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Some Quantum Integral Inequalities for (p, h)-Convex Functions

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Abstract: In this paper, we derive an identity of the q-definite integral of a continuous function f on a finite interval. We then use such identity to prove some new quantum integral inequalities for (p, h)-convex function. The results obtained in this paper generalize previous work in the literature.

Keywords: Hermite–Hadamard inequality; (p, h)-convex function; q-derivative; q-integral; q-calculus

MSC: 05A30; 26A51; 26D10; 26D15

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1. Introduction

Quantum calculus (it can be called q-calculus for short) is known as the study of calculus with no limits. Basically, if we take limit q tends to 1, then q-calculus can be reduced to ordinary calculus. It has been first studied by Euler (1707–1783). In 1910, F. H. Jackson [1] determined the definite q-integral known as the q-Jackson integral. Quantum calculus has many applications in several mathematical areas, such as combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, mechanics, quantum theory, and the theory of relativity (see, for instance, [2–7] and the references therein). The book by V. Kac and P. Cheung [8] covers the fundamental knowledge and basic theoretical concepts of quantum calculus.

Later, in 2013, J. Tariboon and S. K. Ntouyas [9,10] defined the q-derivative and q-integral of a continuous function on finite intervals and proved some of its properties. These definitions are called q_a-calculus. Many well-known integral inequalities such as Hermite–Hadamard, Hölder, trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss, and Grüss–Čebyšev inequalities have been studied in the concept of q_a-calculus.

In 2020, S. Bermudo et al. [11] newly defined the q-derivative and q-integral of a continuous function on finite intervals, which is called q^b-calculus. Moreover, their paper proved Hermite–Hadamard inequalities for convex functions and h-convex functions by using such a new definition. Based on the definitions of q_a-and q^b-calculus, there are many outcomes concerning quantum calculus.

The Hermite–Hadamard inequality is a classical inequality stated as: If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) was introduced by C. Hermite [12] in 1883 and was investigated by J. Hadamard [13] in 1893.

Recently, there have been many works about quantum integral inequalities, especially quantum Hermite–Hadamard type inequalities. Interested readers can see [14–19] and the references therein.

Moreover, in 2014, Z. B. Fang and R. Shi [20] defined a new class of functions which is called (p, h)-convex functions, and also proved some integral inequalities of (p, h)-convex functions. Motivated by the above literature, we propose to establish some new quantum Hermite–Hadamard inequalities for (p, h)-convex function, which is a generalization of the results of [20].

Inspired by the ongoing studies, we aim to prove Hermite–Hadamard inequalities for (p, h)-convex functions via q-calculus. We also show the validity of newly established inequalities with examples for particular choices of $q \in (0, 1)$.

The structure of this paper is as follows: the fundamentals of (p, h)-convex functions and q-calculus are briefly discussed in Section 2. In Section 3, we establish q-integral inequalities for (p, h)-convex functions. We present some examples in Section 4 to illustrate the newly established inequalities. Finally, we conclude our work in Section 5.

2. Preliminaries

Throughout this paper, we let $I = [a, b] \subseteq \mathbb{R}$ be the finite interval with $0 \leq a < b$, $h : (0, 1) \to \mathbb{R}$ be a positive function, p be a real number and $0 < q < 1$ be a constant. The definitions of (p, h)-convex function, q-derivative and q-integral are given in [9,11,20].

Definition 1 ([20]). *A function $f : I \to \mathbb{R}$ is said to be (p, h)-convex function, if f is non-negative and*

$$f\left(\left[tx^p + (1+t)y^p\right]^{\frac{1}{p}}\right) \leq h(t)f(x) + h(1-t)f(y) \tag{2}$$

for all $x, y \in I$ and $t \in (0, 1)$.

Example 1. *Define a function $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number and $h(t) = t^{\frac{1}{2}}$ for $0 < t < 1$. We have f is (p, h)-convex because*

$$\begin{aligned} f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) &= tx^p + (1-t)y^p \\ &= tf(x) + (1-t)f(y) \\ &\leq t^{\frac{1}{2}}f(x) + (1-t)^{\frac{1}{2}}f(y) \\ &= h(t)f(x) + h(1-t)f(y) \end{aligned}$$

for all $x, y \in [0, 1]$ and $0 < t < 1$.

Example 2. *Define a function $f : [0, 1] \to \mathbb{R}$ by $f(x) = 1$, $p \in \mathbb{R} \setminus \{0\}$ and $h(t) = t$ for $0 < t < 1$. We have f is (p, h)-convex because*

$$\begin{aligned} f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) &= 1 \\ &= t + (1-t) \\ &= tf(x) + (1-t)f(y) \\ &= h(t)f(x) + h(1-t)f(y) \end{aligned}$$

for all $x, y \in [0, 1]$ and $0 < t < 1$.

