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## ON THE OSTROWSKI TYPE INTEGRAL INEQUALITY FOR DOUBLE INTEGRALS

**Abstract.** In this note, we establish a new inequality of Ostrowski-type for double integrals involving functions of two independent variables by using fairly elementary analysis.

### 1. Introduction

In 1938, the classical integral inequality was established by Ostrowski [3] as follows:

**THEOREM 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$ .

Then, we have the inequality:

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.

In a recent paper [1], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals

**THEOREM 2.** Let  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be continuous on  $[a, b] \times [c, d]$ ,  $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$  exists on  $(a, b) \times (c, d)$  and is bounded, i.e.,  $\|f''_{x,y}\|_\infty = \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty$ . Then, we have the inequality:

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2000 Mathematics Subject Classification: 26D07, 26D15.  
Key words and phrases: Ostrowski's inequality.

$$\begin{aligned}
 (1.2) \quad & \left| \int_a^b \int_c^d f(s, t) dt ds - (d - c)(b - a)f(x, y) \right. \\
 & \quad \left. - \left[ (b - a) \int_c^d f(x, t) dt + (d - c) \int_a^b f(s, y) ds \right] \right| \\
 & \leq \left[ \frac{1}{4}(b - a)^2 + \left(x - \frac{a + b}{2}\right)^2 \right] \left[ \frac{1}{4}(d - c)^2 + \left(y - \frac{d + c}{2}\right)^2 \right] \left\| f''_{x,y} \right\|_{\infty}
 \end{aligned}$$

for all  $(x, y) \in [a, b] \times [c, d]$ .

In [1], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [5], Pachpatte obtained an inequality in the view (1.2) by using elementary analysis. The interested reader is also referred to ([1], [2], [4]–[8]) for Ostrowski type inequalities in several independent variables.

The main aim of this note is to establish a new Ostrowski type inequality for double integrals involving functions of two independent variables and their partial derivatives.

**2. Main result**

**THEOREM 3.** *Let  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be an absolutely continuous function such that the partial derivative of order 2 exists and is bounded, i.e.,*

$$\left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} = \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| < \infty$$

for all  $(t, s) \in [a, b] \times [c, d]$ . Then, we have

$$\begin{aligned}
 (2.1) \quad & \left| (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) + H(\alpha_1, \alpha_2, \beta_1, \beta_2) + G(\alpha_1, \alpha_2, \beta_1, \beta_2) \right. \\
 & - (\beta_2 - \alpha_2) \int_a^b f\left(t, \frac{c + d}{2}\right) dt - (\beta_1 - \alpha_1) \int_a^d f\left(\frac{a + b}{2}, s\right) ds \\
 & - \int_a^b [(\alpha_2 - c)f(t, c) + (d - \beta_2)f(t, d)] dt - \int_a^d [(\alpha_1 - a)f(a, s) + (b - \beta_1)f(b, s)] ds \\
 & \left. + \int_a^b \int_c^d f(t, s) ds dt \right| \leq \left[ \frac{(\alpha_1 - a)^2 + (b - \beta_1)^2}{2} + \frac{(a + b - 2\alpha_1)^2 + (a + b - 2\beta_1)^2}{8} \right] \\
 & \times \left[ \frac{(\alpha_2 - c)^2 + (d - \beta_2)^2}{2} + \frac{(c + d - 2\alpha_2)^2 + (c + d - 2\beta_2)^2}{8} \right] \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty}
 \end{aligned}$$

for all  $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in [a, b] \times [c, d]$  with  $\alpha_1 < \beta_1, \alpha_2 < \beta_2$  where

$$\begin{aligned}
 (2.2) \quad H(\alpha_1, \alpha_2, \beta_1, \beta_2) &= (\alpha_1 - a)[(\alpha_2 - c)f(a, c) + (d - \beta_2)f(a, d)] \\
 &+ (b - \beta_1)[(\alpha_2 - c)f(b, c) + (d - \beta_2)f(b, d)]
 \end{aligned}$$

