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# New generalization of midpoint type inequalities for fractional integral

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**Abstract** In this paper, we firstly obtain a new generalized identity for Riemann-Liouville fractional integrals. Then, utilizing this equality, we obtain some Midpoint type inequalities for convex and concave functions. We also give several remarks and corollaries as spacial cases.

**Keywords** Hermite-Hadamard inequality · fractional integral operators · convex function · concave function

**Mathematics Subject Classification (2010)** 26D15 · 26B25 · 26D10

## 1 Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention and much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [18, p.137], [6]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if $f$ is concave.

In [15], U. S. Kırmacı gives the following identity and using this identity, obtains some bounds for the left hand side of the inequality (1.1)

**Lemma 1.1** *Let $f : I^\circ \to \mathbb{R}$ be differentiable function on $I^\circ$, $a, b \in I^\circ$ ($I^\circ$ is interior of $I$) with $a < b$. If $f' \in L^1[a, b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \tag{1.2}$$

$$= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (1-t)\,f'(ta+(1-t)b)dt\right].$$

Over the last twenty years, the numerous studies have focused on to obtain new bound for left hand side and right side of the inequality (1.1). For some examples, please refer to ([1], [3], [4], [6], [7], [20], [21], [23], [24], [29]).

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

**Definition 1.2** *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

*and*

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

*respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.*

It is remarkable that Sarikaya et al. [26] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 1.3** *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\,(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

*with $\alpha > 0$.*

Sarıkaya and Yıldırım also give the following Hermite-Hadamard type inequality for the Riemann-Liouville fractional integrals in [25].

**Theorem 1.4** *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If $f$ is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

Whereupon Sarikaya et al. obtain the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, many authors have studied to generalize this inequality and establish Hermite-Hadamard inequality other fractional integrals such as $k$-fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, Conformable fractional integrals, etc. For some of them, please see ([2], [5], [8], [10]-[13], [17], [19], [27], [28], [30]-[33]). For more information about fraction calculus please refer to ([9], [14], [16], [22]).

In the following section, we establish some new generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

## 2 Generalized midpoint inequalities for Riemann-Liouville fractional integral operators

First, we give the following lemma which will be used frequently later.

**Lemma 2.1** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a<b$. If $f'\in L^1[a,b]$, then we have the following equality for fractional integrals*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big)-f(a+b-x)\\&=\frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}f'(t(a+b-x)+(1-t)b)\,dt\\&\quad-\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}f'(t(a+b-x)+(1-t)a)\,dt\end{aligned} \tag{2.1}$$

*for all $x\in[a,b]$.*

*Proof.* By using the integration by parts, we obtain

$$I_1 = \int_0^1 t^\alpha f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)dt$$

$$= \frac{t^\alpha}{(b-x)} f(t\left(a+b-x\right)+(1-t)\,a)\Big|_0^1 - \frac{\alpha}{b-x}\int_0^1 t^{\alpha-1} f\left(t\left(a+b-x\right)+(1-t)\,a\right)dt$$

$$= \frac{1}{(b-x)} f\left(a+b-x\right) - \frac{\alpha}{b-x}\int_0^1 t^{\alpha-1} f\left(t\left(a+b-x\right)+\left(1-t\right)a\right)dt.$$

Using the change of variable, we have

$$I_1 = -\frac{\Gamma\left(\alpha+1\right)}{\left(b-x\right)^{\alpha+1}} J^\alpha_{(a+b-x)-} f\left(a\right) + \frac{1}{(b-x)} f\left(a+b-x\right). \tag{2.2}$$

Similarly, we establish

$$I_2 = \int_0^1 t^\alpha f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)dt \tag{2.3}$$

$$= \frac{\Gamma\left(\alpha+1\right)}{\left(x-a\right)^{\alpha+1}} J^\alpha_{(a+b-x)+} f\left(b\right) - \frac{1}{(x-a)} f\left(a+b-x\right).$$

From the equalities (2.2) and (2.3), we obtain

$$\frac{-\left(b-x\right)^2}{b-a} I_1 + \frac{\left(x-a\right)^2}{b-a} I_2$$

$$= \frac{\Gamma(\alpha+1)}{b-a}\Big(\left(b-x\right)^{1-\alpha} J^\alpha_{(a+b-x)-} f\left(a\right) + \left(x-a\right)^{1-\alpha} J^\alpha_{(a+b-x)+} f\left(b\right)\Big) - f(a+b-x)$$

which completes the proof.

