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New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals

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ABSTRACT

A new identity similar to an identity proved in Alomari et al. (2010) [15] for fractional integrals is established. Then by making use of the established identity, some new Ostrowski type inequalities for Riemann–Liouville fractional integral are established. Our results have some relationships with the results of Alomari et al. (2010), proved in [15] and the analysis used in the proofs is simple.

1. Introduction and preliminary results

In 1938, Ostrowski proved the following interesting and useful integral inequality ([1], see also [2, p. 468]).

Theorem 1. *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $\left|f'(x)\right| \leq M$ for all $x \in [a, b]$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

This inequality gives an upper bound for the approximation of the integral average $\frac{1}{(b-a)}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$. In recent years, such inequalities were studied extensively by many researchers and numerous generalizations, extensions and variants of them appeared in a number of papers (see [3–15]).

The following definition is well known in the literature: let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. Geometrically, this means that if K, L and M are three distinct points on the graph of f with L between K and M, then L is on or below chord KM.

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In [16], the class of functions which are s-convex in the second sense has been introduced by Hudzik and Maligranda as the following.

Definition 1. A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. This class of s-convex functions is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [17], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 2. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f' \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

The following identity is proved by Alomari et al. (see [15]).

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then we have the equality:*

$$f(x) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)dt - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)dt$$

for each $x \in [a, b]$.

Using Lemma 1, Alomari et al. in [15] established the following results which hold for s-convex functions in the second sense.

Theorem 3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \leq M$, $x \in [a, b]$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{s+1}\right], \tag{1.3}$$

for each $x \in [a, b]$.

Theorem 4. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q > 1$, $p = \frac{q}{q-1}$ and $|f'(x)| \leq M$, $x \in [a, b]$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2 + (b-x)^2}{b-a}\right], \tag{1.4}$$

for each $x \in [a, b]$.

Theorem 5. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q \geq 1$ and $|f'(x)| \leq M$, $x \in [a, b]$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right], \tag{1.5}$$

for each $x \in [a, b]$.

Theorem 6. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $q > 1$ and $p = \frac{q}{q-1}$, then we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right| + (b-x)^2\left|f'\left(\frac{b+x}{2}\right)\right|\right], \tag{1.6}$$

for each $x \in [a, b]$.

For other recent results concerning *s*-convex functions, see [15,17,16,18,19].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found [20–30].

Motivated by the recent results given in [15,20–25,29], in the present note, we establish here new Ostrowski type inequalities for *s*-convex functions in the second sense via Riemann–Liouville fractional integral. An interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals.

2. Ostrowski type inequalities via fractional integrals

In order to prove our main results we need the following identity.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then for all $x \in [a, b]$ and $\alpha > 0$ we have:*

$$\begin{aligned}&\left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^\alpha f(a) + J_{x+}^\alpha f(b)\right]\\ &= \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx + (1-t)a)\,dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx+(1-t)b)\,dt\end{aligned} \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$.

Proof. By integration by parts, we can state

$$\begin{aligned}\int_0^1 t^\alpha f'(tx+(1-t)a)\,dt &= t^\alpha \frac{f(tx+(1-t)a)}{x-a}\bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f(tx+(1-t)a)}{x-a}dt\\ &= \frac{f(x)}{x-a} - \frac{\alpha}{x-a}\int_a^x \frac{(a-u)^{\alpha-1}}{(a-x)^{\alpha-1}}\frac{f(u)}{x-a}du\\ &= \frac{f(x)}{x-a} - \frac{\alpha\Gamma(\alpha)}{(x-a)^{\alpha+1}}\frac{1}{\Gamma(\alpha)}\int_a^x (u-a)^{\alpha-1}f(u)du\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\int_0^1 t^\alpha f'(tx+(1-t)b)\,dt &= t^\alpha \frac{f(tx+(1-t)b)}{x-b}\bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f(tx+(1-t)b)}{x-b}dt\\ &= \frac{f(x)}{x-b} - \frac{\alpha}{x-b}\int_b^x \frac{(b-u)^{\alpha-1}}{(b-x)^{\alpha-1}}\frac{f(u)}{x-b}du\\ &= \frac{f(x)}{x-b} + \frac{\alpha\Gamma(\alpha)}{(b-x)^{\alpha+1}}\frac{1}{\Gamma(\alpha)}\int_x^b (b-u)^{\alpha-1}f(u)du.\end{aligned} \tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $\frac{(x-a)^{\alpha+1}}{b-a}$ and $\frac{(b-x)^{\alpha+1}}{b-a}$, respectively, we have

