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GENERALIZATION OF SOME HARDY-TYPE INTEGRAL INEQUALITY WITH NEGATIVE PARAMETER

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Abstract

In 2007, Bicheng Yang [3] presented a new Hardy-type integral inequality with a best constant factor. The aim of this work is to give a direct generalization of these inequalities obtained with negative parameter $p < 0$.

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1 Introduction

Bicheng Yang [3] announced the following Hardy-type integral inequality.

Lemma 1. *If* $p < 0,\ r > 1,\ f(t) \geq 0$ *and* $0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty$, *then*

$$\int_0^\infty x^{-r} \left(\int_x^\infty f(t)dt \right)^p dx \leq \left(\frac{p}{1-r} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt, \tag{1}$$

where the constant factor $\left(\frac{p}{1-r} \right)^p$ *is the best possible.*

Lemma 2. *If* $p < 0,\ r < 1,\ f(t) \geq 0$ *and* $0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty$, *then*

$$\int_0^\infty x^{-r} \left(\int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{r-1} \right)^p \int_0^\infty t^{-r}(tf(t))^p dt, \tag{2}$$

where the constant factor $\left(\frac{p}{r-1} \right)^p$ *is the best possible.*

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These inequalities play an important role in analysis and its applications. Other authors have also studied these type inequalities in more general forms as it may be seen in [1]-[6].

The overall structure of the study takes the form of three sections with an introduction. The remainder of this work is organized as follows: In the second section we prove our results; Theorems 1-3. The third section is for application, we generalize the integral inequalities (1) and (2). In particular a case we obtain the best constant factor.

2 Main Results

Our first result is given in the following theorem.

Theorem 1. *Let* $f, g > 0$, $p < 0$, $r > 1$ *and* $F(x) = \int_x^\infty f(t)dt$. *If* $\frac{x}{g(x)}$ *is non-decreasing, then*

$$\int_0^\infty g^{-r}(x)F^p(x)dx \leq \left(\frac{p}{1-r}\right)^p \int_0^\infty g^{-r}(x)(xf(x))^p dt, \tag{3}$$

where the right hand side is finite.

Proof. By the reverse Hölder inequality for $\frac{1}{p} + \frac{1}{p'} = 1$, it follows that

$$\int_x^\infty f(t)dt = \int_x^\infty t^{-\frac{1+p-r}{p'p}} t^{\frac{1+p-r}{p'p}} f(t)dt$$

$$\geq \left(\int_x^\infty t^{-\frac{1+p-r}{p}} dt\right)^{\frac{1}{p'}} \left(\int_x^\infty t^{\frac{1+p-r}{p'}} f^p(t)dt\right)^{\frac{1}{p}}$$

$$= \left(\frac{p}{1-r} x^{\frac{r-1}{p}}\right)^{\frac{1}{p'}} \left(\int_x^\infty t^{\frac{1+p-r}{p'}} f^p(t)dt\right)^{\frac{1}{p}}$$

then we find

$$F^p(x) \leq \left(\frac{p}{1-r} x^{\frac{r-1}{p}}\right)^{\frac{p}{p'}} \left(\int_x^\infty t^{\frac{1+p-r}{p'}} f^p(t)dt\right)$$

$$= \left(\frac{p}{1-r}\right)^{p-1} x^{\frac{1-r}{p}+r-1} \int_x^\infty t^{\frac{1+p-r}{p'}} f^p(t)dt.$$

Therefore, we get

$$\begin{aligned}
&\int_0^\infty g^{-r}(x)F^p(x)dx \\
&\leq \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty \int_x^\infty g^{-r}(x)x^{\frac{1-r}{p}+r-1}t^{\frac{1+p-r}{p'}} f^p(t)dtdx \\
&= \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t) \int_0^t g^{-r}(x)x^{\frac{1-r}{p}+r-1}dxdt \\
&= \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t) \left(\int_0^t \left(\frac{x}{g(x)}\right)^r x^{\frac{1-r}{p}-1}dx\right) dt.
\end{aligned}$$

From the assumption of the function $\left(\frac{x}{g(x)}\right)^r$ non-decreasing on $(0,t)$, we have

$$\begin{aligned}
R_1(t) &= \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t) \left(\int_0^t \left(\frac{x}{g(x)}\right)^r x^{\frac{1-r}{p}-1}dx\right) dt \\
&\leq \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t) \left(\frac{t}{g(t)}\right)^r \left(\int_0^t x^{\frac{1-r}{p}-1}dx\right) dt \\
&= \left(\frac{p}{1-r}\right)^{p-1} \int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t) \left(\frac{t}{g(t)}\right)^r \left(\frac{p}{1-r}\right) t^{\frac{1-r}{p}} dt \\
&= \left(\frac{p}{1-r}\right)^{p} \int_0^\infty g^{-r}(t)(tf(t))^p dt.
\end{aligned}$$

Thus, we obtain that

$$\int_0^\infty g^{-r}(x)F^p(x)dx \leq \left(\frac{p}{1-r}\right)^{p} \int_0^\infty g^{-r}(x)(xf(x))^p dx$$

which completes the proof. □

Theorem 2. *Let* $f, g > 0,\ p < 0,\ 0 \leq r < 1$ *and* $F(x) = \int_0^x f(t)dt$. *If* $\frac{x}{g(x)}$ *is non-increasing then*

$$\int_0^\infty g^{-r}(x)F^p(x)dx \leq \left(\frac{p}{r-1}\right)^{p} \int_0^\infty g^{-r}(x)(xf(x))^p dx, \tag{4}$$

where the right hand side is finite.

