



## Original article

## Some generalized Ostrowski type inequalities for functions whose second derivatives absolute values are convex and applications

Yusuf Erdem<sup>a</sup>, Hüseyin Budak<sup>b,\*</sup>, Hasan Öğünmez<sup>a</sup><sup>a</sup>Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar, Turkey<sup>b</sup>Department of Mathematics, Faculty of Science and Arts, Düzce University, Turkey

Received 27 December 2016; received in revised form 6 May 2017; accepted 12 June 2017

Available online xxxxx

## Abstract

We first establish some Ostrowski type inequalities for mappings whose second derivatives absolute values are convex. Then we give some special cases of these inequalities which provide extensions of those given in earlier works. Finally, some applications of these inequalities for special means are also provided.

© 2017 Ivane Javakhishvili Tbilisi State University. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

**Keywords:** Ostrowski inequality; Convex functions; Special means

## 1. Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [1] is the following classical integral inequality associated with the differentiable mappings:

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \quad (1.1)$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.

Ostrowski inequality (1.1) has applications in numerical analysis, probability and optimization theory, stochastic, statistics, information and integral operator theory, see for example [2–20].

\* Corresponding author.

E-mail addresses: [mateyus26@gmail.com](mailto:mateyus26@gmail.com) (Y. Erdem), [hsyn.budak@gmail.com](mailto:hsyn.budak@gmail.com) (H. Budak), [hogunmez@aku.edu.tr](mailto:hogunmez@aku.edu.tr) (H. Öğünmez).

Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

The remainder of this work is organized as follows: In this section, we present definition of convex function and give an important identity which will be used to establish our main results. In Section 2, some new Ostrowski type integral inequalities are proved for function whose second derivatives absolute values are convex. These inequalities are provided for special means in Section 3. At the end some conclusions of research are discussed in Section 4.

**Definition 1.** The function  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ , is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

Dragomir and Pearce proved the following identity and using this identity they gave important trapezoid inequalities.

**Lemma 1 ([5]).** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $f'' \in L_1[a, b]$ , then

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta + (1-t)b) dt. \quad (1.2)$$

Sarikaya et al. gave the following identity for twice differentiable mapping:

**Lemma 2 ([17]).** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $f'' \in L_1[a, b]$ , then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{4} \int_0^1 m(t) [f''(ta + (1-t)b) + f''(tb + (1-t)a)] dt, \end{aligned} \quad (1.3)$$

where

$$m(t) = \begin{cases} t^2, & t \in \left[0, \frac{1}{2}\right] \\ (1-t)^2, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

**Theorem 1 ([17]).** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I$  with  $f'' \in L[a, b]$ . If  $|f''|$  is convex on  $[a, b]$ , then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{24} \left[ \frac{|f''(a)| + |f''(b)|}{2} \right]. \quad (1.4)$$

In [10], Erden et al. gave the following important equality for twice differentiable function:

**Lemma 3.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I^\circ$  such that  $f'' \in L[a, b]$ , the interior of the interval  $I$ , where  $a, b \in I^\circ$  with  $a < b$ . Then the following identity holds:

$$\begin{aligned} & \frac{1}{2(b-a)} \int_a^b P_h(x, t) f''(t) dt \\ &= \frac{h-2}{2} \left(x - \frac{a+b}{2}\right) f'(x) + f(x) - \frac{f(b) - f(a)}{2(b-a)} m_h(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ &=: S_{x,h}(f) \end{aligned} \quad (1.5)$$

for

$$P_h(x, t) := \begin{cases} (a-t)(t-a-m_h(x)), & a \leq t < x \\ (b-t)(t-b-m_h(x)), & x \leq t \leq b \end{cases}$$

where  $m_h(x) = (x - \frac{a+b}{2})h$ ,  $h \in [0, 2]$  and  $x \in [a, b]$ .

Using the convexity of  $|f''|$  and identity (1.5), we establish some generalized Ostrowski type inequalities.

## 2. Main results

Now, we establish our main theorems and also give some results related to these theorems.

