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SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR s-CONVEX FUNCTIONS

MEHMET ZEKI SARIKAYA AND MEHMET EYÜP KIRIS

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Abstract. Some new results related of the left-hand side of the Hermite-Hadamard type inequalities for the class of mappings whose second derivatives at certain powers are $s-$convex in the second sense are established. Also, some applications to special means of real numbers are provided.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard inequality [6]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if f is concave. For recent results, generalizations and new inequalities related to the Hermite-Hadamard inequality see [3, 4, 9, 11, 12, 14, 17].

The classical Hermite- Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$.

Definition 1. Let I be on interval in $\mathbb{R}$. Then $f : I \to R$ is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The class of functions which are s-convex in the second sense has been stated as the following (see [7]).

Definition 2. Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex (in the second sense), if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$. Some interesting and important inequalities for s-convex (in the second sense) functions can be found in [1,2,5,8,10,15,16,18,19]. It can be easily seen that convexity means just s-convexity when $s = 1$.

In [13], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt,$$

where

$$m(t) := \begin{cases} t^2 & ,t \in [0,\frac{1}{2}) \\ (1-t)^2 & ,t \in [\frac{1}{2},1]. \end{cases}$$

Also, the main inequalities in [13], pointed out as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{24}\left[\frac{|f''(a)| + |f''(b)|}{2}\right]. \tag{1.1}$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{8(2p+1)^{1/p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{1/q} \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose second derivatives at certain powers are s-convex functions in the second sense.

2. MAIN RESULTS

In order to prove our main results we need the following lemma:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$, then the following equaliy holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)$$

$$= \frac{(b-a)^2}{16}\left\{\int_0^1 t^2 f''\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt + \int_0^1 t^2 f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right\}. \quad (2.1)$$

Proof. By integration by parts, we have the following identity

$$\begin{aligned}
&\int_0^1 t^2 f''\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt + \int_0^1 t^2 f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt \\
&= \frac{4}{b-a}\left\{\int_0^1 t f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt - \int_0^1 t f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right\} \\
&= \frac{4}{b-a}\left\{ t\frac{2}{a-b} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\Big|_0^1 + \frac{2}{b-a}\int_0^1 f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt \right.\\
&\quad \left. + t\frac{2}{b-a} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\Big|_0^1 + \frac{2}{b-a}\int_0^1 f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt \right\} \\
&= -\frac{16}{(b-a)^2} f\left(\frac{a+b}{2}\right) + \frac{8}{(b-a)^2}\int_0^1 f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt \\
&\quad + \frac{8}{(b-a)^2}\int_0^1 f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt.
\end{aligned}$$

Using the change of the variable in last integrals, we get the required identity (2.1). □

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be twice differantiable mapping on I°, $a,b \in I$ with $a < b$. If $|f''|$ is s- convex in the second sense on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequaliy holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2\left(2^{s+4} - (4s+12)\right)}{2^{s+4}(s+1)(s+2)(s+3)}\left[|f''(a)| + |f''(b)|\right]. \quad (2.2)$$

Proof. Using Lemma 2 and the s- convexity in the second sense of $|f''|$, we find

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt$$

$$\leq \frac{(b-a)^2}{16}\int_0^1 t^2\left[\left(\frac{t}{2}\right)^s|f''(a)|+\left(\frac{2-t}{2}\right)^s|f''(b)|\right.$$

$$\left.+\left(\frac{2-t}{2}\right)^s|f''(a)|+\left(\frac{t}{2}\right)^s|f''(b)|\right]dt$$

$$=\frac{(b-a)^2(|f''(a)|+|f''(b)|)}{2^{s+4}}\left[\int_0^1\left[t^{s+2}+t^2(2-t)^s\right]dt\right]$$

$$=\frac{(b-a)^2(2^{s+4}-(4s+12))}{2^{s+4}(s+1)(s+2)(s+3)}\left[|f''(a)|+|f''(b)|\right]$$

where we have used the fact that

$$\int_0^1 t^2(2-t)^s\,dt=\frac{2^{s+4}-\left(s^2+7s+14\right)}{(s+1)(s+2)(s+3)},\quad\text{and}\quad\int_0^1 t^{s+2}dt=\frac{1}{s+3}.$$

The proof is completed. □

Remark 1. If we take $s=1$ in Theorem 3, then inequality (2.2) becomes inequality (1.1).