and

$$\begin{aligned}
 (2.3) \quad G(\alpha_1, \alpha_2, \beta_1, \beta_2) &= (\beta_1 - \alpha_1) \left[ \left( \alpha_2 - c \right) f \left( \frac{a+b}{2}, c \right) + (d - \beta_2) f \left( \frac{a+b}{2}, d \right) \right] \\
 &\quad + (\beta_2 - \alpha_2) \left[ \left( \alpha_1 - a \right) f \left( a, \frac{c+d}{2} \right) + \left( b - \beta_1 \right) f \left( b, \frac{c+d}{2} \right) \right].
 \end{aligned}$$

**Proof.** We define the following functions:

$$p(a, b, \alpha_1, \beta_1, t) = \begin{cases} t - \alpha_1, & t \in [a, \frac{a+b}{2}] \\ t - \beta_1, & t \in (\frac{a+b}{2}, b] \end{cases}$$

and

$$q(c, d, \alpha_2, \beta_2, s) = \begin{cases} s - \alpha_2, & s \in [c, \frac{c+d}{2}] \\ s - \beta_2, & s \in (\frac{c+d}{2}, d] \end{cases}$$

for all  $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in [a, b] \times [c, d]$  with  $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ . Thus, by definitions of  $p(a, b, \alpha_1, \beta_1, t)$  and  $q(c, d, \alpha_2, \beta_2, s)$ , we have

$$\begin{aligned}
 (2.4) \quad &\int_a^b \int_c^d p(a, b, \alpha_1, \beta_1, t) q(c, d, \alpha_2, \beta_2, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 &= \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t - \alpha_1)(s - \alpha_2) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 &\quad + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d (t - \alpha_1)(s - \beta_2) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 &\quad + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} (t - \beta_1)(s - \alpha_2) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 &\quad + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d (t - \beta_1)(s - \beta_2) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt.
 \end{aligned}$$

Integrating by parts, we can state:

$$\begin{aligned}
 (2.5) \quad & \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-\alpha_1)(s-\alpha_2) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 &= \frac{(a+b-2\alpha_1)(c+d-2\alpha_2)}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} f(t,s) ds dt \\
 &\quad - \frac{(a-\alpha_1)(c+d-2\alpha_2)}{2} f\left(a, \frac{c+d}{2}\right) - \frac{(a+b-2\alpha_1)(c-\alpha_2)}{2} f\left(\frac{a+b}{2}, c\right) \\
 &\quad + (a-\alpha_1)(c-\alpha_2)f(a,c) \\
 &\quad - \int_a^{\frac{a+b}{2}} \left[ \frac{(c+d-2\alpha_2)}{2} f\left(t, \frac{c+d}{2}\right) - (c-\alpha_2)f(t,c) \right] dt \\
 &\quad - \int_c^{\frac{c+d}{2}} \left[ \frac{(a+b-2\alpha_1)}{2} f\left(\frac{a+b}{2}, s\right) - (a-\alpha_1)f(a,s) \right] ds.
 \end{aligned}$$

$$\begin{aligned}
 (2.6) \quad & \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d (t-\alpha_1)(s-\beta_2) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 &= -\frac{(a+b-2\alpha_1)(c+d-2\beta_2)}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d f(t,s) ds dt \\
 &\quad + \frac{(a-\alpha_1)(c+d-2\beta_2)}{2} f\left(a, \frac{c+d}{2}\right) + \frac{(a+b-2\alpha_1)(d-\beta_2)}{2} f\left(\frac{a+b}{2}, d\right) \\
 &\quad + (a-\alpha_1)(d-\beta_2)f(a,d) \\
 &\quad + \int_a^{\frac{a+b}{2}} \left[ \frac{(c+d-2\beta_2)}{2} f\left(t, \frac{c+d}{2}\right) - (d-\beta_2)f(t,d) \right] dt \\
 &\quad - \int_{\frac{c+d}{2}}^d \left[ \frac{(a+b-2\alpha_1)}{2} f\left(\frac{a+b}{2}, s\right) - (a-\alpha_1)f(a,s) \right] ds.
 \end{aligned}$$