**Theorem 2.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I^\circ$ , the interior of the interval  $I$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f''|$  is a convex mapping on  $[a, b]$ , then the following inequalities hold:

$$\begin{aligned} & |S_{x,h}(f)| \\ & \leq \frac{1}{2(b-a)^2} \left\{ |f''(a)| \left[ \frac{(b-x)^4 - (x-a)^4}{4} + m_h(x) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a) m_h(x) \frac{(x-a)^2}{2} + \frac{[m_h(x)]^4}{6} \right] \right. \\ & \quad \left. + |f''(b)| \left[ \frac{(x-a)^4 - (b-x)^4}{4} - m_h(x) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(b-x)^3}{3} + (b-a) m_h(x) \frac{(b-x)^2}{2} - \frac{[m_h(x)]^4}{6} - (b-a) \frac{[m_h(x)]^3}{3} \right] \right\} \end{aligned} \quad (2.1)$$

for all  $a \leq x \leq \frac{a+b}{2}$  with  $h \in [0, 2]$  and

$$\begin{aligned} & |S_{x,h}(f)| \\ & \leq \frac{1}{2(b-a)^2} \left\{ |f''(a)| \left[ \frac{(b-x)^4 - (x-a)^4}{4} + m_h(x) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a) m_h(x) \frac{(x-a)^2}{2} - \frac{[m_h(x)]^4}{6} + (b-a) \frac{[m_h(x)]^3}{3} \right] \right. \\ & \quad \left. + |f''(b)| \left[ \frac{(x-a)^4 - (b-x)^4}{4} - m_h(x) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(b-x)^3}{3} + (b-a) m_h(x) \frac{(b-x)^2}{2} + \frac{[m_h(x)]^4}{6} \right] \right\} \end{aligned} \quad (2.2)$$

for all  $\frac{a+b}{2} \leq x \leq b$  with  $h \in [0, 2]$ , where  $m_h(x) = h(x - \frac{a+b}{2})$ .

**Proof.** Taking modulus of equality given in (1.5) and using the triangle inequality for integrals, we find that

$$\begin{aligned} |S_{x,h}(f)| &= \frac{1}{2(b-a)} \left| \int_a^b P_h(x, t) f''(t) dt \right| \\ &\leq \frac{1}{2(b-a)} \int_a^b |P_h(x, t)| |f''(t)| dt \end{aligned}$$

$$= \frac{1}{2(b-a)} \left[ \int_a^x |a-t| |t-a-m_h(x)| |f''(t)| dt + \int_x^b |b-t| |t-b-m_h(x)| |f''(t)| dt \right].$$

Since  $|f''|$  is a convex mapping on  $[a, b]$ , we have

$$|f''(t)| = \left| f'' \left( \frac{b-t}{b-a}a + \frac{t-a}{b-a}b \right) \right| \leq \frac{b-t}{b-a} |f''(a)| + \frac{t-a}{b-a} |f''(b)|. \quad (2.3)$$

Using (2.3), we get

$$\begin{aligned} |S_{x,h}(f)| &\leq \frac{1}{2(b-a)^2} \left[ \int_a^x |a-t| |t-a-m_h(x)| [(b-t)|f''(a)| + (t-a)|f''(b)|] dt \right. \\ &\quad \left. + \int_x^b |b-t| |t-b-m_h(x)| [(b-t)|f''(a)| + (t-a)|f''(b)|] dt \right] \\ &= \frac{1}{2(b-a)^2} \left\{ |f''(a)| \left[ \int_a^x |a-t| |t-a-m_h(x)| (b-t) dt \right. \right. \\ &\quad \left. \left. + \int_x^b |b-t| |t-b-m_h(x)| (b-t) dt \right] \right. \\ &\quad \left. + |f''(b)| \left[ \int_a^x |a-t| |t-a-m_h(x)| (t-a) dt \right. \right. \\ &\quad \left. \left. + \int_x^b |b-t| |t-b-m_h(x)| (t-a) dt \right] \right\} \\ &= \frac{1}{2(b-a)^2} [|f''(a)| (I_1 + I_2) + |f''(b)| (I_3 + I_4)]. \end{aligned} \quad (2.4)$$