Definition 2 ([9]). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the q_a-derivative of f at $x \in (a, b]$ is defined by*

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}.$$

The q_a-integral is defined by

$$\int_a^x f(t)\ {}_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a).$$

Note that, in the case of $a = 0$, we write $D_q f(x) := {}_0D_q f(x)$ and $\int_0^x f(t)\ d_q t := \int_0^x f(t)\ {}_0 d_q t$.

Definition 3 ([11])**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the q^b-derivative of f at $x \in [a,b)$ is defined by*

$${}^bD_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}.$$

The q^b-integral is defined by

$$\int_x^b f(t)\ {}^b d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

3. Main Results

In this section, we shall start with the following key lemma, which is used to derive the main theorems. Then we prove a variant of quantum integral inequalities for (p,h)-convex functions.

Lemma 1. *Let $f : I \to \mathbb{R}$ be a q^{b^p}-integrable function. The q^{b^p}-integral of f on $[a,b]$ is defined by the expression*

$$\frac{1}{(b^p - a^p)}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_q x = \int_0^1 f\left((ta^p + (1-t)b^p)^{\frac{1}{p}}\right) d_q t. \tag{3}$$

Proof. By the definition of q^{b^p}-integral, we directly have

$$\begin{aligned}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_q x &= (1-q)(b^p - a^p)\sum_{n=0}^{\infty} q^n f\left((q^n a^p + (1-q^n)b^p)^{\frac{1}{p}}\right)\\ &= (b^p - a^p)\int_0^1 f\left((ta^p + (1-t)b^p)^{\frac{1}{p}}\right) d_q t.\end{aligned}$$

The proof is completed. □

Similarly,

$$\frac{1}{(b^p - a^p)}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}_{a^p}d_q x = \int_0^1 f\left((tb^p + (1-t)a^p)^{\frac{1}{p}}\right) d_q t. \tag{4}$$

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be (p,h)-convex and q-integrable function. Then, we have*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) &\le \frac{1}{(b^p - a^p)}\left\{\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}_{a^p}d_q x + \int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_q x\right\}\\ &\le (f(a) + f(b))\left(\int_0^1 h(t) d_q t + \int_0^1 h(1-t)\ d_q t\right).\end{aligned} \tag{5}$$

Proof. Since f is a (p,h)-convex function, we have

$$f\left(\left[\frac{x^p + y^p}{2}\right]^{\frac{1}{p}}\right) \le h\left(\frac{1}{2}\right)(f(x) + f(y)),$$

for any $x, y \in [a, b]$.

Setting $x = (tb^p + (1-t)a^p)^{\frac{1}{p}}$ and $y = (ta^p + (1-t)b^p)^{\frac{1}{p}}$. Then, we obtain

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2}\right)\left[f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)+f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)\right],$$

that is

$$\frac{1}{h\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)+f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right). \tag{6}$$

q-Integrating (6) over $[0, 1]$ and then using (3) and (4), we have

$$\begin{aligned}\int_0^1 \frac{1}{h\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) d_qt &\leq \int_0^1 f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right) d_qt + \int_0^1 f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) d_qt \\ &= \frac{1}{(b^p-a^p)}\left\{\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}_{a^p}d_qx + \int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx\right\}.\end{aligned} \tag{7}$$

Moreover, since f is (p, h)-convex function, we get

$$f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right) \leq h(t)f(b)+h(1-t)f(a) \tag{8}$$

and

$$f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \leq h(t)f(a)+h(1-t)f(b). \tag{9}$$

From (8) and (9), we get

$$\begin{aligned}&f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)+f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \\ &\qquad \leq h(t)f(a)+h(1-t)f(b)+h(t)f(b)+h(1-t)f(a).\end{aligned} \tag{10}$$

Furthermore, q-integrating (10) over $[0, 1]$, we have

$$\begin{aligned}&\int_0^1 f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right) d_qt + \int_0^1 f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) d_qt \\ &\leq \int_0^1 h(t)f(a)+h(1-t)f(b)\ d_qt + \int_0^1 h(t)f(b)+h(1-t)f(a)\ d_qt \\ &= f(a)\int_0^1 h(t)\ d_qt + f(b)\int_0^1 h(1-t)\ d_qt + f(b)\int_0^1 h(t)\ d_qt + f(a)\int_0^1 h(1-t)\ d_qt \\ &= (f(a)+f(b))\left(\int_0^1 h(t)d_qt + \int_0^1 h(1-t)\ d_qt\right).\end{aligned} \tag{11}$$

