

On Generalizations of the Hadamard Inequality for (α, m) -Convex Functions

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ABSTRACT. In this paper we establish several Hadamard-type integral inequalities for (α, m) -convex functions.

1. Introduction

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature.

Theorem 1. *If f is convex function on $[a, b]$, then*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

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Received March 10, 2010; accepted September 23, 2011.

2010 Mathematics Subject Classification: 26A51, 26D15.

Key words and phrases: Hadamard inequality, convex functions, (α, m) -convex function.

Proof. See [1]. □

If the function f is concave, (1.1) can be written as following:

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f\left(\frac{a+b}{2}\right).$$

For recent results related to the Hadamard inequality are given in [9], [10] and [17].

In the literature, the concepts of m -convexity and (α, m) -convexity are well known. The concept of m -convexity was first introduced by G. Toader in [18] (see also [5], [6]) and it is defined as follows:

The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have:

$$(1.2) \quad f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

The class of (α, m) -convex functions was also first introduced in [8] and it is defined as follows:

The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$(1.3) \quad f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

It can be easily seen that for $(\alpha, m) \in \{(0, 0), (1, 1), (1, m)\}$ one obtains the following classes of functions: increasing, convex and m -convex functions respectively. The interested reader can find more about partial ordering of convexity in [15, P. 8,280]. For many papers connected with m -convex and (α, m) -convex functions see ([2], [3], [6], [11], [12], [13], [14], [19]) and the references therein. There are similar inequalities for s -convex and h -convex functions in [7] and [16], respectively.

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequality for m -convex functions.

Theorem 2. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a m -convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:*

$$(1.4) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2} \right\}.$$

Some generalizations of this result can be found in [2], [3].

In [4] S. S. Dragomir established two new Hadamard-type inequalities for m -convex functions. They are given in the following Theorems.

Theorem 3. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a m -convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, then the following inequality holds:

$$(1.5) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} dx.$$

Theorem 4. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a m -convex function with $m \in (0, 1]$. If $f \in L_1[am, b]$ where $0 \leq a < b$, then the following inequality holds:

$$(1.6) \quad \frac{1}{m+1} \left[\frac{1}{mb-a} \int_a^{mb} f(x) dx + \frac{1}{b-ma} \int_{ma}^b f(x) dx \right] \leq \frac{f(a) + f(b)}{2}.$$

The goal of this paper is to obtain new inequalities like those given in Theorems 1, 2, 3, 4, but now for the class of (α, m) -convex functions.

2. Inequalities for (α, m) -convex functions

The following theorem is a generalization of the Hadamard inequality.

Theorem 5. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $0 \leq a < b$ and $(\alpha, m) \in [0, 1] \times (0, 1]$. If $f \in L_1[m^2a, (2-m)b] \cap L_1\left[ma, \frac{(2-m)b}{m}\right]$, then the following inequalities hold:

$$(2.1) \quad \begin{aligned} & f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) \\ & \leq \frac{1}{2^\alpha} \frac{1}{b(2-m)-m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1)m \right. \\ & \quad \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m \frac{x-m^2a}{2b-mb-m^2a} a\right)] dx \Big\} \\ & \leq \frac{1}{2^\alpha(\alpha+1)} [f((2-m)b) \\ & \quad + (\alpha + (2^\alpha - 1))mf(am) + (2^\alpha - 1)\alpha m^2 f\left(\frac{(2-m)b}{m^2}\right)]. \end{aligned}$$

Proof. Let $U_1 = t(2-m)b + (1-t)m^2a$ and $U_2 = (1-t)(2-m)b + tm^2a$, where $t \in [0, 1]$ is arbitrary. Then we get

$$f\left(\frac{U_1 + U_2}{2}\right) = f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right).$$

By the (α, m) -convexity of f we can write the following inequality:

$$\begin{aligned} f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) &= f\left(\frac{U_1+U_2}{2}\right) \\ &\leq \frac{1}{2^\alpha} f(U_1) + \left(1 - \frac{1}{2^\alpha}\right) m f\left(\frac{U_2}{m}\right) \\ &= \frac{1}{2^\alpha} \left[f(U_1) + (2^\alpha - 1) m f\left(\frac{U_2}{m}\right) \right], \end{aligned}$$

or

$$\begin{aligned} f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) &\leq \frac{1}{2^\alpha} \left[f(t(2-m)b + (1-t)m^2a) \right. \\ &\quad \left. + (2^\alpha - 1) m f\left(\frac{(1-t)(2-m)b}{m} + tma\right) \right]. \end{aligned}$$