We calculate integrals  $I_i, i = 1, \dots, 4$ , for the cases  $a \leq x \leq \frac{a+b}{2}$  and  $\frac{a+b}{2} \leq x \leq b$ ; In case of  $a \leq x \leq \frac{a+b}{2}$ , using the fact that  $m_h(x) \leq 0$ , we get

$$\begin{aligned} I_1 &= \int_a^x (t-a)(t-a-m_h(x))(b-t) dt \\ &= (b-a) \int_a^x (t-a)(t-a-m_h(x)) dt - \int_a^x (t-a)^2 (t-a-m_h(x)) dt \\ &= (b-a) \frac{(x-a)^3}{3} - (b-a)m_h(x) \frac{(x-a)^2}{2} - \frac{(x-a)^4}{4} + m_h(x) \frac{(x-a)^3}{3}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} I_2 &= \int_x^b (b-t)^2 |t-b-m_h(x)| dt \\ &= \int_x^{b+m_h(x)} (b-t)^2 (m_h(x) + b-t) dt + \int_{b+m_h(x)}^b (b-t)^2 (t-b-m_h(x)) dt \\ &= \frac{[m_h(x)]^4}{6} + m_h(x) \frac{(b-x)^3}{3} + \frac{(b-x)^4}{4}, \end{aligned} \quad (2.6)$$

$$I_3 = \int_a^x (t-a)^2 (t-a-m_h(x)) dt = \frac{(x-a)^4}{4} - m_h(x) \frac{(x-a)^3}{3} \quad (2.7)$$

and

$$I_4 = \int_x^b (b-t) |t-b-m_h(x)| (t-a) dt$$

$$\begin{aligned}
&= \int_x^{b+m_h(x)} (b-t)(m_h(x)+b-t)(t-a) dt \\
&\quad + \int_x^b (b-t)(t-b-m_h(x))(t-a) dt \\
&= -\frac{[m_h(x)]^4}{6} - (b-a)\frac{[m_h(x)]^3}{3} - \frac{(b-x)^4}{4} - m_h(x)\frac{(b-x)^3}{3} \\
&\quad + (b-a)m_h(x)\frac{(b-x)^2}{2} + (b-a)\frac{(b-x)^3}{3}.
\end{aligned} \tag{2.8}$$

If we substitute the equalities (2.5)–(2.6) in (2.4), then we obtain the required inequality (2.1).

In case of  $\frac{a+b}{2} \leq x \leq b$ , using the fact that  $m_h(x) \geq 0$ , we get

$$\begin{aligned}
I_1 &= \int_a^x (t-a)|t-a-m_h(x)|(b-t) dt \\
&= -\frac{[m_h(x)]^4}{6} + (b-a)\frac{[m_h(x)]^3}{3} - \frac{(x-a)^4}{4} + m_h(x)\frac{(x-a)^3}{3} \\
&\quad - (b-a)m_h(x)\frac{(x-a)^2}{2} + (b-a)\frac{(x-a)^3}{3}
\end{aligned} \tag{2.9}$$

$$I_2 = \int_x^b (b-t)^2(m_h(x)+b-t) dt = m_h(x)\frac{(b-x)^3}{3} + \frac{(b-x)^4}{4}, \tag{2.10}$$

$$I_3 = \int_a^x (t-a)^2|t-a-m_h(x)| dt = \frac{[m_h(x)]^4}{6} + \frac{(x-a)^4}{4} - m_h(x)\frac{(x-a)^3}{3} \tag{2.11}$$

and

$$\begin{aligned}
I_4 &= \int_x^b (b-t)(m_h(x)+b-t)(t-a) dt \\
&= -m_h(x)\frac{(b-x)^3}{3} - \frac{(b-x)^4}{4} + (b-a)m_h(x)\frac{(b-x)^2}{2} + (b-a)\frac{(b-x)^3}{3}.
\end{aligned} \tag{2.12}$$

If we substitute the equalities (2.9)–(2.12) in (2.4), then we obtain the desired inequality (2.2). The proof is thus completed.  $\square$

**Remark 1.** If we choose  $x = \frac{a+b}{2}$  in Theorem 2, then the inequalities (2.1) and (2.2) reduce to the inequality (1.4).

