the functions f and g are α -differentiable as needed. Then,

- (i) $D^\alpha(\ln t) = t^{-\alpha}$ for $t > 0$
- (ii) $D^\alpha \left[\int_a^t f(t, s) d_\alpha s \right] = f(t, t) + \int_a^t D^\alpha [f(t, s)] d_\alpha s$
- (iii) $\int_a^b f(x) D^\alpha(g)(x) d_\alpha x = fg|_a^b - \int_a^b g(x) D^\alpha(f)(x) d_\alpha x.$

In this paper, using the Katugampola-type conformable fractional calculus, we introduced retarded Gronwall–Bellman- and Bihari-like conformable fractional integrals inequalities.

2. Main findings and cumulative results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M, S)$ and $C^1(M, S)$ denote the class of all continuous functions and the first-order conformable derivative, respectively, defined on set M with range in the set S . Additionally, \mathbb{R} denotes the set of real numbers such that $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^1 = [1, \infty)$ and $\mathbb{Q} = [0, T)$ are the given subset of \mathbb{R} .

THEOREM 2 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies

$$u(t) \leq m + \int_0^t x(s)u(s) d_\alpha s + \int_0^{r(t)} y(s)u(s) d_\alpha s, \quad t \in \mathbb{Q}, \tag{2.1}$$

where $m \geq 0$ is constant, then

$$u(t) \leq me^{X(t)+Y(t)}$$

where

$$X(t) = \int_0^t x(s) d_\alpha s, \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s. \tag{2.2}$$

Proof Let us first assume that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.1). Then, $u(t) \leq z(t)$ and $z(0) = m$, and

$$\begin{aligned} D^\alpha z(t) &= x(t)u(t) + y(r(t))u(r(t))D^\alpha r(t) \\ &\leq x(t)z(t) + y(r(t))z(r(t))D^\alpha r(t) \\ &\leq x(t)z(t) + y(r(t))z(t)D^\alpha r(t) \end{aligned}$$

as $r(t) \leq t$. Then, the solution of the above fractional order differential equation by taking integration from 0 to t , we get

$$z(t) \leq me^{\int_0^t x(s) d_\alpha s + \int_0^{r(t)} y(s) d_\alpha s}$$

Since $u(t) \leq z(t)$, we get the desired inequality, that is

$$u(t) \leq me^{X(t)+Y(t)}$$

where

$$X(t) = \int_0^t x(s) d_\alpha s, \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s \tag{2.3}$$

□

THEOREM 3 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{Q}, \mathbb{R}^1)$ satisfies

$$u(t) \leq n + \int_0^t x(s)u(s) \log(u(s))d_\alpha s + \int_0^{r(t)} y(s)u(s) \log(u(s))d_\alpha s, \quad t \in \mathbb{Q}, \quad (2.4)$$

where $n \geq 1$ is constant, then

$$u(t) \leq n e^{X(t)+Y(t)}$$

where $X(t)$ and $Y(t)$ are defined in (2.3).

Proof Let us first assume that $n > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.4). Then, $u(t) \leq z(t)$ and $z(0) = n$, and as in the same steps with above proof, we get

$$D^\alpha z(t) \leq x(t)z(t) \log z(t) + y(r(t))z(t) \log z(r(t))D^\alpha r(t)$$

Then, the solution of the above fractional order differential equation by taking integration from 0 to t , we get

$$\log z(t) \leq \log n + \int_0^t x(s) \log z(s) d_\alpha s + \int_0^{r(t)} y(s) \log z(s) d_\alpha s$$

Now using the result of Theorem 2, we obtain

$$\log z(t) \leq (\log n) e^{X(t)+Y(t)} \quad (2.5)$$

In other words, we get

$$z(t) \leq n e^{X(t)+Y(t)} \quad (2.6)$$

Since $u(t) \leq z(t)$, we get the desired inequality, that is

$$u(t) \leq n e^{X(t)+Y(t)}$$

where $X(t)$ and $Y(t)$ are defined in (2.3). □

THEOREM 4 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies

$$u^q(t) \leq m + \int_0^t x(s)u(s)d_\alpha s + \int_0^{r(t)} y(s)u(s)d_\alpha s, \quad t \in \mathbb{Q}, \quad (2.7)$$

where $m \geq 0$ and $q > 1$ are constant, then

$$u(t) \leq \left(m^{\frac{q-1}{q}} + \frac{q-1}{q} [X(t) + Y(t)] \right)^{\frac{1}{q-1}}$$

where $X(t)$ and $Y(t)$ are defined in 2.3.