Integrating over $t \in [0, 1]$, we get

$$\begin{aligned} &f\left(\frac{2-m}{2}b + \frac{m}{2}(ma)\right) \\ &\leq \frac{1}{2^\alpha} \int_0^1 \left[f(t(2-m)b + (1-t)m^2a) \right. \\ (2.2) \quad &\quad \left. + (2^\alpha - 1) m f\left(\frac{(1-t)(2-m)b}{m} + tma\right) \right] dt \\ &= \frac{1}{2^\alpha} \frac{1}{b(2-m) - m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1) m \right. \\ &\quad \left. \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m \frac{x-m^2a}{2b-mb-m^2a} a\right) \right] dx \right\}, \end{aligned}$$

where we used the change of the variable $x = t(2-m)b + (1-t)m^2a$ or $t = \frac{x-m^2a}{2b-mb-m^2a}$ and so

$$\int_0^1 f(t(2-m)b + (1-t)m^2a) dt = \frac{1}{(2-m)b - m^2a} \int_{m^2a}^{(2-m)b} f(x) dx$$

and

$$\begin{aligned} &\int_0^1 f\left(\frac{(1-t)(2-m)b}{m} + tma\right) dt \\ &= \frac{1}{(2-m)b - m^2a} \\ &\quad \times \int_{m^2a}^{(2-m)b} f\left(\frac{(2-m)b}{m} \left(1 - \frac{x-m^2a}{2b-mb-m^2a}\right) + m \frac{x-m^2a}{2b-mb-m^2a} a\right) dx. \end{aligned}$$

This completes the proof of the first inequality in (2.1).

Next, by the (α, m) -convexity of f , we also have

$$\begin{aligned} f(t(2-m)b + (1-t)m^2a) &= f(t(2-m)b + m(1-t)ma) \\ &\leq t^\alpha f((2-m)b) + m(1-t^\alpha) f(ma) \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{(1-t)(2-m)b}{m} + tma\right) &= f\left(t(ma) + m(1-t)\left(\frac{(2-m)b}{m^2}\right)\right) \\ &\leq t^\alpha f(ma) + m(1-t^\alpha) f\left(\frac{(2-m)b}{m^2}\right). \end{aligned}$$

So

$$\begin{aligned} &\frac{1}{2^\alpha} \left[f\left(t(2-m)b + (1-t)m^2a\right) + (2^\alpha - 1)m f\left(\frac{(1-t)(2-m)b}{m} + tma\right) \right] \\ (2.3) \quad &\leq \frac{1}{2^\alpha} \left\{ t^\alpha f((2-m)b) + m(1-t^\alpha) f(ma) \right. \\ &\quad \left. + (2^\alpha - 1)m \left[t^\alpha f(ma) + m(1-t^\alpha) f\left(\frac{(2-m)b}{m^2}\right) \right] \right\}. \end{aligned}$$

Integrating (2.3) over t on $[0, 1]$, we get

$$\begin{aligned} &\frac{1}{2^\alpha} \frac{1}{(2-m)b - m^2a} \left\{ \int_{m^2a}^{(2-m)b} [f(x) + (2^\alpha - 1)m \right. \\ &\quad \left. \times f\left(\frac{(2-m)b}{m} \left(1 - \frac{x - m^2a}{2b - mb - m^2a}\right) + m \frac{x - m^2a}{2b - mb - m^2a} a\right)] dx \right\} \\ &\leq \frac{1}{2^\alpha(\alpha+1)} \left\{ f((2-m)b) \right. \\ &\quad \left. + (\alpha + (2^\alpha - 1))mf(am) + (2^\alpha - 1)\alpha m^2 f\left(\frac{(2-m)b}{m^2}\right) \right\}. \end{aligned}$$

This completes the proof of the second inequality in (2.1). $\square$

Remark 1. Choosing $(\alpha, m) = (1, 1)$ in (2.1), from the first and the second inequalities of (2.1), respectively, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \frac{1}{b-a} \left[\int_a^b [f(x) + f(a+b-x)] dx \right] \\ &= \frac{1}{2} \frac{1}{b-a} \left[\int_a^b f(x) dx + \int_a^b f(x) dx \right] \\ &= \frac{1}{b-a} \int_a^b f(x) dx \end{aligned}$$

and

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \frac{1}{4} [f(b) + 2f(a) + f(b)] \\ &= \frac{f(a) + f(b)}{2}. \end{aligned}$$

Note that, we used

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

and so

$$\int_a^b [f(x) + f(a+b-x)] dx = 2 \int_a^b f(x) dx.$$

Clearly, we can drop the assumption $f \in L_1 [m^2a, (2-m)b] \cap L_1 [ma, \frac{(2-m)b}{m}] = L_1 [a, b]$, and in this case (2.1) exactly becomes the Hermite-Hadamard inequalities for $(\alpha, m) = (1, 1)$.