□

**Theorem 2.2** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $0 \le a < b$ and $f' \in L^1[a,b]$. If $|f'|$ is convex on $[a,b]$, then we have the following inequality for fractional integrals*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\Big(\left(b-x\right)^{1-\alpha} J^\alpha_{(a+b-x)-} f(a) + \left(x-a\right)^{1-\alpha} J^\alpha_{(a+b-x)^+} f(b)\Big) - f(a+b-x)\right|$$

$$\le \frac{1}{(b-a)(\alpha+2)}\left[\frac{\left(b-x\right)^2\left|f'(a)\right| + \left(x-a\right)^2\left|f'(b)\right|}{\alpha+1} + \left[\left(x-a\right)^2 + \left(b-x\right)^2\right]\left|f'(a+b-x)\right|\right]$$

for all $x \in [a,b]$.

*Proof.* Taking the modulus in Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)^{+}}f(b)\Big)-f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|dt \\
&\quad+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|dt.
\end{aligned}$$

Since $\left|f'\right|$ is convex, we get

$$\begin{aligned}
&\left|\frac{\Gamma\left(\alpha+1\right)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)^{+}}f(b)\Big)-f(a+b-x)\right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1\left[t^{\alpha+1}\left|f'(a+b-x)\right|+t^{\alpha}(1-t)\left|f'(b)\right|\right]dt \\
&\quad+\frac{(b-x)^2}{b-a}\int_0^1\left[t^{\alpha+1}\left|f'(a+b-x)\right|+t^{\alpha}(1-t)\left|f'(a)\right|\right]dt \\
&=\frac{(x-a)^2}{b-a}\left[\left|f'(a+b-x)\right|\int_0^1 t^{\alpha+1}dt+\left|f'(b)\right|\int_0^1 t^{\alpha}(1-t)dt\right] \\
&\quad+\frac{(b-x)^2}{b-a}\left[\left|f'(a+b-x)\right|\int_0^1 t^{\alpha+1}dt+\left|f'(a)\right|\int_0^1 t^{\alpha}(1-t)dt\right] \\
&=\frac{1}{(b-a)(\alpha+2)}\left[\frac{(b-x)^2\left|f'(a)\right|+(x-a)^2\left|f'(b)\right|}{\alpha+1}+\left[(x-a)^2+(b-x)^2\right]\left|f'(a+b-x)\right|\right].
\end{aligned}$$

This completes the proof. □

**Corollary 2.3** *Under assumptions of Theorem 2.2 with* $x = \frac{a+b}{2}$, *we have the following inequality*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\left(\alpha+2\right)}\left[\frac{\left|f^{'}(a)\right|+\left|f^{'}(b)\right|}{\alpha+1}+2\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right] \\
&\leq \frac{b-a}{4(\alpha+1)}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]
\end{aligned}$$

*which is given by Sarikaya and Yıldırım in [25, Theorem 5 (for* $q = 1$*)].*

**Corollary 2.4** *If we choose* $\alpha = 1$ *in Theorem 2.2, then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(t)dt-f(a+b-x)\right| \\
&\leq \frac{1}{3(b-a)}\left[\frac{(b-x)^2\left|f'(a)\right|+(x-a)^2\left|f'(b)\right|}{2}+\left[(x-a)^2+(b-x)^2\right]\left|f^{'}(a+b-x)\right|\right].
\end{aligned}$$

**Remark 2.1** If we take $x = \frac{a+b}{2}$ in Corollary 2.4, then we obtain

$$\left|\frac{1}{b-a}\int_{a}^{b}f(t)dt-f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$

which is given by Kırmacı in [15].

**Theorem 2.5** *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* $(a,b)$ *with* $0 \leq a < b$ *and* $f' \in L^1[a,b]$. *If* $|f'|^q$, $q > 1$, *is convex on* $[a,b]$, *then we have the following inequality for fractional integrals*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right| \\
&\leq \frac{1}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[\frac{|f'(a+b-x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}}\right. \\
&\quad\left.+(b-x)^2\left[\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right]
\end{aligned}$$

*for all* $x \in [a,b]$, *where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* By the Lemma 2.1, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]\right. \tag{2.4}$$
$$\left.-f(a+b-x)\right|$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|dt$$
$$+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|dt.$$

Using the Hölder's inequality and convexity of $|f'|^q$, we obtain

$$\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|dt \tag{2.5}$$
$$\leq \left(\int_0^1 |t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|^q\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 \left(t^{\alpha p}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[t\left|f'\left(a+b-x\right)\right|^q+\left(1-t\right)\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(a+b-x\right)\right|^q+\left|f'\left(b\right)\right|^q}{2}\right)^{\frac{1}{q}}$$

Similarly, we have

$$\int_0^1 t^{\alpha}\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|dt \tag{2.6}$$
$$\leq \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(a+b-x\right)\right|^q+\left|f'\left(a\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (2.5) and (2.6) in (2.4), then we obtain the desired result.