$$\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+(1-t)a)\,dt = \frac{(x-a)^{\alpha} f(x)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a} J_{x-}^{\alpha} f(a) \tag{2.4}$$

and

$$\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx+(1-t)b)\,dt = -\frac{(b-x)^{\alpha} f(x)}{b-a} + \frac{\Gamma(\alpha+1)}{b-a} J_{x+}^{\alpha} f(b). \tag{2.5}$$

From (2.4) to (2.5), we obtain the desired result. □

Using this lemma, we can obtain the following fractional integral inequalities.

Theorem 7. *Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \le M, x \in [a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right]\right| \\ &\quad \le \frac{M}{b-a}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+s+1}\right]\end{aligned} \tag{2.6}$$

where Γ is Euler gamma function.

Proof. From (2.1) and since $|f'|$ is a s-convex mapping in the second sense on $[a, b]$, we have

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right]\right| \\ &\quad \le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right| dt + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right| dt \\ &\quad \le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha+s}\left|f'(x)\right| + t^{\alpha}(1-t)^s\left|f'(a)\right| dt + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha+s}\left|f'(x)\right| + t^{\alpha}(1-t)^s\left|f'(b)\right| dt \\ &\quad \le \frac{M}{b-a}\left(\frac{1}{\alpha+s+1} + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\end{aligned}$$

where we have used the fact that

$$\int_0^1 t^{\alpha+s} dt = \frac{1}{\alpha+s+1} \quad \text{and} \quad \int_0^1 t^{\alpha}(1-t)^s dt = \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}.$$

So using the reduction formula $\Gamma(n+1) = n\Gamma(n)$ $(n > 0)$ for Euler gamma function, the proof is complete. □

Corollary 1. *Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|$ is convex on $[a, b]$ and $|f'(x)| \le M, x \in [a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right]\right| \le \frac{M}{b-a}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right]$$

where Γ is Euler gamma function.

Proof. Setting $s = 1$ in (2.6), we get the required result. □

Remark 1. In Theorem 7, if we choose $\alpha = 1$, then (2.6) reduces inequality (1.3) of Theorem 3.

Theorem 8. *Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, $p, q > 1$ and $|f'(x)| \le M, x \in [a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right]\right| \\ &\quad \le \frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\end{aligned} \tag{2.7}$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $\alpha > 0$ and Γ is Euler gamma function.

Proof. From Lemma 2 and using the well known Hölder inequality (see for example [31]), we have

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right|\\&\quad\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt\\&\quad\le\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\&\qquad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|f'\right|^q$ is s-convex in the second sense on $[a,b]$ and $\left|f'(x)\right|\le M$, we get (see [15, p. 1073])

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\le\frac{2M^q}{s+1}\quad\text{and}\quad\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\le\frac{2M^q}{s+1}$$

and by simple computation

$$\int_0^1 t^{p\alpha}dt=\frac{1}{p\alpha+1}.$$

Hence, we have

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right|\\&\quad\le\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].\end{aligned}$$

which completes the proof. □

Corollary 2. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $\left|f'\right|^q$ is convex on $[a,b]$, $p,q>1$ and $\left|f'(x)\right|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right|\le\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler gamma function.

Proof. Setting $s=1$ in (2.7), we get the required result. □

Remark 2. In Theorem 8, if we choose $\alpha=1$, then (2.7) reduces inequality (1.4) of Theorem 4.

Theorem 9. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$ such that $f'\in L[a,b]$. If $\left|f'\right|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, $q\ge1$, and $\left|f'(x)\right|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right|\le M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\\&\quad\times\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\end{aligned}\tag{2.8}$$

where $\alpha>0$ and Γ is Euler gamma function.