Proof Let us first assume that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.7). Then, $u^q(t) \leq z(t)$ and $z(0) = m$, and as in the same steps with the above proof, we get

$$D^\alpha z(t) \leq x(t)z^{1/q}(t) + y(r(t))z^{1/q}(t)D^\alpha r(t)$$

Then, the solution of the above fractional order differential equation by taking integration from 0 to t , we get

$$z(t) \leq \left(m^{\frac{q-1}{q}} + \frac{q-1}{q} \left[\int_0^t x(s) d_\alpha s + \int_0^{r(t)} y(s) d_\alpha s \right] \right)^{\frac{q}{q-1}}$$

Since $u^q(t) \leq z(t)$, we get the desired inequality, that is

$$u(t) \leq \left(m^{\frac{q-1}{q}} + \frac{q-1}{q} [X(t) + Y(t)] \right)^{\frac{1}{q-1}}$$

where $X(t)$ and $Y(t)$ are defined in 2.3. □

THEOREM 5 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, assume that r and ψ are non-decreasing with $r(t) \leq t$ for $t \geq 0$ and $\psi_i(\xi) > 0$ for $\xi > 0$, respectively. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies

$$u(t) \leq m + \int_0^t x(s) \psi_1(u(s)) d_\alpha s + \int_0^{r(t)} y(s) \psi_2(u(s)) d_\alpha s, \quad t \in \mathbb{Q}, \tag{2.8}$$

where $m \geq 0$ is constant, then

$$z(t) \leq \mathcal{G}^{-1}(\mathcal{G}(m) + X(t) + Y(t)) \tag{2.9}$$

where

$$X(t) = \int_0^t x(s) d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s \tag{2.10}$$

and \mathcal{G}^{-1} is the inverse function defined by

$$\mathcal{G}^{-1}(\xi) = \int_0^\xi \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s$$

so that

$$\mathcal{G}(m) + X(t) + Y(t) \in \text{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof Let us first suppose that $m > 0$. Define the non-decreasing positive function $z(t)$ by the right-hand side of (2.8). Then, $u(t) \leq z(t)$ and $z(0) = m$, and as in the same steps with the above proofs, we get

$$\begin{aligned} D^\alpha z(t) &\leq x(t) \psi_1(z(t)) + y(r(t)) \psi_2(z(t)) y(r(t)) \\ &\leq \max(\psi_1(z(t)), \psi_2(z(t))) [x(t) + y(r(t)) D^\alpha r(t)] \end{aligned}$$

Then, from the definition of \mathcal{G} , we have

$$\mathcal{G}(z(t)) = \int_0^{z(t)} \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s. \tag{2.11}$$

Then, taking α -th order of conformable derivative of $\mathcal{G}(z(t))$, we obtain

$$\begin{aligned} D^\alpha \mathcal{G}(z(t)) &= \frac{1}{\max(\psi_1(z(t)), \psi_2(z(t)))} D^\alpha z(t) \\ &\leq x(t) + y(r(t)) D^\alpha r(t) \end{aligned}$$

Then, by taking integration from 0 to t , we get

$$\mathcal{G}(z(t)) \leq \mathcal{G}(m) + \int_0^t x(s) d_\alpha s + \int_0^{r(t)} y(s) d_\alpha s. \quad (2.12)$$

Because $\mathcal{G}^{-1}(z(t))$ is increasing on $\text{Dom}(\mathcal{G}^{-1}(z(t)))$, we get

$$z(t) \leq \mathcal{G}^{-1} \left(\mathcal{G}(m) + \int_0^t x(s) d_\alpha s + \int_0^{r(t)} y(s) d_\alpha s \right) \quad (2.13)$$

As $u(t) \leq z(t)$, we get the required inequality. \square

THEOREM 6 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, assume that r and ψ are non-decreasing with $r(t) \leq t$ for $t \geq 0$ and $\psi_i(\xi) > 0$ for $\xi > 0$, respectively. If $u \in C(\mathbb{Q}, \mathbb{R}^1)$ satisfies

$$u(t) \leq n + \int_0^t x(s) u(s) \psi_1(\log(u(s))) d_\alpha s + \int_0^{r(t)} y(s) u(s) \psi_2(\log(u(s))) d_\alpha s, \quad t \in \mathbb{Q}, \quad (2.14)$$

where $n \geq 1$ is constant, then

$$z(t) \leq e^{\mathcal{G}^{-1}(\mathcal{G}(\log(n)) + X(t) + Y(t))} \quad (2.15)$$

where

$$X(t) = \int_0^t x(s) d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s \quad (2.16)$$

and \mathcal{G}^{-1} is the inverse function of

$$\mathcal{G}^{-1}(\xi) = \int_0^\xi \frac{1}{\max(\psi_1(s), \psi_2(s))} d_\alpha s$$

so that

$$\mathcal{G}(\log(n)) + X(t) + Y(t) \in \text{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof The proof of Theorem 6 can be done following the similar steps of proof of Theorems 5 and 3. \square