**Remark 2.** If we choose  $h = 0$  in the inequalities (2.1) and (2.2), then we have the following inequality

$$\begin{aligned}
&\left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
&\leq \frac{b-a}{2} \left\{ |f''(a)| \left[ \left[ \frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2 \right] \left( \frac{a+b}{2} - x \right) + \frac{(x-a)^3}{3(b-a)^2} \right] \right. \\
&\quad \left. + |f''(b)| \left[ \left[ \frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2 \right] \left( x - \frac{a+b}{2} \right) + \frac{(b-x)^3}{3(b-a)^2} \right] \right\}
\end{aligned}$$

for  $x \in [a, b]$ .

**Corollary 1.** Let us substitute  $x = a$  and  $x = b$  in Theorem 2. Subsequently, if we add the obtained result and use the triangle inequality for the modulus, we get the inequality

$$\left| \frac{h-2b-a}{2} \frac{f'(b)-f'(a)}{4} + \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{8} \left[ \frac{2}{3} - \frac{h}{2} + \frac{h^3}{12} \right] [|f''(a)| + |f''(b)|].$$

**Remark 3.** If we take  $h = 0$  in Corollary 1, then we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{b-a}{4} (f'(b)-f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{6} \left[ \frac{|f''(a)| + |f''(b)|}{2} \right]. \quad (2.13)$$

Particularly, if  $|f(x)| < M$ ,  $x \in [a, b]$ , then the inequality reduces the inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{b-a}{4} (f'(b)-f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M(b-a)^2}{6}$$

which was given by Liu in [12].

**Remark 4.** If we take  $h = 2$  in Corollary 1, then we have the trapezoid inequality

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{12} \left[ \frac{|f''(a)| + |f''(b)|}{2} \right] \quad (2.14)$$

which was given by Kiris and Sarikaya in [11].

**Theorem 3.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be twice differentiable function on  $I^\circ$ , the interior of the interval  $I$ , where  $a, b \in I^\circ$  with  $a < b$ . If  $|f''|^q$ ,  $q > 1$ , is a convex mapping on  $[a, b]$ , then the following inequalities hold:

$$\begin{aligned} & |S_{x,h}(f)| \\ & \leq \frac{1}{2(b-a)^{1+\frac{1}{q}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ ((x-a-m_h(x))^{p+1} + (-1)^p [m_h(x)]^{p+1})^{\frac{1}{p}} \right. \\ & \quad \times \left[ \left( \frac{(b-a)(x-a)^{q+1}}{q+1} - \frac{(x-a)^{q+2}}{q+2} \right) |f''(a)|^q + \frac{(x-a)^{q+2}}{q+2} |f''(b)|^q \right]^{\frac{1}{q}} \\ & \quad + ((m_h(x)+b-x)^{p+1} + (-1)^{p+1} [m_h(x)]^{p+1})^{\frac{1}{p}} \\ & \quad \times \left[ \frac{(b-x)^{q+2}}{q+2} |f''(a)|^q + \left( \frac{(b-a)(b-x)^{q+1}}{q+1} - \frac{(b-x)^{q+2}}{q+2} \right) |f''(b)|^q \right]^{\frac{1}{q}} \Big\} \end{aligned} \quad (2.15)$$