Finally, by (7) and (11), we derive

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{1}{(b^p-a^p)}\left\{\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}_{a^p}d_qx + \int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx\right\} \\ &\leq (f(a)+f(b))\left(\int_0^1 h(t)d_qt + \int_0^1 h(1-t)\ d_qt\right).\end{aligned}$$

The proof is completed. □

Remark 1. *If $q \to 1$, then (5) reduces to ([20], Theorem 5).*

Remark 2. *If $q \to 1$, $p = 1$ and $h(t) = t$ for all $t \in [0,1]$, then (5) reduces to (1).*

Theorem 2. *Let f, g be (p, h_1)-convex and (p, h_2)-convex functions, respectively. Suppose that f, g are q-integrable, $fg \in [a,b]$ and $h_1 h_2 \in [0,1]$. Then, we have*

$$\begin{aligned}\frac{1}{(b^p - a^p)} \int_{a^p}^{b^p} f(x^{\frac{1}{p}}) g(x^{\frac{1}{p}}) \ {}^{b^p} d_q x \\ \leq f(a)g(a) \int_0^1 h_1(t)h_2(t)\ d_q t + f(a)g(b) \int_0^1 h_1(t)h_2(1-t)\ d_q t \\ + f(b)g(a) \int_0^1 h_1(1-t)h_2(t)\ d_q t + f(b)g(b) \int_0^1 h_1(1-t)h_2(1-t)\ d_q t.\end{aligned} \tag{12}$$

Proof. Since f, g are (p, h_1)-convex and (p, h_2)-convex functions, respectively, we get

$$f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \leq h_1(t) f(a) + h_1(1-t) f(b) \tag{13}$$

and

$$g\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \leq h_2(t) g(a) + h_2(1-t) g(b), \tag{14}$$

for any $t \in [0,1]$. Then we have

$$\begin{aligned} f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) g\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \\ \leq (h_1(t)f(a) + h_1(1-t)f(b))(h_2(t)g(a) + h_2(1-t)g(b)) \\ = h_1(t)h_2(t)f(a)g(a) + h_1(t)h_2(1-t)f(a)g(b) \\ + h_1(1-t)h_2(t)f(b)g(a) + h_1(1-t)h_2(1-t)f(b)g(b). \end{aligned} \tag{15}$$

q-Integrating (15) over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) g\left([ta^p + (1-t)b^p]^{\frac{1}{p}} \right) d_q t \\ &\leq \int_0^1 (h_1(t)f(a) + h_1(1-t)f(b))(h_2(t)g(a) + h_2(1-t)g(b))\ d_q t \\ &= f(a)g(a) \int_0^1 h_1(t)h_2(t)\ d_q t + f(a)g(b) \int_0^1 h_1(t)h_2(1-t)\ d_q t \\ &\quad + f(b)g(a) \int_0^1 h_1(1-t)h_2(t)\ d_q t + f(b)g(b) \int_0^1 h_1(1-t)h_2(1-t)\ d_q t. \end{aligned}$$

Using (3), we finally have

$$\begin{aligned}&\frac{1}{(b^p - a^p)} \int_{a^p}^{b^p} f(x^{\frac{1}{p}}) g(x^{\frac{1}{p}}) \ {}^{b^p} d_q x \\ &\leq f(a)g(a) \int_0^1 h_1(t)h_2(t)\ d_q t + f(a)g(b) \int_0^1 h_1(t)h_2(1-t)\ d_q t \\ &\quad + f(b)g(a) \int_0^1 h_1(1-t)h_2(t)\ d_q t + f(b)g(b) \int_0^1 h_1(1-t)h_2(1-t)\ d_q t.\end{aligned}$$

The proof is completed. □

Remark 3. *If $q \to 1$, then (12) reduces to ([20], Theorem 6).*

Theorem 3. *Let f, g be (p, h_1)-convex and (p, h_2)-convex functions, respectively. Suppose that f, g are q-integrable, $fg \in [a, b]$ and $h_1 h_2 \in [0, 1]$. Then we have*

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&\qquad - \frac{1}{(b^p-a^p)}\left[\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}_{a^p}d_q x + \int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_q x\right] \\
&\qquad \leq M(a,b)\int_0^1 h_1(t)h_2(1-t) + h_1(1-t)h_2(t)\ d_q t \\
&\qquad + N(a,b)\int_0^1 (h_1(t)h_2(t) + h_1(1-t)h_2(1-t)\ d_q t,
\end{aligned} \tag{16}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f is a (p, h_1)-convex function and g is a (p, h_2)-convex function, we have