Theorem 6. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1 [a, b]$, then the following inequality holds:

$$(2.4) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha + 1}, \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha + 1} \right\}.$$

Proof. Since f is (α, m) -convex, we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

for all $x, y \geq 0$, which gives:

$$f(ta + (1-t)b) \leq t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right)$$

and

$$f(tb + (1-t)a) \leq t^\alpha f(b) + m(1-t^\alpha) f\left(\frac{a}{m}\right)$$

for all $t \in [0, 1]$. Integrating on $[0, 1]$, we obtain

$$\int_0^1 f(ta + (1-t)b) dt \leq \frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha + 1}$$

and

$$\int_0^1 f(tb + (1-t)a) dt \leq \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha + 1}.$$

However,

$$\int_0^1 f(ta + (1-t)b) dt = \int_0^1 f(tb + (1-t)a) dt = \frac{1}{b-a} \int_a^b f(x) dx$$

and the inequality (2.4) is obtained. $\square$

Remark 2. The inequality (2.4) yields inequality (1.4) for $\alpha = 1$.

Theorem 7. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $(\alpha, m) \in$

$(0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, then the following inequalities hold:

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha(b-a)} \int_a^b [f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)] dx \\
 &\leq \frac{1}{2^{\alpha+1}(\alpha+1)} [(f(a) + f(b)) \\
 (2.5) \quad &+ m(\alpha + 2^\alpha - 1)(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)) \\
 &+ \alpha m^2(2^\alpha - 1)(f\left(\frac{a}{m^2}\right) + f\left(\frac{b}{m^2}\right))].
 \end{aligned}$$

Proof. By the (α, m) -convexity of f , we have

$$\begin{aligned}
 f\left(\frac{x+y}{2}\right) &= f\left(\frac{x}{2} + m\frac{y}{2m}\right) \\
 &\leq \frac{1}{2^\alpha} f(x) + m\left(1 - \frac{1}{2^\alpha}\right) f\left(\frac{y}{m}\right) \\
 &= \frac{1}{2^\alpha} \left[f(x) - m f\left(\frac{y}{m}\right) \right] + m f\left(\frac{y}{m}\right)
 \end{aligned}$$

for all $x, y \in [0, \infty)$.

Now, if we choose $x = ta + (1-t)b$ and $y = (1-t)a + tb$, we deduce

$$\begin{aligned}
 f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha} \left[f(ta + (1-t)b) - m f\left(\frac{(1-t)a + tb}{m}\right) \right] + m f\left(\frac{(1-t)a + tb}{m}\right) \\
 &= \frac{1}{2^\alpha} \left[f(ta + (1-t)b) + m(2^\alpha - 1) f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) \right]
 \end{aligned}$$

for all $t \in [0, 1]$.

Integrating over $t \in [0, 1]$, we get

$$\begin{aligned}
 (2.6) \quad f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^\alpha} \left[\int_0^1 f(ta + (1-t)b) dt \right. \\
 &\quad \left. + m(2^\alpha - 1) \int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt \right].
 \end{aligned}$$

Taking into account that

$$\int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x) dx$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m} + t\frac{b}{m}\right) dt = \frac{1}{b-a} \int_a^b f\left(\frac{x}{m}\right) dx,$$

we deduce from (2.6) the first inequality in (2.5).

Next, by the (α, m) -convexity of f , we also have

$$\begin{aligned}
 (2.7) \quad & \frac{1}{2^\alpha} \left[f(ta + (1-t)b) + m(2^\alpha - 1) f\left(t \frac{b}{m} + (1-t) \frac{a}{m}\right) \right] \\
 & \leq \frac{1}{2^\alpha} \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right. \\
 & \quad \left. + m(2^\alpha - 1) \left(t^\alpha f\left(\frac{b}{m}\right) + m(1-t^\alpha) f\left(\frac{a}{m^2}\right) \right) \right].
 \end{aligned}$$

Integrating over t on $[0, 1]$, we get

$$\begin{aligned}
 (2.8) \quad & \frac{1}{2^\alpha(b-a)} \int_a^b \left(f(x) + m(2^\alpha - 1) f\left(\frac{x}{m}\right) \right) dx \\
 & \leq \frac{1}{2^\alpha} \left[f(a) \int_0^1 t^\alpha dt + m f\left(\frac{b}{m}\right) \int_0^1 (1-t^\alpha) dt + \right. \\
 & \quad \left. + m(2^\alpha - 1) f\left(\frac{b}{m}\right) \int_0^1 t^\alpha dt \right. \\
 & \quad \left. + m^2(2^\alpha - 1) f\left(\frac{a}{m^2}\right) \int_0^1 (1-t^\alpha) dt \right] \\
 & = \frac{1}{2^\alpha(\alpha+1)} \left[f(a) + m(\alpha+2^\alpha-1) f\left(\frac{b}{m}\right) + \alpha m^2(2^\alpha-1) f\left(\frac{a}{m^2}\right) \right].
 \end{aligned}$$