for  $a \leq x \leq \frac{a+b}{2}$ , and

$$\begin{aligned} & |S_{x,h}(f)| \\ & \leq \frac{1}{2(b-a)^{1+\frac{1}{q}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ ([m_h(x)]^{p+1} + (x-a-m_h(x))^{p+1})^{\frac{1}{p}} \right. \\ & \quad \times \left[ \left( \frac{(b-a)(x-a)^{q+1}}{q+1} - \frac{(x-a)^{q+2}}{q+2} \right) |f''(a)|^q + \frac{(x-a)^{q+2}}{q+2} |f''(b)|^q \right]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & + \left( (m_h(x) + b - x)^{p+1} - [m_h(x)]^{p+1} \right)^{\frac{1}{p}} \\
 & \times \left[ \frac{(b-x)^{q+2}}{q+2} |f''(a)|^q + \left( \frac{(b-a)(b-x)^{q+1}}{q+1} - \frac{(b-x)^{q+2}}{q+2} \right) |f''(b)|^q \right]^{\frac{1}{q}} \Bigg\}
 \end{aligned} \tag{2.16}$$

for  $\frac{a+b}{2} \leq x \leq b$  with  $h \in [0, 2]$ , where  $m_h(x) = h(x - \frac{a+b}{2})$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Proof.** Taking the modulus of equality given in Lemma 3 and then using the well-known Hölder's inequality, we have

$$\begin{aligned}
 & |S_{x,h}(f)| \\
 & \leq \frac{1}{2(b-a)} \int_a^b |P_h(x, t)| |f''(t)| dt \\
 & = \frac{1}{2(b-a)} \left[ \int_a^x |a-t| |t-a-m_h(x)| |f''(t)| dt \right. \\
 & \quad \left. + \int_x^b |b-t| |t-b-m_h(x)| |f''(t)| dt \right] \\
 & \leq \frac{1}{2(b-a)} \left[ \left( \int_a^x |t-a-m_h(x)|^p dt \right)^{\frac{1}{p}} \left( \int_a^x (t-a)^q |f''(t)|^q dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left( \int_x^b |t-b-m_h(x)|^p dt \right)^{\frac{1}{p}} \left( \int_x^b (b-t)^q |f''(t)|^q dt \right)^{\frac{1}{q}} \right].
 \end{aligned} \tag{2.17}$$

Since  $|f''|^q$  is a convex mapping on  $[a, b]$ , we get

$$|f''(t)|^q = \left| f'' \left( \frac{b-t}{b-a}a + \frac{t-a}{b-a}b \right) \right|^q \leq \frac{b-t}{b-a} |f''(a)|^q + \frac{t-a}{b-a} |f''(b)|^q. \tag{2.18}$$

Using (2.18), we have

$$\begin{aligned}
 & \int_a^x (t-a)^q |f''(t)|^q dt \\
 & \leq \frac{1}{b-a} \int_a^x (t-a)^q [(b-t) |f''(a)|^q + (t-a) |f''(b)|^q] dt \\
 & = \frac{1}{b-a} \left\{ \left[ \frac{(b-a)(x-a)^{q+1}}{q+1} - \frac{(x-a)^{q+2}}{q+2} \right] |f''(a)|^q + \frac{(x-a)^{q+2}}{q+2} |f''(b)|^q \right\}
 \end{aligned} \tag{2.19}$$

and similarly,

$$\begin{aligned}
 & \int_x^b (b-t)^q |f''(t)|^q dt \\
 & \leq \frac{1}{b-a} \int_x^b (b-t)^q [(b-t) |f''(a)|^q + (t-a) |f''(b)|^q] dt \\
 & = \frac{1}{b-a} \left\{ \frac{(b-x)^{q+2}}{q+2} |f''(a)|^q + \left[ \frac{(b-a)(b-x)^{q+1}}{q+1} - \frac{(b-x)^{q+2}}{q+2} \right] |f''(b)|^q \right\}.
 \end{aligned} \tag{2.20}$$

Moreover, we obtain

$$\int_a^x |t-a-m_h(x)|^p dt = \frac{(x-a-m_h(x))^{p+1} + (-1)^p [m_h(x)]^{p+1}}{p+1} \tag{2.21}$$

for  $a \leq x \leq \frac{a+b}{2}$ , and

$$\int_a^x |t - a - m_h(x)|^p dt = \frac{[m_h(x)]^{p+1} + (x - a - m_h(x))^{p+1}}{p+1} \quad (2.22)$$

for  $\frac{a+b}{2} \leq x \leq b$ .