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq h_1\left(\frac{1}{2}\right)\left(f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) + f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)\right)$$

and

$$g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \leq h_2\left(\frac{1}{2}\right)\left(g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) + g\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)\right),$$

respectively, for any $t \in [0, 1]$. Then we get

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \\
&\quad + f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)g\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right) \\
&\quad + f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \\
&\quad + f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)g\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)\Big\} \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \\
&\quad + f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)g\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right) \\
&\quad + h_1(t)h_2(t)f(a)g(b) + h_1(t)h_2(1-t)f(a)g(a) + h_1(1-t)h_2(t)f(b)g(b) \\
&\quad + h_1(1-t)h_2(1-t)f(b)g(a) + h_1(t)h_2(t)f(b)g(a) + h_1(t)h_2(1-t)f(b)g(b) \\
&\quad + h_1(1-t)h_2(t)f(a)g(a) + h_1(1-t)h_2(1-t)f(a)g(b)\} \\
&= h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right) \\
&\quad + f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)g\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)\Big\} \\
&\quad + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\{(f(a)g(b)+f(b)g(a))(h_1(t)h_2(t)+h_1(1-t)h_2(1-t)) \\
&\quad + (f(a)g(a)+f(b)g(b))(h_1(t)h_2(1-t)+h_1(1-t)h_2(t))\}.
\end{aligned} \tag{17}$$

q-Integrating (17) over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\,d_qt \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left\{\int_0^1 f\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)g\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\,d_qt\right. \\
&\quad\left.+\int_0^1 f\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}}\right)g\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}}\right)\,d_qt\right\} \\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left\{M(a,b)\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\,d_qt\right. \\
&\quad\left.+N(a,b)\int_0^1 h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\,d_qt\right\}.
\end{aligned}$$

Finally, by using (3), we obtain

$$\begin{aligned}
&\int_0^1 f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\,d_qt \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx+\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}_{a^p}d_qx\right] \\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[M(a,b)\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\,d_qt\right. \\
&\quad\left.+N(a,b)\int_0^1(h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\,d_qt\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&\qquad-\frac{1}{(b^p-a^p)}\left[\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}_{a^p}d_qx+\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx\right] \\
&\qquad\leq M(a,b)\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\,d_qt \\
&\qquad\quad+N(a,b)\int_0^1(h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\,d_qt.
\end{aligned}$$

The proof is completed. □

Remark 4. *If $q\to 1$, then (16) reduces to ([20], Theorem 7).*

Theorem 4. *Let f,g be (p,h_1)-convex and (p,h_2)-convex functions, respectively, and let r be a function, is defined by $r(x)=x^{\frac{1}{p}}$. Suppose that $fg\in[a,b]$ and $h_1h_2\in[0,1]$. Then, we have*

$$\begin{aligned}
&\frac{1}{(b^p-a^p)^2}\int_{a^p}^{b^p}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p+(1-t)r(y)^p\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)r(y)^p\right]^{\frac{1}{p}}\right)\,d_qt\ {}^{b^p}d_qy\ {}^{b^p}d_qx \\
&\leq I_1\left(\int_0^1 h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\,d_qt\right) \\
&\quad+I_2\left(\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\,d_qt\right),
\end{aligned}\tag{18}$$

where

$$I_1 = f(a)g(a)\int_0^1 h_1(t)h_2(t)\ d_qt + f(a)g(b)\int_0^1 h_1(t)h_2(1-t)\ d_qt$$
$$+ f(b)g(a)\int_0^1 h_1(1-t)h_2(t)\ d_qt + f(b)g(b)\int_0^1 h_1(1-t)h_2(1-t)\ d_qt$$

and

$$\begin{aligned}I_2 = {} & f(a)g(a)\int_0^1 h_1(t)\ d_qt\int_0^1 h_2(t)\ d_qt\\ & + f(a)g(b)\int_0^1 h_1(t)\ d_qt\int_0^1 h_2(1-t)\ d_qt\\ & + f(b)g(a)\int_0^1 h_1(1-t)\ d_qt\int_0^1 h_2(t)\ d_qt\\ & + f(b)g(b)\int_0^1 h_1(1-t)\ d_qt\int_0^1 h_2(1-t)\ d_qt.\end{aligned}$$

Proof. Since f, g are (p, h_1)-convex and (p, h_2)-convex functions, respectively, we have

$$\begin{aligned}f\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right) &\le h_1(t)f(r(x))) + h_1(1-t)f(r(y))\\ &= h_1(t)f(x^{\frac{1}{p}}) + h_1(1-t)f(y^{\frac{1}{p}})\end{aligned}$$

and

$$\begin{aligned}g\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right) &\le h_2(t)g(r(x))) + h_2(1-t)g(r(y))\\ &\le h_2(t)g(x^{\frac{1}{p}}) + h_2(1-t)g(y^{\frac{1}{p}})\end{aligned}$$