Similarly, changing the roles of a and b , we get

$$\begin{aligned}
 (2.9) \quad & \frac{1}{2^\alpha(b-a)} \int_a^b \left(f(x) + m(2^\alpha - 1) f\left(\frac{x}{m}\right) \right) dx \\
 & \leq \frac{1}{2^\alpha(\alpha+1)} \left[f(b) + m(\alpha+2^\alpha-1) f\left(\frac{a}{m}\right) + \alpha m^2(2^\alpha-1) f\left(\frac{b}{m^2}\right) \right].
 \end{aligned}$$

Now adding (2.8) and (2.9) with each other, we obtain the second inequality in (2.5). $\square$

Remark 3. Choosing $\alpha = 1$ in the first part of (2.5), we get (1.5).

Remark 4. The inequality (2.5) yields the Hadamard inequality (1.1) for $\alpha = 1$ and $m = 1$.

Theorem 8. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:

$$(2.10) \quad \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \left[\frac{f(a) + f(b) + \alpha m \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right)}{\alpha + 1} \right].$$

Proof. By the (α, m) -convexity of f , we can write

$$f(ta + (1-t)b) \leq t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right)$$

and

$$f(tb + (1-t)a) \leq t^\alpha f(b) + m(1-t^\alpha) f\left(\frac{a}{m}\right)$$

for all $t \in [0, 1]$.

Adding the above inequalities, we get

$$\begin{aligned} & f(ta + (1-t)b) + f(tb + (1-t)a) \\ & \leq t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) + t^\alpha f(b) + m(1-t^\alpha) f\left(\frac{a}{m}\right). \end{aligned}$$

Integrating over $t \in [0, 1]$, we obtain

$$\begin{aligned} (2.11) \quad & \int_0^1 f(ta + (1-t)b) dt + \int_0^1 f(tb + (1-t)a) dt \\ & \leq \int_0^1 t^\alpha (f(a) + f(b)) dt + \int_0^1 m(1-t^\alpha) \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) dt \\ & = \frac{f(a) + f(b)}{\alpha + 1} + \frac{m\alpha}{\alpha + 1} \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) \\ & = \frac{f(a) + f(b) + m\alpha \left(f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right)}{\alpha + 1}. \end{aligned}$$

As it is easy to see that

$$\int_0^1 f(ta + (1-t)b) dt = \int_0^1 f(tb + (1-t)a) dt = \frac{1}{b-a} \int_a^b f(x) dx,$$

from (2.11) we deduce the desired result, namely, the inequality (2.10). $\square$

Remark 5. The inequality (2.10) yields the right side of the Hadamard inequality (1.1) for $\alpha = 1$ and $m = 1$.

Theorem 9. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be an (α, m) -convex function with $(\alpha, m) \in (0, 1]^2$. If $f \in L_1[am, b]$ where $0 \leq a < b$, then the following inequality holds:

$$\begin{aligned} (2.12) \quad & \frac{1}{mb-a} \int_a^{mb} f(x) dx + \frac{1}{b-ma} \int_{ma}^b f(x) dx \\ & \leq \frac{1}{(\alpha+1)} [(f(a) + f(b)) (1 + m\alpha)]. \end{aligned}$$

Proof. By (α, m) -convexity of f , for all $t \in [0, 1]$, we can write:

$$\begin{aligned} f(ta + m(1-t)b) &\leq t^\alpha f(a) + m(1-t^\alpha)f(b), \\ f(tb + m(1-t)a) &\leq t^\alpha f(b) + m(1-t^\alpha)f(a), \\ f((1-t)a + mtb) &\leq (1-t)^\alpha f(a) + m(1-(1-t)^\alpha)f(b), \\ f((1-t)b + mta) &\leq (1-t)^\alpha f(b) + m(1-(1-t)^\alpha)f(a). \end{aligned}$$

Adding the above inequalities with each other, we get:

$$\begin{aligned} &f(ta + m(1-t)b) + f(tb + m(1-t)a) \\ &+ f((1-t)a + mtb) + f((1-t)b + mta) \\ &\leq [t^\alpha + m(1-t^\alpha) + (1-t)^\alpha + m(1-(1-t)^\alpha)](f(a) + f(b)). \end{aligned}$$

Now integrating over $t \in [0, 1]$ and taking into account that:

$$\int_0^1 f(ta + m(1-t)b) dt = \int_0^1 f((1-t)a + mtb) dt = \frac{1}{mb-a} \int_a^{mb} f(x) dx$$

and

$$\int_0^1 f(tb + m(1-t)a) dt = \int_0^1 f((1-t)b + mta) dt = \frac{1}{b-ma} \int_{ma}^b f(x) dx,$$

we obtain the inequality (2.12). $\square$

Remark 7. Choosing $\alpha = 1$ in (2.12), we obtain (1.6).

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