Using the similar way we also have,

$$\int_x^b |t - b - m_h(x)|^p dt = \frac{(m_h(x) + b - x)^{p+1} + (-1)^{p+1} [m_h(x)]^{p+1}}{p+1} \quad (2.23)$$

for  $a \leq x \leq \frac{a+b}{2}$ , and

$$\int_x^b |t - b - m_h(x)|^p dt = \frac{(m_h(x) + b - x)^{p+1} - [m_h(x)]^{p+1}}{p+1} \quad (2.24)$$

for  $\frac{a+b}{2} \leq x \leq b$ .

Using the identities (2.19)–(2.21) and (2.23) for the case  $a \leq x \leq \frac{a+b}{2}$  and using the identities (2.19), (2.20), (2.22) and (2.24) for the case  $\frac{a+b}{2} \leq x \leq b$ , we obtain required results (2.15) and (2.16).  $\square$

**Corollary 2.** If we choose  $x = \frac{a+b}{2}$  in Theorem 3, then we have the inequality

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^2}{2^{4+\frac{1}{q}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ \left[ \frac{(q+3)|f''(a)|^q + (q+1)|f''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} + \left[ \frac{(q+1)|f''(a)|^q + (q+3)|f''(b)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

**Corollary 3.** If we choose  $h = 0$  in Theorem 3, then we have the following inequality for  $a \leq x \leq b$

$$\begin{aligned} & \left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2(b-a)^{1+\frac{1}{q}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \quad \times \left\{ (x-a)^3 \left[ \left( \frac{b-a}{q+1} - \frac{x-a}{q+2} \right) |f'(a)|^q + \frac{x-a}{q+2} |f'(b)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + (b-x)^3 \left[ \frac{b-x}{q+2} |f'(a)|^q + \left( \frac{b-a}{q+1} - \frac{b-x}{q+2} \right) |f'(b)|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Corollary 4.** Let us  $x = a$  and  $x = b$  in Theorem 3. Subsequently, if we add the obtained result and use the triangle inequality for the modulus, we get the inequality for  $h \in [0, 2]$

$$\begin{aligned} & \left| \frac{h-2}{2} \frac{b-a}{4} (f'(b) - f'(a)) + \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^2}{2^{3+\frac{1}{p}}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} ((2-h)^{p+1} + h^{p+1})^{\frac{1}{p}} \end{aligned}$$



$$\times \left\{ \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} + \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right\}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Remark 5.** If we take  $h = 0$  in [Corollary 4](#), then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{(b-a)}{4} [f'(a) + f'(b)] - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

$$\leq \frac{(b-a)^2}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} + \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right\}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Remark 6.** If we take  $h = 2$  in [Corollary 4](#), then we have following inequality

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

$$\leq \frac{(b-a)^2}{4} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} + \left[ \frac{|f''(b)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)} \right]^{\frac{1}{q}} \right\}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

### 3. Applications to some special means

Let us recall the following means:

(a) The *Arithmetic mean*:

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0$$

(b) The *Geometric mean*:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0$$

(c) The *Harmonic mean*:

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0$$

(d) The *Logarithmic mean*:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

(e) The *Identric mean*:

$$I = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

(f) The *p-logarithmic mean*:

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad a, b > 0$$

where  $p \in \mathbb{R} \setminus \{-1, 0\}$ .

The following simple relationships are known in literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that  $L_p$  is monotonically increasing in  $p \in \mathbb{R}$  with  $L_0 = I$  and  $L_{-1} = L$ .