for any $t \in [0, 1]$. Then we have inequality

$$\begin{aligned}& f\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right)\\ &\le h_1(t)h_2(t)f(x^{\frac{1}{p}})g(x^{\frac{1}{p}}) + h_1(t)h_2(1-t)f(x^{\frac{1}{p}})g(y^{\frac{1}{p}})\\ &\quad + h_1(1-t)h_2(t)f(y^{\frac{1}{p}})g(x^{\frac{1}{p}}) + h_1(1-t)h_2(1-t)f(y^{\frac{1}{p}})g(y^{\frac{1}{p}}).\end{aligned}$$

q-Integrating both sides of above inequality over $[0, 1]$ and over $[a^p, b^p]$, then multiplying by $\frac{1}{(b^p - a^p)^2}$, we get

$$\begin{aligned}&\frac{1}{(b^p-a^p)^2}\int_{a^p}^{b^p}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p + (1-t)r(y)^p\right]^{\frac{1}{p}}\right)\ d_qt\ {}^{b^p}d_qy\ {}^{b^p}d_qx\\ &\le \frac{1}{(b^p-a^p)^2}\int_0^{b^p}\int_0^{b^p}\int_0^1 h_1(t)h_2(t)f(x^{\frac{1}{p}})g(x^{\frac{1}{p}}) + h_1(t)h_2(1-t)f(x^{\frac{1}{p}})g(y^{\frac{1}{p}})\\ &\quad + h_1(1-t)h_2(t)f(y^{\frac{1}{p}})g(x^{\frac{1}{p}}) + h_1(1-t)h_2(1-t)f(y^{\frac{1}{p}})g(y^{\frac{1}{p}})\ d_qt\ {}^{b^p}d_qy\ {}^{b^p}d_qx\\ &= \frac{1}{(b^p-a^p)^2}\left\{\int_0^1 h_1(t)h_2(t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx\int_{a^p}^{b^p} 1\ {}^{b^p}d_qy\right.\\ &\quad \left. + \int_0^1 h_1(1-t)h_2(1-t)\ d_qt\int_{a^p}^{b^p} f(y^{\frac{1}{p}})g(y^{\frac{1}{p}})\ {}^{b^p}d_qy\int_{a^p}^{b^p} 1\ {}^{b^p}d_qx\right\}\\ &\quad + \frac{1}{(b^p-a^p)^2}\left\{\int_0^1 h_1(t)h_2(1-t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx\int_{a^p}^{b^p} g(y^{\frac{1}{p}})\ {}^{b^p}d_qy\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 h_1(1-t)h_2(t)\ d_qt \int_{a^p}^{b^p} f(y^{\frac{1}{p}})\ {}^{b^p}d_qx \int_{a^p}^{b^p} g(x^{\frac{1}{p}})\ {}^{b^p}d_qy\Big\} \\
&= \frac{1}{(b^p-a^p)^2}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx \int_{a^p}^{b^p} 1\ {}^{b^p}d_qy\left(\int_0^1 h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\ d_qt\right) \\
&+\frac{1}{(b^p-a^p)^2}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx \int_{a^p}^{b^p} g(y^{\frac{1}{p}})\ {}^{b^p}d_qy\left(\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\ d_qt\right) \\
&\leq I_1\left(\int_0^1 h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\ d_qt\right)+I_2\left(\int_0^1 h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\ d_qt\right).
\end{aligned}
$$

The proof is completed. □

Theorem 5. *Suppose that f,g are (p,h_1)-covex and (p,h_2)-convex functions, respectively, and let r be a function, is defined by $r(x)=x^{\frac{1}{p}}$. If $fg\in[a,b]$ and $h_1h_2\in[0,1]$, then we have*

$$
\begin{aligned}
&\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)d_qt\ {}^{b^p}d_qx \\
&\leq I_1\left[\int_0^1 h_1(t)h_2(t)\ d_qt\right]+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(S(a,b)+T(a,b))\left[\int_0^1 h_1(1-t)h_2(1-t)\ d_qt\right] \\
&+I_3\ S(a,b)+I_4\ T(a,b),
\end{aligned}
\tag{19}
$$

where I_1 is defined in Theorem 4 and

$$
\begin{aligned}
S(a,b)&=f(a)g(a)+f(b)g(a), \qquad T(a,b)=f(b)g(a)+f(b)g(b), \\
I_3&=h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)d_qt\int_0^1 h_1(t)h_2(1-t)\ d_qt \\
&\quad+h_1\left(\frac{1}{2}\right)\int_0^1 h_2(t)\ d_qt\int_0^1 h_1(1-t)h_2(t)\ d_qt
\end{aligned}
$$

and

$$
\begin{aligned}
I_4&=h_1\left(\frac{1}{2}\right)\int_0^1 h_2(1-t)\ d_qt\int_0^1 h_1(1-t)h_2(t)\ d_qt \\
&\quad+h_2\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)\ d_qt\int_0^1 h_1(t)h_2(1-t)\ d_qt.
\end{aligned}
$$