The next theorem gives a new upper bound of the left-hand side Hermite-Hadamard inequality for s-convex functions:

Theorem 4. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be twice differantiable mapping on I°, $a,b\in I$ with $a<b$. If $|f''|^q$ is s- convex in the second sense on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequaliy holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f''(a)|^q+\left(2^{s+1}-1\right)|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left(2^{s+1}-1\right)|f''(a)|^q+|f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right] \tag{2.3}$$

$$\leq\frac{(b-a)^2}{8}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\frac{1}{(s+1)^{\frac{1}{q}}}\left[|f''(a)|+|f''(b)|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using Hölder's inequality and the s-convexity in the second sense of $|f''|^q$, we find

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{16}\left(\int_0^1 |t|^{2p}\,dt\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1 \left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left\{\left(\int_0^1 \left[\left(\frac{t}{2}\right)^s |f''(a)|^q + \left(\frac{2-t}{2}\right)^s |f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 \left[\left(\frac{2-t}{2}\right)^s |f''(a)|^q + \left(\frac{t}{2}\right)^s |f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$
$$\le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{|f''(a)|^q + \left(2^{s+1}-1\right)|f''(b)|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left(2^{s+1}-1\right)|f''(a)|^q + |f''(b)|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

Let $a_1 = |f''(a)|^q$, $b_1 = \left(2^{s+1}-1\right)|f''(b)|^q$, $a_2 = \left(2^{s+1}-1\right)|f''(a)|^q$, $b_2 = |f''(b)|^q$. Here , $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact that

$$\sum_{k=1}^{n}(a_k+b_k)^s \le \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s.$$

for $(0 \le s < 1)$, $a_1, a_2, ..., a_n \ge 0$, $b_1, b_2, ..., b_n \ge 0$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left[\left(\left|f''(a)\right| + \left(2^{s+1}-1\right)^{\frac{1}{q}}\left|f''(b)\right|\right) + \left(\left(2^{s+1}-1\right)^{\frac{1}{q}}\left|f''(a)\right| + \left|f''(b)\right|\right)\right]$$
$$= \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(1+\left(2^{s+1}-1\right)^{\frac{1}{q}}\right)\left(\left|f''(a)\right| + \left|f''(b)\right|\right)\right]$$
$$\le \frac{(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(2^{s+1}\right)\left(\left|f''(a)\right| + \left|f''(b)\right|\right).$$

This completes the proof. □

Corollary 1. *Under assumption Theorem 4, if we take $s=1$, then the inequality (2.3) becomes the following inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left[\left(\frac{|f''(a)|^q+3|f''(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f''(a)|^q+|f''(b)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^2}{2^{\frac{2}{q}+2}(2p+1)^{\frac{1}{p}}}\left[|f''(a)|+|f''(b)|\right].
\end{aligned}$$

Theorem 5. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be twice differantiable mapping on I°, $a,b\in I$ with $a<b$. If $|f''|^q$ is s-convex in the second sense on $[a,b]$, for some fixed $s\in(0,1]$ and $q\geq 1$, then the following inequaliy holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \qquad (2.4)\\
&\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^s(s+3)}|f''(a)|^q+\frac{2^{s+4}-(s^2+7s+14)}{2^s(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{2^{s+4}-(s^2+7s+14)}{2^s(s+1)(s+2)(s+3)}|f''(a)|^q+\frac{1}{2^s(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 2, using the well known power mean inequality for $q\geq 1$ and the s- convexity in the second sense of $|f''|^q$, we find

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{16}\left\{\left(\int_0^1 t^2\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^2\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\right. \\
&\left.\left.+\left(\int_0^1 t^2\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]\right\} \\
&\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 t^2\left[\left(\frac{t}{2}\right)^s|f''(a)|^q+\left(\frac{2-t}{2}\right)^s|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^1 t^2\left[\left(\frac{2-t}{2}\right)^s\left|f''(a)\right|^q+\left(\frac{t}{2}\right)^s\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg]$$

$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2^s(s+3)}\left|f''(a)\right|^q+\frac{2^{s+4}-\left(s^2+7s+14\right)}{2^s(s+1)(s+2)(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{2^{s+4}-\left(s^2+7s+14\right)}{2^s(s+1)(s+2)(s+3)}\left|f''(a)\right|^q+\frac{1}{2^s(s+3)}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

which completes the proof of Theorem 5. □

Corollary 2. *Under assumption Theorem 5, if we take $s=1$, then inequality (2.4) becomes the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^2}{48}\left[\left(\frac{3\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{5\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right].$$

3. Applications to special means

In [1], the following result is given.

Let $g: I \to I_1 \subseteq [0,\infty)$ be a non-negative convex functions on I. Then $g^s(x)$ is $s-$convex on I, $0<s<1$.