$$\begin{aligned}
 (2.7) \quad & \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} (t-\beta_1)(s-\alpha_2) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \\
 &= -\frac{(a+b-2\beta_1)(c+d-2\alpha_2)}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} f(t,s) ds dt \\
 &\quad + \frac{(b-\beta_1)(c+d-2\alpha_2)}{2} f\left(b, \frac{c+d}{2}\right) + \frac{(a+b-2\beta_1)(c-\alpha_2)}{2} f\left(\frac{a+b}{2}, c\right)
 \end{aligned}$$

$$\begin{aligned}
 & - (b - \beta_1)(c - \alpha_2)f(b, c) \\
 & - \int_{\frac{a+b}{2}}^b \left[ \frac{(c+d-2\alpha_2)}{2} f\left(t, \frac{c+d}{2}\right) - (c - \alpha_2)f(t, c) \right] dt \\
 & + \int_c^{\frac{c+d}{2}} \left[ \frac{(a+b-2\beta_1)}{2} f\left(\frac{a+b}{2}, s\right) - (b - \beta_1)f(b, s) \right] ds.
 \end{aligned}$$

$$\begin{aligned}
 (2.8) \quad & \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d (t - \beta_1)(s - \beta_2) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 & = \frac{(a+b-2\beta_1)(c+d-2\beta_2)}{4} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d f(t, s) ds dt \\
 & - \frac{(b-\beta_1)(c+d-2\beta_2)}{2} f\left(b, \frac{c+d}{2}\right) - \frac{(a+b-2\beta_1)(d-\beta_2)}{2} f\left(\frac{a+b}{2}, d\right) \\
 & + (b-\beta_1)(d-\beta_2)f(b, d) \\
 & + \int_{\frac{a+b}{2}}^b \left[ \frac{(c+d-2\beta_2)}{2} f\left(t, \frac{c+d}{2}\right) - (d-\beta_2)f(t, d) \right] dt \\
 & + \int_c^{\frac{c+d}{2}} \left[ \frac{(a+b-2\beta_1)}{2} f\left(\frac{a+b}{2}, s\right) - (b-\beta_1)f(b, s) \right] ds.
 \end{aligned}$$

Adding (2.5)–(2.8) and rewriting, we easily deduce:

$$\begin{aligned}
 (2.9) \quad & \int_a^b \int_c^d p(a, b, \alpha_1, \beta_1, t)q(c, d, \alpha_2, \beta_2, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 & = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + H(\alpha_1, \alpha_2, \beta_1, \beta_2) \\
 & + G(\alpha_1, \alpha_2, \beta_1, \beta_2) - (\beta_2 - \alpha_2) \int_a^b f\left(t, \frac{c+d}{2}\right) dt \\
 & - (\beta_1 - \alpha_1) \int_c^d f\left(\frac{a+b}{2}, s\right) ds - \int_a^b [(\alpha_2 - c)f(t, c) + (d - \beta_2)f(t, d)] dt \\
 & - \int_c^d [(\alpha_1 - a)f(a, s) + (b - \beta_1)f(b, s)] ds + \int_a^b \int_c^d f(t, s) ds dt
 \end{aligned}$$

where  $H(\alpha_1, \alpha_2, \beta_1, \beta_2)$  and  $G(\alpha_1, \alpha_2, \beta_1, \beta_2)$  defined by (2.2) and (2.3), respectively. Now, using the identity (2.9), it follows that