Proof. By the reverse Hölder inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\begin{aligned}\int_0^x f(t)dt &= \int_0^x t^{-\frac{1+p-r}{p'p}} t^{\frac{1+p-r}{p'p}} f(t)dt\\ &\geq \left(\int_0^x t^{-\frac{1+p-r}{p}}dt\right)^{\frac{1}{p'}} \left(\int_0^x t^{\frac{1+p-r}{p'}} f^p(t)dt\right)^{\frac{1}{p}}\\ &= \left(\frac{p}{r-1}x^{\frac{r-1}{p}}\right)^{\frac{1}{p'}} \left(\int_0^x t^{\frac{1+p-r}{p'}} f^p(t)dt\right)^{\frac{1}{p}}\end{aligned}$$

then we get

$$\begin{aligned}F^p(x) &\leq \left(\frac{p}{r-1}x^{\frac{r-1}{p}}\right)^{\frac{p}{p'}} \left(\int_0^x t^{\frac{1+p-r}{p'}} f^p(t)dt\right)\\ &= \left(\frac{p}{r-1}\right)^{p-1} x^{\frac{1-r}{p}+r-1}\int_0^x t^{\frac{1+p-r}{p'}} f^p(t)dt.\end{aligned}$$

Thus, we find that

$$\begin{aligned}&\int_0^\infty g^{-r}(x)F^p(x)dx\\ &\leq \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty\int_0^x g^{-r}(x)x^{\frac{1-r}{p}+r-1}t^{\frac{1+p-r}{p'}} f^p(t)dtdx\\ &= \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t)\int_t^\infty g^{-r}(x)x^{\frac{1-r}{p}+r-1}dxdt\\ &= \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t)\left(\int_t^\infty \left(\frac{x}{g(x)}\right)^r x^{\frac{1-r}{p}-1}dx\right)dt.\end{aligned}$$

By the assumption of the function $\left(\dfrac{x}{g(x)}\right)^r$ non-increasing on (t,∞), we have

$$\begin{aligned}
R_2(t) &= \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t)\left(\int_t^\infty \left(\frac{x}{g(x)}\right)^r x^{\frac{1-r}{p}-1}dx\right)dt\\
&\le \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t)\left(\frac{t}{g(t)}\right)^r\left(\int_t^\infty x^{\frac{1-r}{p}-1}dx\right)dt\\
&= \left(\frac{p}{r-1}\right)^{p-1}\int_0^\infty t^{\frac{1+p-r}{p'}} f^p(t)\left(\frac{t}{g(t)}\right)^r\left(\frac{p}{r-1}\right)t^{\frac{1-r}{p}}dt\\
&= \left(\frac{p}{1-r}\right)^p\int_0^\infty g^{-r}(t)(tf(t))^p dt.
\end{aligned}$$

We obtain that

$$\int_0^\infty h^{-r}(x)F^p(x)dx \le \left(\frac{p}{1-r}\right)^p\int_0^\infty g^{-r}(x)(xf(x))^p dx$$

which completes the proof. □

Theorem 3. *Let $f, g > 0,\ p < 0,\ r < 0$ and $F(x) = \int_0^x f(t)dt$. If $\dfrac{x}{g(x)}$ is non-decreasing then*

$$\int_0^\infty g^{-r}(x)F^p(x)dx \le \left(\frac{p}{r-1}\right)^p\int_0^\infty g^{-r}(x)(xf(x))^p dx, \tag{5}$$

where the right hand side is finite.

Proof. The proof of Theorem 3 is similar to Theorem 2. □

3 Applications

We believe that the above three theorems should have many applications especially in the theory of weights and other fields. In this paper we give some applications of theorems.

Corollary 1. *Let $f > 0,\ p < 0,\ r > 1,\ m \le 1$ and $F(x) = \int_x^\infty f(t)dt$, then*

$$\int_0^\infty x^{-mr}F^p(x)dx \le \left(\frac{p}{1-r}\right)^p\int_0^\infty x^{-mr}(xf(x))^p dx. \tag{6}$$

Proof. This follows from Theorem 1 where $g(x) = x^m$. □

Remark 1. *We can get the particular cases*
(i) If we take $m = 1$ *we get*

$$\int_0^\infty x^{-r}\left(\int_x^\infty f(t)dt\right)^p dx \le \left(\frac{p}{1-r}\right)^p \int_0^\infty x^{-r}(xf(x))^p dx. \tag{7}$$

(ii) for $r = 2$*, one has*

$$\int_0^\infty x^{-2m}\left(\int_x^\infty f(t)dt\right)^p dx \le (-p)^p \int_0^\infty x^{p-2m} f^p(x)dx. \tag{8}$$