**Proposition 1.** Let  $a, b \in \mathbb{R}$ ,  $0 < a < b$ ,  $n \in \mathbb{Z}$  and  $|n(n-1)| \geq 3$ . Then, we have

$$\begin{aligned} & \left| \frac{n(h-2)}{2} (x-A)x^{n-1} + x^n - \frac{n \cdot h}{2} L_{n-1}^{n-1}(x-A) - L_n^n \right| \\ & \leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)| a^{n-2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a) h(x-A) \frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \right. \\ & \quad \left. + |n(n-1)| b^{n-2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} - h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(b-x)^3}{3} + (b-a) h(x-A) \frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a) \frac{[h(x-A)]^3}{3} \right] \right\} \end{aligned}$$

for all  $a \leq x \leq A$  with  $h \in [0, 2]$  and

$$\begin{aligned} & \left| \frac{n(h-2)}{2} (x-A)x^{n-1} + x^n - \frac{n \cdot h}{2} L_{n-1}^{n-1}(x-A) - L_n^n \right| \\ & \leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)| a^{n-2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a) h(x-A) \frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a) \frac{[h(x-A)]^3}{3} \right] \right. \\ & \quad \left. + |n(n-1)| b^{n-2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} - h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. \times (b-a) \frac{(b-x)^3}{3} + (b-a) h(x-A) \frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \right\} \end{aligned}$$

for all  $A \leq x \leq b$  with  $h \in [0, 2]$ .

**Proof.** The proof is immediate from [Theorem 2](#) applied for  $f(x) = x^n$ ,  $x \in \mathbb{R}$ ,  $n \in \mathbb{Z}$ ,  $|n(n-1)| \geq 3$ .  $\square$

**Remark 7.** If we choose  $h = 0$  in [Proposition 1](#), then we have the inequality

$$\begin{aligned} & |x^n - n(x-A)x^{n-1} - L_n^n| \\ & \leq \frac{1}{2(b-a)^2} \left\{ |n(n-1)| a^{n-2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + (b-a) \frac{(x-a)^3}{3} \right] \right. \\ & \quad \left. + |n(n-1)| b^{n-2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} + (b-a) \frac{(b-x)^3}{3} \right] \right\} \end{aligned}$$

for  $x \in [a, b]$ .

**Proposition 2.** Let  $a, b \in (0, \infty)$  and  $a < b$ . Then, we have

$$\begin{aligned} & \left| \ln I + \frac{h(x-A)}{2L} - \frac{(h-2)(x-A)}{2x} - \ln x \right| \\ & \leq \frac{1}{2(b-a)^2} \left\{ \frac{1}{a^2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a)h(x-A) \frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \right. \\ & \quad \left. + \frac{1}{b^2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} - h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(b-x)^3}{3} + (b-a)h(x-A) \frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a) \frac{[h(x-A)]^3}{3} \right] \right\} \end{aligned}$$

for all  $a \leq x \leq A$  with  $h \in [0, 2]$  and

$$\begin{aligned} & \left| \ln I + \frac{h(x-A)}{2L} - \frac{(h-2)(x-A)}{2x} - \ln x \right| \\ & \leq \frac{1}{2(b-a)^2} \left\{ \frac{1}{a^2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(x-a)^3}{3} - (b-a)h(x-A) \frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a) \frac{[h(x-A)]^3}{3} \right] \right. \\ & \quad \left. + \frac{1}{b^2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} - h(x-A) \frac{(x-a)^3 + (b-x)^3}{3} \right. \right. \\ & \quad \left. \left. + (b-a) \frac{(b-x)^3}{3} + (b-a)h(x-A) \frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6} \right] \right\} \end{aligned}$$

for all  $A \leq x \leq b$  with  $h \in [0, 2]$ .

**Proof.** The assertion follows from [Theorem 2](#) applied to the mapping  $f : (0, \infty) \rightarrow (-\infty, 0)$ ,  $f(x) = -\ln x$  and the details are omitted.  $\square$

**Remark 8.** If we choose  $h = 0$  in [Proposition 2](#), then we have the inequality,

$$\begin{aligned} & \left| \ln I + \frac{(x-A)}{x} - \ln x \right| \\ & \leq \frac{1}{2(b-a)^2} \left\{ \frac{1}{a^2} \left[ \frac{(b-x)^4 - (x-a)^4}{4} + (b-a) \frac{(x-a)^3}{3} \right] \right. \\ & \quad \left. + \frac{1}{b^2} \left[ \frac{(x-a)^4 - (b-x)^4}{4} + (b-a) \frac{(b-x)^3}{3} \right] \right\} \end{aligned}$$

for  $x \in [a, b]$ .