$$\begin{aligned}
 (2.10) \quad & \left| (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + H(\alpha_1, \alpha_2, \beta_1, \beta_2) + \int_a^b \int_c^d f(t, s) ds dt \right. \\
 & + G(\alpha_1, \alpha_2, \beta_1, \beta_2) - (\beta_2 - \alpha_2) \int_a^b f\left(t, \frac{a+b}{2}\right) dt - (\beta_1 - \alpha_1) \int_c^d f\left(x, \frac{c+d}{2}\right) ds \\
 & \left. - \int_a^b [(\alpha_2 - c)f(t, c) + (d - \beta_2)f(t, d)] dt - \int_c^d [(\alpha_1 - a)f(a, s) + (b - \beta_1)f(b, s)] ds \right| \\
 & \leq \int_a^b \int_c^d |p(a, b, \alpha_1, \beta_1, t)| |q(c, d, \alpha_2, \beta_2, s)| \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right| ds dt \\
 & \leq \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \int_a^b \int_c^d |p(a, b, \alpha_1, \beta_1, t)| |q(c, d, \alpha_2, \beta_2, s)| ds dt.
 \end{aligned}$$

On the other hand, we get

$$\begin{aligned}
 (2.11) \quad & \int_a^b |p(a, b, \alpha_1, \beta_1, t)| dt = \int_a^{\frac{a+b}{2}} |t - \alpha_1| dt + \int_{\frac{a+b}{2}}^b |t - \beta_1| dt \\
 & = \int_a^{\alpha_1} (\alpha_1 - t) dt + \int_{\alpha_1}^{\frac{a+b}{2}} (t - \alpha_1) dt + \int_{\frac{a+b}{2}}^{\beta_1} (\beta_1 - t) dt + \int_{\beta_1}^b (t - \beta_1) dt \\
 & = \frac{(\alpha_1 - a)^2 + (b - \beta_1)^2}{2} + \frac{(a + b - 2\alpha_1)^2 + (a + b - 2\beta_1)^2}{8}
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 (2.12) \quad & \int_c^d |q(a, b, \alpha_1, \beta_1, t)| dt = \int_c^{\frac{c+d}{2}} |s - \alpha_2| ds + \int_{\frac{c+d}{2}}^d |s - \beta_2| ds \\
 & = \frac{(\alpha_2 - c)^2 + (d - \beta_2)^2}{2} + \frac{(c + d - 2\alpha_2)^2 + (c + d - 2\beta_2)^2}{8}.
 \end{aligned}$$

Using (2.11) and (2.12) in (2.10), we see that (2.1) holds. ■

**COROLLARY 1.** Under the assumptions of Theorem 3, we have

$$(2.13) \quad \left| (b-a)(d-c)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - (d-c) \int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a) \int_c^d f\left(\frac{a+b}{2}, s\right) ds \right| \leq \frac{1}{16} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} (b-a)^2 (d-c)^2.$$

**Proof.** We choose  $\alpha_1 = a$ ,  $\beta_1 = b$ ,  $\alpha_2 = c$  and  $\beta_2 = d$  in (2.1), then we see that (2.13) holds. ■

**COROLLARY 2.** Under the assumptions of Theorem 3, we have

$$(2.14) \quad \left| \frac{(b-a)(d-c)}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] - \frac{(d-c)}{2} \int_a^b [f(t, c) + f(t, d)] dt - \frac{(b-a)}{2} \int_c^d [f(a, s) + f(b, s)] ds + \int_a^b \int_c^d f(t, s) ds dt \right| \leq \frac{1}{16} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} (b-a)^2 (d-c)^2.$$

**Proof.** We choose  $\alpha_1 = \beta_1 = \frac{a+b}{2}$ ,  $\alpha_2 = \beta_2 = \frac{c+d}{2}$  in (2.1), then we see that (2.14) holds. ■

**REMARK 1.** If we take  $x = \frac{a+b}{2}$  and  $y = \frac{c+d}{2}$  in Theorem 2, then the inequality (1.2) reduces (2.13). So, our result is generalization of the corresponding result of Theorem 2.

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*Received March 10, 2010; revised version January 11, 2011.*