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Some (p, q) -Integral Inequalities of Hermite–Hadamard Inequalities for (p, q) -Differentiable Convex Functions

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Abstract: In this paper, we establish a new $(p, q)^b$ -integral identity involving the first-order $(p, q)^b$ -derivative. Then, we use this result to prove some new $(p, q)^b$ -integral inequalities related to Hermite–Hadamard inequalities for $(p, q)^b$ -differentiable convex functions. Furthermore, our main results are used to study some special cases of various integral inequalities. The newly presented results are proven to be generalizations of some integral inequalities of already published results. Finally, some examples are given to illustrate the investigated results.



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1. Introduction

In mathematics, the study of calculus without limits is called quantum calculus (briefly called q -calculus), and was first studied by Euler (1707–1783), introducing the number in the q -infinite series defined by Newton (also called Newton's infinite series). In the early 20th century, Jackson [1] relied on the concept of Euler to define the q -integral and q -derivative (well-known q -Jackson integral and q -Jackson derivative) over the interval $(0, \infty)$. In q -calculus, we obtain the q -analogues of mathematical objects that can be recaptured by taking $q \rightarrow 1$. In recent years, q -calculus has had numerous applications in various disciplines of physics and mathematics; see [2–10] and the references cited therein for more details.

In 2013, Tariboon and Ntouyas [11] presented the q_a -integral and q_a -derivative over finite intervals and also investigated the existence and uniqueness results of initial value problems for the first- and second-order impulsive q_a -difference equations. In 2020, Bermudo et al. [12] introduced the q^b -integral and q^b -derivative over finite intervals and also proved some of their basic properties. Recently, the topic of q -calculus has been applied in various integral inequalities, for example, Simpson- and Newton-type inequalities [13], Hanh inequalities [14], Ostrowski inequalities [15], Fejér-type inequalities [16], Hermite–Hadamard-like inequalities [17], Hermite–Hadamard inequalities [18], and the references cited therein. In particular, Hermite–Hadamard inequalities have also been studied by using q -calculus for convex functions by many researchers; see [12,19–25] and the references cited therein for more details.

The q -calculus generalization is called post-quantum calculus (briefly called (p, q) -calculus). In (p, q) -calculus, two independent parameters, p - and q -number, are included. It is commonly known that q -calculus cannot be recaptured by taking $q \rightarrow q/p$ in q -calculus, but it can be recaptured by taking $p = 1$ in (p, q) -calculus. Then, the classical formula can be gained by taking $q \rightarrow 1$. The concept of (p, q) -integral and (p, q) -derivative over the interval $(0, \infty)$ was first studied by Chakrabarti and Jagannathan [26] in 1991. Later on, the concept of the $(p, q)_a$ -integral and $(p, q)_a$ -derivative over finite intervals was proposed by Tunç and Gv [27,28] in 2016. Recently, the concept of the $(p, q)^b$ -integral and $(p, q)^b$ -derivative over finite intervals was proposed by Vivas-Cortez et al. [29] in 2021. In the past few years, the topic of (p, q) -calculus has become interesting in various integral inequalities for many researchers, and the results of (p, q) -calculus can be found in [30–45], and the references cited therein.

In 2021, Li et al. [46] presented a new generalization of q^b -integral inequalities related to Hermite–Hadamard inequalities for q^b -differentiable convex functions. Inspired by the above-mentioned literature, we propose establishing a new generalization of $(p, q)^b$ -integral inequalities related to Hermite–Hadamard inequalities for $(p, q)^b$ -differentiable convex functions to extend and generalize the results given in the above-mentioned literature. Moreover, we study some special cases of various integral inequalities. Finally, we give two examples to investigate the main results.

The rest of the paper is organized as follows: In Section 2, we give some definitions and notations of (p, q) -calculus. In Section 3, we present the $(p, q)^b$ -integral inequalities related to Hermite–Hadamard inequalities for $(p, q)^b$ -differentiable convex functions and display some special cases of various integral inequalities. In Section 4, we show two examples to investigate our main results. In Section 5, we summarize our results.