THEOREM 7 Let $x, y \in C(\mathbb{Q}, \mathbb{R}^+)$, $r \in C^1(\mathbb{Q}, \mathbb{Q})$, $\psi_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, assume that r and ψ are non-decreasing with $r(t) \leq t$ for $t \geq 0$ and $\psi_i(u) > 0$ for $u > 0$, respectively. If $u \in C(\mathbb{Q}, \mathbb{R}^+)$ satisfies

$$u(t)^q \leq m + \int_0^t x(s) \psi_1(u(s)) d_\alpha s + \int_0^{r(t)} y(s) \psi_2(u(s)) d_\alpha s, \quad t \in \mathbb{Q}, \quad (2.17)$$

where $m \geq 0$ and $q > 1$ are constant, then

$$z(t) \leq (\mathcal{G}^{-1}(\mathcal{G}(m) + X(t) + Y(t)))^{1/q} \quad (2.18)$$

where

$$X(t) = \int_0^t x(s) d_\alpha s \quad Y(t) = \int_0^{r(t)} y(s) d_\alpha s \quad (2.19)$$

and \mathcal{G}^{-1} is the inverse function of

$$\mathcal{G}^{-1}(\xi) = \int_0^{\xi} \frac{1}{\max(\psi_1(s^{1/q}), \psi_2(s^{1/q}))} d_a s$$

so that

$$\mathcal{G}(m) + X(t) + Y(t) \in \text{Dom}(\mathcal{G}^{-1})$$

for all $t > 0$.

Proof The proof of Theorem 7 can be done following the similar steps of proof of Theorems 5 and 4. □

3. Concluding remark

In this study, we established the explicit bounds on retarded integral inequalities with the help of conformable fractional calculus. We take the advantage of Katugampola-type conformable fractional derivatives and integrals.

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References

- Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of Computational and Applied Mathematics*, 279, 57–66.
- Anderson, D. R., & Ulness, D. J. (in press). *Results for conformable differential equations*.
- Atangana, A., Baleanu, D., & Alsaedi, A. (2015). New properties of conformable derivative. *Open Mathematics*, 13, 889–898.
- Bellman, R. (1943). The stability of the solution of linear differential equations. *Duke Mathematical Journal*, 10, 643–647.
- Bihari, I. (1965). OA generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Mathematica Hungarica*, 7, 81–94.
- Dragomir, S. S. (1987). On Volterra integral equations with kernels of (L)-type. *Ann. Univ. Timisoara Facult de Math. Infor*, 25, 21–41.
- Dragomir, S. S. (2002). *Some Gronwall type inequalities and applications, RGMIA monographs*. Victoria University, Australia.
- Gronwall, T. H. (1919). Note on derivatives with respect to a parameter of the solutions of a system of differential equations. *Annals of Mathematics*, 20, 292–296.
- Hammad, M. A., & Khalil, R. (2014a). Abel's formula and wronskian for conformable fractional differential equations. *International Journal of Differential Equations and Applications*, 13, 177–183.
- Hammad, M. A., & Khalil, R. (2014b). Conformable fractional heat differential equations. *International Journal of Differential Equations and Applications*, 13, 177–183.
- Iyiola, O. S., & Nwaeze, E. R. (2016). Some new results on the new conformable fractional calculus with application using D'Alambert approach. *Progress in Fractional Differentiation and Applications*, 2, 115–122.
- Katugampola, U. (2014). A new fractional derivative with classical properties. ArXiv:1410.6535v2.
- Khalil, R., Al horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational Applied Mathematics*, 264, 65–70.
- Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Amsterdam: Elsevier B.V.
- Pachpatte, B. G. (1995). On some new inequalities related to certain inequalities in the theory of differential equations. *Journal of Mathematical Analysis and Applications*, 189, 128–144.
- Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional integrals and derivatives: Theory and applications*. Gordonand Breach: Yverdon et alibi.
- Sarikaya, M. Z. (2016). Gronwall type inequality for conformable fractional integrals. *RGMIA Research Report Collection*, 19. Article 122.
- Usta, F., & Sarikaya, M. Z. (2016). On generalization conformable fractional integral inequalities. *RGMIA Research Report Collection*, 19. Article 123.
- Zheng, A., Feng, Y., & Wang, W. (2015). The Hyers-Ulam stability of the conformable fractional differential equation. *Mathematica Aeterna*, 5, 485–492.



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