□

**Corollary 2.6** *Under assumption of Theorem 2.5 with* $x = \frac{a+b}{2}$*, we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(J^{\alpha}_{(\frac{a+b}{2})^{-}}f(a)+J^{\alpha}_{(\frac{a+b}{2})^{+}}f(b)\right)-f\left(\frac{a+b}{2}\right)\right| \tag{2.7}$$

$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].$$

*Proof.* The proof of the first inequality in (2.7) is obvious from convexity of $|f'|^q$. For the proof of second inequality, let $a_1 = 3|f'(a)|^q, b_1 = 3|f'(b)|^q$ and $a_2 = |f'(a)|^q, b_2 = |f'(a)|^q$. Using the fact that

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s\ ,\ 0\leq s<1,$$

the desired result can be obtained straightforwardly.

□

**Remark 2.2** If we choose $\alpha = 1$ in Theorem 2.5, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(a+b-x)\right|$$

$$\leq \frac{1}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left[\frac{|f'(a+b-x)|^q+|f'(a)|^q}{2}\right]^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^2\left[\frac{|f'(a+b-x)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right]$$

which is proved by Qaisar and Hussain in [23, Theorem 4].

**Theorem 2.7** *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable mapping on* $(a,b)$ *with* $0\leq a<b$ *and* $f'\in L^1[a,b]$*. If* $|f'|^q$*,* $q\geq 1$*, is convex on* $[a,b]$*, then we have the following inequality for fractional integrals*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\Big[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big]-f(a+b-x)\right|$$

$$\leq \frac{1}{b-a}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg[(x-a)^2\left(\frac{1}{\alpha+2}\left|f'(a+b-x)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$

$$+(b-x)^2\left(\frac{1}{\alpha+2}\left|f'(a+b-x)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q\right)^{1-\frac{1}{q}}\Bigg]$$

for all $x\in[a,b]$

*Proof.* By the Lemma 2.1 and the power mean inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\Big[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big]-f(a+b-x)\right| \tag{2.8}$$

$$\leq \frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|dt$$

$$+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|dt$$

$$\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.$$

Using the convexity of $|f'|^q$, we obtain

$$\int_0^1 t^{\alpha}\left|f'(t(a+b-x)+(1-t)b)\right|^q dt$$

$$\leq \int_0^1 t^{\alpha}\left[t\left|f'(a+b-x)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt$$

$$=\frac{1}{\alpha+2}\left|f'(a+b-x)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q$$

and similary we have

$$\int_0^1 t^\alpha \left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|^q dt$$

$$\leq \frac{1}{\alpha+2}\left|f'\left(a+b-x\right)\right|^q + \frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)}\left|f'(a)\right|^q.$$

This completes the proof.

□

**Corollary 2.8** *Under assumption of Theorem 2.7 with* $x = \frac{a+b}{2}$, *we have the following inequality*

$$\left|\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)_-}f\left(a\right)+J^\alpha_{\left(\frac{a+b}{2}\right)_+}f\left(b\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{\alpha+12}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{\left(\alpha+1\right)\left(\alpha+2\right)}\left|f'\left(a\right)\right|^p\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{1}{2\left(\alpha+2\right)}\left|f'\left(a\right)\right|^q+\frac{\alpha+3}{2\left(\alpha+1\right)\left(\alpha+2\right)}\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2\left(\alpha+2\right)}\left|f'\left(b\right)\right|^q+\frac{\alpha+3}{2\left(\alpha+1\right)\left(\alpha+2\right)}\left|f'\left(a\right)\right|^q\right)^{\frac{1}{q}}\right].$$

**Remark 2.3** If we choose $\alpha = 1$ in Theorem 2.7, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(a+b-x)\right|$$

$$\leq \frac{1}{2\left(b-a\right)}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(x-a\right)^2\left(2\left|f'\left(a+b-x\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(b-x\right)^2\left(2\left|f'\left(a+b-x\right)\right|^q+\left|f'(a)\right|^p\right)^{1-\frac{1}{q}}\right]$$

which is proved by Qaisar and Hussain in [23, Theorem 5].

**Theorem 2.9** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a differentiable mapping on $(a,b)$ with $0\leq a<b$ and $f'\in L^1[a,b]$. If $|f'|^q$, $q>1$, is concave on $[a,b]$, then we have the following inequality for fractional integrals*

$$\left|\frac{\Gamma(\alpha+1)}{b-a}\left[(b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\right]-f(a+b-x)\right|$$
$$\leq\frac{1}{b-a}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left|f'\left(\frac{a+2b-x}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{2a+b-x}{2}\right)\right|\right]$$