2. Preliminaries

In this section, we provide some definitions and notations of (p, q) -calculus used in our work. Throughout this paper, we assume that $0 < q < p \leq 1$ are constants and $[a, b] \subseteq \mathbb{R}$ is an interval with $a < b$. The (p, q) -number of λ is given by

$$[\lambda]_{p,q} = \frac{p^\lambda - q^\lambda}{p - q} = p^{\lambda-1} + p^{\lambda-2}q + \dots + pq^{\lambda-2} + q^{\lambda-1}, \quad \lambda \in \mathbb{N}. \tag{1}$$

If $p = 1$ in (1), then (1) is reduced as follows:

$$[\lambda]_q = \frac{1 - q^\lambda}{1 - q} = 1 + q + q^2 + \dots + q^{\lambda-1}, \quad \lambda \in \mathbb{N},$$

which is called the q -analogue or q -number of λ ; see [47] for more details.

Definition 1 ([27,28]). Let $\Psi : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the $(p, q)_a$ -derivative of Ψ at x is given by

$${}_aD_{p,q}\Psi(x) = \begin{cases} \frac{\Psi(px + (1 - p)a) - \Psi(qx + (1 - q)a)}{(p - q)(x - a)}, & \text{if } x \neq a; \\ \lim_{x \rightarrow a} {}_aD_{p,q}\Psi(x), & \text{if } x = a. \end{cases} \tag{2}$$

The function Ψ is called $(p, q)_a$ -differentiable function on $[a, b]$ if ${}_aD_{p,q}\Psi(x)$ exists for all $x \in [a, (b - a)/p + a]$.

Note that if $p = 1$ and ${}_aD_{1,q}\Psi(x) = {}_aD_q\Psi(x)$, then (2) is reduced as follows:

$${}_aD_q\Psi(x) = \begin{cases} \frac{\Psi(x) - \Psi(qx + (1 - q)a)}{(1 - q)(x - a)}, & \text{if } x \neq a; \\ \lim_{x \rightarrow a} {}_aD_q\Psi(x), & \text{if } x = a, \end{cases} \tag{3}$$

which is the well-known q_a -derivative of Ψ on $[a, b]$; see [48,49] for more details. Moreover, if $a = 0$ and ${}_0D_q\Psi(x) = D_q\Psi(x)$, then (3) is reduced as follows:

$$D_q\Psi(x) = \begin{cases} \frac{\Psi(x) - \Psi(qx)}{(1-q)x}, & \text{if } x \neq a; \\ \lim_{x \rightarrow 0} D_q\Psi(x), & \text{if } x = a, \end{cases}$$

which is the well-known q_a -derivative of Ψ on $[0, b]$; see [47] for more details.

Definition 2 ([29]). Let $\Psi : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the $(p, q)^b$ -derivative of Ψ at x is given by

$${}^bD_{p,q}\Psi(x) = \begin{cases} \frac{\Psi(qx + (1-q)b) - \Psi(px + (1-p)b)}{(p-q)(b-x)}, & \text{if } x \neq b; \\ \lim_{x \rightarrow b} {}^bD_{p,q}\Psi(x), & \text{if } x = b. \end{cases} \tag{4}$$

The function Ψ is called $(p, q)^b$ -differentiable function on $[a, b]$ if ${}^bD_{p,q}\Psi(x)$ exists for all $x \in [b - (b-a)/p, b]$.

Note that if $p = 1$ and ${}^bD_{1,q}\Psi(x) = {}^bD_q\Psi(x)$, then (4) is reduced as follows:

$${}^bD_q\Psi(x) = \begin{cases} \frac{\Psi(qx + (1-q)b) - \Psi(x)}{(1-q)(b-x)}, & \text{if } x \neq b; \\ \lim_{x \rightarrow b} {}^bD_q\Psi(x), & \text{if } x = b, \end{cases}$$

which is the well-known q^b -derivative of Ψ on $[a, b]$; see [12,24] for more details.