Proof. From Lemma 2 and using the well known power mean inequality, we have

$$\begin{aligned}&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right|\\&\quad\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt\end{aligned}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^q$ is s-convex in the second sense on $[a, b]$ and $\left|f'(x)\right| \leq M$, we get (see [15, p. 1073])

$$\begin{aligned}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|^q dt &\leq \int_0^1\left[t^{s+\alpha}\left|f'(x)\right|^q + t^{\alpha}(1-t)^s\left|f'(a)\right|^q\right]dt\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(a)\right|^q\int_0^1 t^{\alpha}(1-t)^s dt\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(a)\right|^q\beta(\alpha+1, s+1)\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(a)\right|^q\frac{\Gamma(\alpha+1)\Gamma(s+1)}{(\alpha+s+1)\,\Gamma(\alpha+s+1)}\\ &\leq \frac{M^q}{\alpha+s+1}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\end{aligned}$$

and similarly

$$\begin{aligned}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|^q dt &\leq \int_0^1\left[t^{s+\alpha}\left|f'(x)\right|^q + t^{\alpha}(1-t)^s\left|f'(b)\right|^q\right]dt\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(b)\right|^q\int_0^1 t^{\alpha}(1-t)^s dt\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(b)\right|^q\beta(\alpha+1, s+1)\\ &= \frac{\left|f'(x)\right|^q}{\alpha+s+1} + \left|f'(b)\right|^q\frac{\Gamma(\alpha+1)\Gamma(s+1)}{(\alpha+s+1)\,\Gamma(\alpha+s+1)}\\ &\leq \frac{M^q}{\alpha+s+1}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\end{aligned}$$

where β is Euler beta function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x, y > 0)$$

and we used the fact that

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(n+1) = n\Gamma(n)(n > 0).$$

Hence, we have

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a) + J_{x+}^{\alpha}f(b)\right]\right|$$
$$\leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

which completes the proof. □

Corollary 3. *Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $\left|f'\right|^q$ is convex on $[a, b]$, $q \geq 1$, and $\left|f'(x)\right| \leq M$, $x \in [a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a) + J_{x+}^{\alpha}f(b)\right]\right| \leq M\left(\frac{1}{1+\alpha}\right)\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]$$

where $\alpha > 0$ and Γ is Euler gamma function.

Proof. Setting $s = 1$ in (2.8), we get the required result. □

Remark 3. In Theorem 9, if we choose $\alpha = 1$, then (2.8) reduces inequality (1.5) of Theorem 5.

The following result holds for s-concavity.

Theorem 10. *Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. If $|f'|^q$ is $s-$ concave in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $p, q > 1$, then the following inequality for fractional integrals holds:*

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \le \frac{2^{(s-1)/q}}{(1+p\alpha)^{\frac{1}{p}}(b-a)}\left[(x-a)^{\alpha+1}\left|f'\left(\frac{x+a}{2}\right)\right|+(b-x)^{\alpha+1}\left|f'\left(\frac{b+x}{2}\right)\right|\right] \tag{2.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha > 0$ and Γ is Euler gamma function.

Proof. From Lemma 2 and using the well known Hölder inequality, we have

$$\begin{aligned}
&\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}\left|f'(tx+(1-t)b)\right|dt \\
&\le \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.10}$$

Since $|f'|^q$ is $s-$concave, using inequality (1.2) we get (see [15, p. 1074])

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q dt \le 2^{s-1}\left|f'\left(\frac{x+a}{2}\right)\right|^q \tag{2.11}$$

and

$$\int_0^1\left|f'(tx+(1-t)b)\right|^q dt \le 2^{s-1}\left|f'\left(\frac{b+x}{2}\right)\right|^q. \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we have

$$\left|\left(\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right)f(x)-\frac{\Gamma(\alpha+1)}{(b-a)}\left[J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right]\right| \le \frac{2^{(s-1)/q}}{(1+p\alpha)^{\frac{1}{p}}(b-a)}\left[(x-a)^{\alpha+1}\left|f'\left(\frac{x+a}{2}\right)\right|+(b-x)^{\alpha+1}\left|f'\left(\frac{b+x}{2}\right)\right|\right].$$

which completes the proof. □

Remark 4. In Theorem 10, if we choose $\alpha = 1$, then (2.9) reduces inequality (1.6) of Theorem 6.

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