Proof. Since f,g are (p,h_1)-covex and (p,h_2)-convex functions, respectively, we get

$$
\begin{aligned}
f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)&\leq h_1(t)f(r(x))+h_1(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&=h_1(t)f(x^{\frac{1}{p}})+h_1(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)&\leq h_2(t)g(r(x))+h_2(1-t)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \\
&=h_2(t)g(x^{\frac{1}{p}})+h_2(1-t)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right),
\end{aligned}
$$

for any $t\in[0,1]$. Then, we have inequality

$$f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)$$
$$\leq h_1(t)h_2(t)f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})+h_1(1-t)h_2(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)$$
$$+h_1(t)h_2(1-t)f(x^{\frac{1}{p}})g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)+h_1(1-t)h_2(t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g(x^{\frac{1}{p}}).$$

q-Integrating both sides of above expression over $[0,1]$ and over $[a^p,b^p]$, then multiplying by $\frac{1}{(b^p-a^p)}$, we obtain

$$\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)\,d_qt\ {}^{b^p}d_qx$$
$$\leq\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p}\int_0^1 h_1(t)h_2(t)f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})$$
$$+h_1(1-t)h_2(1-t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)$$
$$+h_1(t)h_2(1-t)f(x^{\frac{1}{p}})g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)$$
$$+h_1(1-t)h_2(t)f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g(x^{\frac{1}{p}})\,d_qt\ {}^{b^p}d_qx$$
$$=\frac{1}{(b^p-a^p)}\int_0^1 h_1(t)h_2(t)\,d_qt\int_{a^p}^{b^p}f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx$$
$$+\frac{1}{(b^p-a^p)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 h_1(1-t)h_2(1-t)\,d_qt\int_{a^p}^{b^p}1\ {}^{b^p}d_qx$$
$$+\frac{1}{(b^p-a^p)}g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 h_1(t)h_2(1-t)\,d_qt\int_{a^p}^{b^p}f(x^{\frac{1}{p}})\ {}^{b^p}d_qx$$
$$+\frac{1}{(b^p-a^p)}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 h_1(1-t)h_2(t)\,d_qt\int_{a^p}^{b^p}g(x^{\frac{1}{p}})\ {}^{b^p}d_qx.$$

Since

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\leq h_1\left(\frac{1}{2}\right)(f(a)+f(b))$$

and

$$g\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\leq h_2\left(\frac{1}{2}\right)(g(a)+g(b)),$$

we have

$$\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)\,d_qt\ {}^{b^p}d_qx$$
$$\leq\frac{1}{(b^p-a^p)}\int_0^1 h_1(t)h_2(t)\,d_qt\int_{a^p}^{b^p}f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx$$
$$+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(f(a)+f(b))(g(a)+g(b))\int_0^1 h_1(1-t)h_2(1-t)\,d_qt$$

$$+\frac{1}{(b^p-a^p)}h_2\left(\frac{1}{2}\right)(g(a)+g(b))\int_0^1 h_1(t)h_2(1-t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx$$
$$+\frac{1}{(b^p-a^p)}h_1\left(\frac{1}{2}\right)(f(a)+f(b))\int_0^1 h_1(1-t)h_2(t)\ d_qt\int_0^{b^p} g(x^{\frac{1}{p}})\ {}^{b^p}d_qx.$$