#### 4. Concluding Remarks

In this study, first of all, using practical identity for twice differentiable functions proved by Erden et al., we present some new upper bounds for generalized Ostrowski type inequalities by taking advantage of mappings whose second derivatives absolute values are convex. Moreover, we provide these inequalities for special means.

## References

- [1] A.M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert, *Comment. Math. Helv.* 10 (1938) 226–227.
- [2] N.S. Barnett, S.S. Dragomir, An ostrowski type inequality for double integrals and applications for cubature formulae, *Soochow J. Math.* 27 (1) (2001) 109–114.
- [3] Huseyin Budak, M. Zeki Sarikaya, On generalized ostrowski type inequalities for functions whose first derivatives absolute value are convex, *Turkish J. Math.* 40 (2016) 1193–1210.
- [4] P. Cerone, S.S. Dragomir, J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications, in: *RGMIA Research Report Collection*, vol. 1, No. 1, 1998 Art. 4.
- [5] S.S. Dragomir, C.E.M. Pearce, Selected topics on hermite–hadamard inequalities and applications, in: *RGMIA Monographs*, Victoria University, 2000. Online: [http://www.sta.vu.edu.au/RGMIA/monographs/hermite\\_hadamard.html](http://www.sta.vu.edu.au/RGMIA/monographs/hermite_hadamard.html).
- [6] S.S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.* 11 (5) (1998) 91–95.
- [7] S.S. Dragomir, P. Cerone, J. Roumeliotis, A new generalization of Ostrowski’s integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, in: *RGMIA Research Report Collection*, vol. 2, No. 1, 1999.
- [8] S.S. Dragomir, A. Sofo, An integral inequality for twice differentiable mappings and applications, *Tamk. J. Math.* 31 (4) (2000).
- [9] S.S. Dragomir, P. Cerone, A. Sofo, Some remarks on the midpoint rule in numerical integration, in: *RGMIA Research Report Collection*, vol. 1, No. 2, 1998 Art. 4.
- [10] S. Erden, H. Budak, M.Z. Sarikaya, An Ostrowski type inequality for twice differentiable mappings and applications, *Math. Model. Anal.* 21 (4) (2016) 522–532.
- [11] M.E. Kiriş, M.Z. Sarikaya, On Ostrowski type inequalities and Čebyšev type inequalities with applications, *Filomat* 29 (8) (2015) 1695–1713.
- [12] Z. Liu, Some Ostrowski type inequalities, *Math. Comput. Modelling* 48 (2008) 949–960.
- [13] B.G. Pachpatte, *Analytic Inequalities: Recent Advances*, Atlantis Pres, Paris, France, 2012.
- [14] A. Qayyum, A generalized inequality of Ostrowski type for twice differentiable bounded mappings and applications, *Appl. Math. Sci.* 8 (38) (2014) 1889–1901.
- [15] M.Z. Sarikaya, On the Ostrowski type integral inequality, *Acta Math. Univ. Comenian.* LXXIX (1) (2010) 129–134.
- [16] M.Z. Sarikaya, On the Ostrowski type integral inequality for double integrals, *Demonstratio Math.* XLV (3) (2012) 533–540.
- [17] M.Z. Sarikaya, A. Saglam, H. Yildirim, New inequalities of hermite–hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, *Int. J. Open Probl. Comput. Sci. Math.* 5 (3) (2012) 1–14.
- [18] M.Z. Sarikaya, H. Yildirim, Some new integral inequalities for twice differentiable convex mappings, *Nonlinear Ana. Forum* 17 (2012) 1–14.
- [19] F. Zafar, F.A. Mir, A generalized integral inequality for twice differentiable mappings, *Kragujevac J. Math.* 32 (2009) 81–96.
- [20] S.S. Dragomir, N.S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, in: *RGMIA Research Report Collection*, V.U.T., vol. 1, 1999, pp. 67–76.