(iii)for $r = 1-p$*, one has*

$$\int_0^\infty x^{m(1-p)}\left(\int_x^\infty f(t)dt\right)^p dx \le \int_0^\infty x^{m(p-1)+p} f^p(x)dx. \tag{9}$$

Remark 2. *If* $m = 1$ *then the constant factor* $\left(\frac{p}{1-r}\right)^p$ *is the best possible.*

Proof. For $0 < \theta < mr - 1$, we put

$$f_\theta(x) = \begin{cases} x^{\frac{mr-1-\theta}{p}-1}, & x \in [1,\infty) \\ 0, & x \in (0,1) \end{cases}$$

then we get

$$\begin{aligned} &\int_0^\infty x^{-mr}\left(\int_x^\infty f_\theta(t)dt\right)^p dx \\ &= \int_1^\infty x^{-mr}\left(\int_x^\infty t^{\frac{mr-1-\theta}{p}-1}dt\right)^p dx \\ &= \left(\frac{p}{1-mr+\theta}\right)^p \int_1^\infty x^{-\theta-1}dx \\ &= \left(\frac{p}{1-mr+\theta}\right)^p \frac{1}{\theta}. \end{aligned}$$

$$\int_0^\infty x^{-mr}(xf_\theta(x))^p dx = \int_1^\infty x^{-\theta-1}dx = \frac{1}{\theta}.$$

We have $0 < mr - 1$ and $r > 1$ then $0 < m \le 1$. For $\theta \longrightarrow 0$, we get that the constant factor $\left(\frac{p}{1-mr}\right)^p$ is positive if $1-mr < 0$. We obtain $r > \frac{1}{m}$ then $m = 1$. Therefore,

$$\int_0^\infty x^{-mr}\left(\int_x^\infty f_\theta(t)dt\right)^p dx = \left(\frac{p}{1-r}\right)^p \int_0^\infty x^{-mr}(xf_\theta(x))^p dx$$

we deduct if $m = 1$ then the constant factor $\left(\frac{p}{1-r}\right)^p$ is the best possible. □

With $m = 1$, the constant factors in inequalities (7),(8) and (9) are the best possible.

Corollary 2. *Let* $f > 0,\ p < 0,\ r < 0,\ m \leq 1$ *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty x^{-mr} F^p(x)dx \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{-mr}(xf(x))^p dx, \tag{10}$$

Proof. This follows from Theorem 3 where $g(x) = x^m$. □

Remark 3. *We can get the particular cases*
(i) For $r = p$, *one has*

$$\int_0^\infty x^{-mp} \left(\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty x^{p(1-m)} f^p(x)dx. \tag{11}$$

(ii)for $r = p+1$, *one has*

$$\int_0^\infty x^{-m} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \int_0^\infty x^{p(1-m)-m} f^p(x)dx. \tag{12}$$

Corollary 3. *Let* $f > 0,\ p < 0,\ 0 \leq r < 1,\ m \geq 1$ *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty x^{-mr} F^p(x)dx \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{-mr}(xf(x))^p dx, \tag{13}$$

Proof. This follows from Theorem 2 where $g(x) = x^m$. □

Remark 4. *We can get the particular cases*
(i) For $r = 0$, *one has*

$$\int_0^\infty \left(\int_0^x f(t)dt\right)^p dx \leq (-p)^p \int_0^\infty (xf(x))^p dx. \tag{14}$$

(ii)For $r = \frac{1}{m}$ *and* $m \neq 1$, *one has*

$$\int_0^\infty x^{-1} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{mp}{1-m}\right)^p \int_0^\infty x^{p-1} f^p(x)dx. \tag{15}$$

Remark 5. *If we take* $m = 1$ *in Corollary 2 and Corollary 3, we get;*

$$for\ r < 1, \quad \int_0^\infty x^{-r}(\int_0^x f(t)dt)^p dx \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{-r}(xf(x))^p dx. \tag{16}$$

where the constant factor $\left(\frac{p}{r-1}\right)^p$ *is the best possible.*

Proof. For $0 < \theta < 1-r$, we put

$$f_\theta(x) = \begin{cases} x^{\frac{r-1+\theta}{p}-1}, & x \in (0,1] \\ 0, & x \in (1,\infty) \end{cases},$$

then we get

$$\begin{aligned}&\int_0^\infty x^{-r}\left(\int_0^x f_\theta(t)dt\right)^p dx\\&=\int_0^1 x^{-r}\left(\int_0^x t^{\frac{r-1+\theta}{p}-1}dt\right)^p dx\\&=\left(\frac{p}{r-1+\theta}\right)^p\int_0^1 x^{\theta-1}dx\\&=\left(\frac{p}{r-1+\theta}\right)^p\frac{1}{\theta}.\end{aligned}$$

$$\int_0^\infty x^{-r}(xf_\theta(x))^p dx=\int_0^1 x^{\theta-1}dx=\frac{1}{\theta}.$$

For $\theta \longrightarrow 0$, we get that the constant factor $\left(\frac{p}{r-1}\right)^p$ is the best possible in (16). □

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