Moreover, since

$$\begin{aligned}\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p} f(x^{\frac{1}{p}})\ {}^{b^p}d_qx &= \int_0^1 f\left((ta^p+(1-t)b^p)^{\frac{1}{p}}\right)d_qt\\ &\le f(a)\int_0^1 h_1(t)d_qt+f(b)\int_0^1 h_1(1-t)\ d_qt,\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p} g(x^{\frac{1}{p}})\ {}^{b^p}d_qx &= \int_0^1 g\left((ta^p+(1-t)b^p)^{\frac{1}{p}}\right)d_qt\\ &\le g(a)\int_0^1 h_2(t)d_qt+g(b)\int_0^1 h_2(1-t)\ d_qt,\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{(b^p-a^p)}\int_{a^p}^{b^p}\int_0^1 f\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)g\left(\left[tr(x)^p+(1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right)d_qt\ {}^{b^p}d_qx\\
&\le \frac{1}{(b^p-a^p)}\int_0^1 h_1(t)h_2(t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(f(a)+f(b))(g(a)+g(b))\int_0^1 h_1(1-t)h_2(1-t)\ d_qt\\
&\quad+h_2\left(\frac{1}{2}\right)(g(a)+g(b))\int_0^1 h_1(t)h_2(1-t)\ d_qt\left(f(a)\int_0^1 h_1(t)\ d_qt+f(b)\int_0^1 h_1(1-t)\ d_qt\right)\\
&\quad+h_1\left(\frac{1}{2}\right)(f(a)+f(b))\int_0^1 h_1(1-t)h_2(t)\ d_qt\left(g(a)\int_0^1 h_2(t)\ d_qt+g(b)\int_0^1 h_2(1-t)\ d_qt\right).\\
&= \frac{1}{(b^p-a^p)}\int_0^1 h_1(t)h_2(t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx\\
&\quad+h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(f(a)+f(b))(g(a)+g(b))\int_0^1 h_1(1-t)h_2(1-t)\ d_qt\\
&\quad+h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)h_2(1-t)\ d_qt\left((f(a)g(a)+f(b)g(a))\int_0^1 h_1(t)\ d_qt+(f(b)g(a)+f(b)g(b))\right.\\
&\quad\times\left.\int_0^1 h_1(1-t)\ d_qt\right)+h_1\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)h_2(t)\ d_qt\left((f(a)g(a)+f(b)g(a))\int_0^1 h_2(t)\ d_qt\right.\\
&\quad+\left.(f(b)g(a)+f(b)g(b))\int_0^1 h_2(1-t)\ d_qt\right)\\
&= \frac{1}{(b^p-a^p)}\int_0^1 h_1(t)h_2(t)\ d_qt\int_{a^p}^{b^p} f(x^{\frac{1}{p}})g(x^{\frac{1}{p}})\ {}^{b^p}d_qx\\
&\quad+(f(a)g(a)+f(b)g(a)+f(a)g(b)+f(b)g(b))h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)h_2(1-t)\ d_qt\\
&\quad+(f(a)g(a)+f(b)g(a))\left(h_2\left(\frac{1}{2}\right)\int_0^1 h_1(t)\ d_qt\int_0^1 h_1(t)h_2(1-t)\ d_qt+h_1\left(\frac{1}{2}\right)\int_0^1 h_2(t)\ d_qt\right.\\
&\quad\times\left.\int_0^1 h_1(1-t)h_2(t)\ d_qt\right)+(f(b)g(a)+f(b)g(b))\left(h_1\left(\frac{1}{2}\right)\int_0^1 h_2(1-t)\ d_qt\int_0^1 h_1(1-t)h_2(t)\ d_qt\right.\\
&\quad+\left.h_2\left(\frac{1}{2}\right)\int_0^1 h_1(1-t)\ d_qt\int_0^1 h_1(t)h_2(1-t)\ d_qt\right).\end{aligned}$$

By Theorem 2, we have

$$\begin{aligned}
&\frac{1}{(b^p - a^p)} \int_{a^p}^{b^p} \int_0^1 f\left(\left[tr(x)^p + (1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right) g\left(\left[tr(x)^p + (1-t)\left(\frac{a^p+b^p}{2}\right)\right]^{\frac{1}{p}}\right) d_qt \,{}^{b^p}d_qx \\
&\le I_1\left[\int_0^1 h_1(t)h_2(t)\, d_qt\right] + h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)(S(a,b)+T(a,b))\left[\int_0^1 h_1(1-t)h_2(1-t)\, d_qt\right] \\
&+ I_3\, S(a,b) + I_4\, T(a,b).
\end{aligned}$$

The proof is completed. □

4. Examples

In this section, we give some examples to demonstrate our main results.

Example 3. *Define a function $f : [0,1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number and $h(t) = t^{1/2}$ for $0 < t < 1$. We have f is (p,h)-convex, by Example 1. By applying Theorem 1 with $q = \frac{1}{4}$ and $p = 3$, the first inequality of (5) becomes*

$$0.70710 \approx \frac{1}{\left(\frac{1}{2}\right)^{1/2}}\left(\frac{0+1}{2}\right) \le \frac{1}{1-0}\left(\int_0^1 x\; {}_0d_qx + \int_0^1 x\; {}^1d_qx\right) = 1.$$

The second inequality of (5) becomes

$$\begin{aligned}
1 &= \frac{1}{1-0}\left(\int_0^1 x\; {}_0d_qx + \int_0^1 x\; {}^1d_qx\right) \\
&\le (0+1)\left(\int_0^1 t^{1/2}\, d_qt + \int_0^1 (1-t)^{1/2}\, d_qt\right) \approx 1.08044.
\end{aligned}$$

It is clear that

$$0.70710 \le 1 \le 1.08044,$$

which demonstrates the result described in Theorem 1.