For arbitrary positive real numbers a,b $(a \neq b)$, we shall consider the following special means:

(a) The arithmetic mean: $A=A(a,b):=\dfrac{a+b}{2}$, $a,b>0$,

(b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b>0,$$

(c) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b \end{cases},\ a,b>0,$$

(d) The $p-$logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \ p \in \mathbb{R} \diagdown \{-1,0\};\ a,b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

(1) Let $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^{s+1}$, $s \in (0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)dx = L_{s+1}^{s+1}(a,b),$$

$$\frac{f(a)+f(b)}{2} = A(a^{s+1}, b^{s+1}),$$

$$f\left(\frac{a+b}{2}\right) = A^{s+1}(a,b).$$

(a) From Theorem 3, we obtain

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{(b-a)^2 s(2^{s+4} - (4s+12))}{2^{s+3}(s+2)(s+3)} A(a^{s-1}, b^{s-1}).$$

For instance, if $s = 1$ then we get

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{1}{12}(b-a)^2.$$

(b) From Theorem 4, we have

$$\begin{aligned} &\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \\ &\leq \frac{s(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{(s-1)q} + \left(2^{s+1}-1\right)b^{(s-1)q}\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\left(2^{s+1}-1\right)a^{(s-1)q} + b^{(s-1)q}\right)^{\frac{1}{q}}\right] \\ &\leq \frac{s(b-a)^2}{4}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}-1}} A(a^{s-1}, b^{s-1}), \end{aligned}$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For instance, if $s = 1$ then we have

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4}\left(\frac{1}{4p+2}\right)^{\frac{1}{p}},\ p > 1.$$

(c) From Theorem 5, we get

$$\left|L_{s+1}^{s+1}(a,b)-A^{s+1}(a,b)\right| \leq \frac{(b-a)^2}{2^{s+4}(s+2)(s+3)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}$$
$$\left[\left(s(s+1)(s+2)a^{(s-1)q}+\left(2^{s+4}s-s^3-7s^2-14s\right)b^{(s-1)q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(2^{s+4}s-s^3-7s^2-14s\right)a^{(s-1)q}+s(s+1)(s+2)b^{(s-1)q}\right)^{\frac{1}{q}}\right]$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\left|L_2^2(a,b)-A^2(a,b)\right| \leq \frac{(b-a)^2(48)^{\frac{1}{q}}}{576}, \quad q>1.$$

(2) Let $f:[a,b]\subseteq[0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\dfrac{1}{x^s}\in K_s^2$, $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_a^b f(x)dx = L_{-s}^{-s}(a,b),$$
$$\frac{f(a)+f(b)}{2} = A(a^{-s},b^{-s}),$$
$$f\left(\frac{a+b}{2}\right) = A^{-s}(a,b).$$

(a) From Theorem 3, we obtain

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right| \leq \frac{(b-a)^2\left(2^{s+4}-(4s+12)\right)}{2^{s+4}(s+1)(s+2)(s+3)}\left[a^{(-s-2)}+b^{(-s-2)}\right].$$

For instance, if $s=1$ then we get

$$\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right| \leq \frac{(b-a)^2}{24}A\left(a^{-3},b^{-3}\right).$$

(b) From Theorem 4, we have

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right|$$
$$\leq \frac{s(b-a)^2}{2^{s+4}}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{(-s-2)q}+\left(2^{s+1}-1\right)b^{(-s-2)q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(2^{s+1}-1\right)a^{(-s-2)q}+b^{(-s-2)q}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{8}\left(\frac{2^s}{2p+1}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}-1}}\left[a^{(-s-2)}+b^{(-s-2)}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\begin{aligned}&\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right|\\&\leq\frac{(b-a)^2}{32}\left(\frac{2}{2p+1}\right)^{\frac{1}{p}}\left[\left(a^{-3q}+3b^{-3q}\right)^{\frac{1}{q}}+\left(3a^{-3q}+b^{-3q}\right)^{\frac{1}{q}}\right]\\&\leq\frac{(b-a)^2}{8}\left(\frac{4}{2p+1}\right)^{\frac{1}{p}}A\left(a^{-3},b^{-3}\right)\end{aligned}$$

where $q>1$.

(c) From Theorem 5, we get

$$|L_{-s}^{-s}(a,b)-A^{-s}(a,b)|\leq\frac{(b-a)^2}{2^{s+4}(s+2)(s+3)}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}$$

$$\left[\left(s(s+1)(s+2)a^{(-s-2)q}+\left(2^{s+4}s-s^3-7s^2-14s\right)b^{(-s-2)q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(2^{s+4}s-s^3-7s^2-14s\right)a^{(-s-2)q}+s(s+1)(s+2)b^{(-s-2)q}\right)^{\frac{1}{q}}\right]$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. For instance, if $s=1$ then we have

$$\left|L_{-1}^{-1}(a,b)-A^{-1}(a,b)\right|\leq\frac{(b-a)^2}{384}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left[\left(6a^{-3q}+10b^{-3q}\right)^{\frac{1}{q}}+\left(10a^{-3q}+6b^{-3q}\right)^{\frac{1}{q}}\right],$$

where $q>1$.

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Authors' addresses

Mehmet Zeki Sarıkaya
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail address: sarikayamz@gmail.com

Mehmet Eyüp Kiris
Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon, Turkey
E-mail address: mkiris@gmail.com, kiris@aku.edu.tr