*for all $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* By Lemma 2.1 and the Hölder inequality, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big)+f(a+b-x)\\
&\leq\frac{(x-a)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t(a+b-x)+(1-t)b\right)\right|dt \qquad (2.9)\\
&\quad+\frac{(b-x)^2}{b-a}\int_0^1 t^{\alpha}\left|f'\left(t(a+b-x)+(1-t)a\right)\right|dt\\
&\leq\frac{(x-a)^2}{b-a}\left(\int_0^1(t^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t(a+b-x)+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1(t^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t(a+b-x)+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is concave on $[a,b]$, by using Jensen integrals inequality, we obtain

$$\int_0^1 \left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|^q dt \tag{2.10}$$

$$\leq \left|f'\left(\int_0^1 \left(t\left(a+b-x\right)+\left(1-t\right)b\right)dt\right)\right|^q$$

$$= \left|f'\left(\frac{a+2b-x}{2}\right)\right|^q$$

and similarly,

$$\int_0^1 \left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|^q dt \leq \left|f'\left(\frac{2a+b-x}{2}\right)\right|^q. \tag{2.11}$$

By using the inequality (2.10) and (2.11) and the equality

$$\int_0^1 (t^\alpha)^p \, dt = \frac{1}{\alpha p+1},$$

in (2.9), we obtain the desired result.

□

**Corollary 2.10** *Under assumptions of Theorem 2.9, if we choose* $x = \frac{a+b}{2}$, *then we have the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)_-}f\left(b\right)+J^\alpha_{\left(\frac{a+b}{2}\right)_+}f\left(a\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{b-a}{4}\right)\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right].$$

**Remark 2.4** If we choose $\alpha = 1$ in Theorem 2.9, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(a+b-x)\right|$$

$$\leq \frac{1}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(x-a)^2\left|f'\left(\frac{a+2b-x}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{2a+b-x}{2}\right)\right|\right]$$

which is proved by Qaisar and Hussain in [23, Theorem 7].

**Theorem 2.11** *Let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on (a,b) with $0\le a<b$ and $f'\in L^1[a,b]$. If $|f'|^q$, $q\ge 1$, is concave on $[a,b]$, then we have the following inequality for fractional integrals*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big)+f(a+b-x)\right|\\
&\le \frac{1}{b-a}\left(\frac{1}{\alpha+1}\right)\left[(x-a)^2\left|f'\left(\frac{(\alpha+1)}{(\alpha+2)}(a-x)+b\right)\right|\right.\\
&\quad\left.+(b-x)^2\left|f'\left(\frac{(\alpha+1)}{(\alpha+2)}(b-x)+a\right)\right|\right]
\end{aligned}$$

for all $x\in[a,b]$

*Proof.* From the inequality (2.8) we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{b-a}\Big((b-x)^{1-\alpha}J^{\alpha}_{(a+b-x)-}f(a)+(x-a)^{1-\alpha}J^{\alpha}_{(a+b-x)+}f(b)\Big)+f(a+b-x)\right|\\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^{\alpha}|\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1|t^{\alpha}|\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By the Jensen inequality, we obtain

$$\begin{aligned}
&\int_0^1|t^{\alpha}|\left|f'\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right|^q dt\\
&\le\left(\int_0^1 t^{\alpha}dt\right)f'\left|\left(\frac{1}{\int_0^1 t^{\alpha}dt}\int_0^1 t^{\alpha}\left(t\left(a+b-x\right)+\left(1-t\right)b\right)\right)\right|^q\\
&=\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{(\alpha+1)}{(\alpha+2)}\left(a+b-x\right)+\frac{1}{\alpha+2}b\right)\right|^q\\
&=\left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{(\alpha+1)}{(\alpha+2)}\left(a-x\right)+b\right)\right|^q
\end{aligned}$$

and similarly,

$$\int_0^1 |t^\alpha| \left| f'\left(t\left(a+b-x\right)+\left(1-t\right)a\right)\right|^q dt$$
$$\leq \left(\frac{1}{\alpha+1}\right)\left|f'\left(\frac{(\alpha+1)}{(\alpha+2)}\left(b-x\right)+a\right)\right|^q.$$

The proof is completed.

□

**Corollary 2.12** *Under assumptions of Theorem 2.11 with* $x=\frac{a+b}{2}$, *then we have the inequality*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f\left(a\right)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f\left(b\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha+1}\right)^{2-\frac{1}{q}}\left[\left|f'\left(\frac{\left(\alpha+3\right)b+\left(\alpha+1\right)a}{2\left(\alpha+2\right)}\right)\right|\right.$$
$$\left.+\left|f'\left(\frac{\left(\alpha+3\right)a+\left(\alpha+1\right)b}{2\left(\alpha+2\right)}\right)\right|\right].$$

**Remark 2.5** If we choose $\alpha=1$ in Theorem 2.11, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-f(a+b-x)\right|$$
$$\leq \frac{1}{2\left(b-a\right)}\left[\left(x-a\right)^2\left|f'\left(\frac{2a+3b-2x}{3}\right)\right|\right.$$
$$\left.+\left(b-x\right)^2\left|f'\left(\frac{3a+2b-2x}{3}\right)\right|\right]$$

which is proved by Qaisar and Hussain in [23, Theorem 8].

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