Example 4. *Define function $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number, $g(x) = 1$ and $h_1(t) = t^{1/2}, h_2(t) = t$ for $0 < t < 1$. Examples 1 and 2, we have f, g is $(p,h_1),(p,h_2)$-convex, respectively. By applying Theorem 2 with $q = \frac{1}{4}$ and $p = 3$, we get*

$$\begin{aligned}
0.2 = \frac{1}{(1-0)}\int_0^1 x\; {}^1d_qx &\le f(0)g(0)\int_0^1 t^{1/2}t\, d_qt + f(0)g(1)\int_0^1 t^{1/2}(1-t)\, d_qt \\
&+ f(1)g(0)\int_0^1 (1-t)^{1/2}t\, d_qt \\
&+ f(1)g(1)\int_0^1 (1-t)^{1/2}(1-t)\, d_qt. \\
&\approx 0.04362 + 0.17966 = 0.22328.
\end{aligned}$$

It is clear that

$$0.2 \le 0.22328,$$

which demonstrates the result described in Theorem 2.

Example 5. *Define function $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number, $g(x) = 1$ and $h_1(t) = t^{1/2}, h_2(t) = t$ for $0 < t < 1$. Examples 1 and 2, we have f, g is $(p,h_1),(p,h_2)$-convex, respectively. By applying Theorem 3 with $q = \frac{1}{4}$ and $p = 3$, we get*

$$
\begin{aligned}
0.41421 &\approx \frac{1}{\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{2}\right)}\left(\frac{0+1}{2}\right)\left(\frac{0+1}{2}\right) - \frac{1}{(1-0)}\left[\int_0^1 x\ {}_0d_qx + \int_0^1 x\ {}^1d_qx\right] \\
&\leq M(0,1)\left(\int_0^1 t^{1/2}(1-t) + (1-t)^{1/2}t\ d_qt\right) \\
&\quad + N(0,1)\left(\int_0^1 t^{1/2}t + (1-t)^{1/2}(1-t)\ d_qt\right) \\
&\approx 0.12657 + 0.95386 = 1.14652.
\end{aligned}
$$

It is clear that

$$0.41421 \leq 1.14652,$$

which demonstrates the result described in Theorem 3.

Example 6. *Define function $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number, $g(x) = 1$ and $h_1(t) = t^{1/2}, h_2(t) = t$ for $0 < t < 1$. Examples 1 and 2, we have f, g is $(p, h_1), (p, h_2)$-convex, respectively. By applying Theorem 4 with $q = \frac{1}{4}$ and $p = 3$, we get*

$$
\begin{aligned}
0.2 &= \int_0^1\int_0^1\int_0^1 tx + (1-t)y\ d_qt\ {}^1d_qy\ {}^1d_qx \\
&\leq I_1\left(\int_0^1 t^{1/2}t + (1-t)^{1/2}(1-t)\ d_qt\right) \\
&\quad + I_2\left(\int_0^1 (1-t)^{1/2}\ d_qt \int_0^1 t\ d_qt + \int_0^1 (1-t)^{1/2}\ d_qt \int_0^1 (1-t)\ d_qt\right) \\
&\approx 0.22329(0.77419 + 0.17967) + 0.22329(0.08295 + 0.04362) \\
&= 0.24125.
\end{aligned}
$$

It is clear that

$$0.2 \leq 0.24125,$$

which demonstrates the result described in Theorem 4.

Example 7. *Define function $f, g : [0,1] \to \mathbb{R}$ by $f(x) = x^p$, where p is an odd natural number, $g(x) = 1$ and $h_1(t) = t^{1/2}, h_2(t) = t$ for $0 < t < 1$. Examples 1 and 2, we have f, g is $(p, h_1), (p, h_2)$-convex, respectively. By applying Theorem 5 with $q = \frac{1}{4}$ and $p = 3$, we get*

$$
\begin{aligned}
0.26 &= \frac{1}{(1-0)}\int_0^1\int_0^1 tx + (1-t)\left(\frac{0+1}{2}\right)\ d_qt\ {}^1d_qx \\
&\leq I_3\int_0^1 t^{1/2}t\ d_qt + \left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{2}\right)(S(0,1) + T(0,1)) \\
&\quad \times \int_0^1 (1-t)^{\frac{1}{2}}(1-t)\ d_qt + (S(0,1) \times I_4) + (T(0,1) \times I_5) \\
&\approx (0.22329 \times 0.77419) + \left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{2}\right)(3 \times 0.17967) \\
&\quad + (1 \times 0.06023) + (2 \times 0.01539) \\
&= 0.28294
\end{aligned}
$$

It is clear that

$$0.26 \leq 0.28294,$$

which demonstrates the result described in Theorem 5.

5. Conclusions

In this paper, we prove a variant of inequalities for (p,h)-convex function via q-calculus. We start with the necessary lemma and then use the key lemma to derive the main theorems. The results obtained in this paper generalize previous work in the earlier work if we take limit q tends to 1.

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