Definition 3 ([27]). Let $\Psi : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the $(p, q)_a$ -integral of Ψ at x is given by

$$\int_a^b \Psi(x) {}_a d_{p,q}x = (p-q)(b-a) \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^{\lambda+1}} \Psi\left(\frac{q^\lambda}{p^{\lambda+1}}b + \left(1 - \frac{q^\lambda}{p^{\lambda+1}}\right)a\right). \tag{5}$$

The function Ψ is called $(p, q)_a$ -integrable function on $[a, b]$ if $\int_a^b \Psi(x) {}_a d_{p,q}x$ exists for all $x \in [a, a + p(b-a)]$.

Note that if $a = 0$, then (5) is reduced as follows:

$$\int_0^b \Psi(x) d_{p,q}x = (p-q)b \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^{\lambda+1}} \Psi\left(\frac{q^\lambda}{p^{\lambda+1}}b\right), \tag{6}$$

which appears in [28]. Moreover, if $p = 1$, then (6) is reduced as follows:

$$\int_0^b \Psi(x) d_qx = (1-q)b \sum_{\lambda=0}^{\infty} q^\lambda \Psi(q^\lambda b),$$

which is the well-known q -Jackson integral; see [1] for more details.

Definition 4 ([29]). Let $\Psi : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the $(p, q)^b$ -integral of Ψ at x is given by

$$\int_a^b \Psi(x) {}^b d_{p,q}x = (p-q)(b-a) \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^{\lambda+1}} \Psi\left(\frac{q^\lambda}{p^{\lambda+1}}a + \left(1 - \frac{q^\lambda}{p^{\lambda+1}}\right)b\right). \tag{7}$$

The function Ψ is called $(p, q)^b$ -integrable function on $[a, b]$ if $\int_a^b \Psi(x) {}^b d_{p,q}x$ exists for all $x \in [b - p(b - a), b]$.

Lemma 1 ([27]). For $\alpha \in \mathbb{R} \setminus \{-1\}$, the following inequality holds:

$$\int_a^b (x - a)^\alpha {}_a d_{p,q}x = \frac{(b - a)^{\alpha+1}}{[\alpha + 1]_{p,q}}. \tag{8}$$

Theorem 1 ([28]). Suppose that $\Psi, \Phi : [a, b] \rightarrow \mathbb{R}$ are continuous functions and $r > 0$ with $1/s + 1/r = 1$, then

$$\int_a^b |\Psi(x)\Phi(x)| {}_a d_{p,q}x \leq \left(\int_a^b |\Psi(x)|^s {}_a d_{p,q}x \right)^{1/s} \left(\int_a^b |\Phi(x)|^r {}_a d_{p,q}x \right)^{1/r}. \tag{9}$$

3. Main Results

In this section, we prove $(p, q)^b$ -integral inequalities related to Hermite–Hadamard inequalities for which the first-order $(p, q)^b$ -derivatives in absolute value are convex functions. We define $I_1 = [b - (b - a)/p, b]$ and $I_2 = [b - p(b - a), b]$. The (p, q) -integral identity is as follows:

Theorem 2. Suppose that $\Psi : [a, b] \rightarrow \mathbb{R}$ is a $(p, q)^b$ -differentiable function on I_1 such that ${}^b D_{p,q}\Psi$ is continuous and integrable functions on I_2 with $\gamma, \nu \in [0, 1]$, then

$$\begin{aligned} & (b - a) \left[\int_0^\nu (qt + \gamma\nu - \gamma) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t + \int_\nu^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t \right] \\ &= \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma[\nu\Psi(a) + (1 - \nu)\Psi(b)] - (1 - \gamma)\Psi(va + (1 - \nu)b). \end{aligned} \tag{10}$$

Proof. Using Definition 2, we have

$$\begin{aligned} {}^b D_{p,q}\Psi(ta + (1 - t)b) &= \frac{\Psi(q(ta + (1 - t)b) + (1 - q)b) - \Psi(p(ta + (1 - t)b) + (1 - p)b)}{(p - q)(b - (ta + (1 - t)b))} \\ &= \frac{\Psi(qta + (1 - qt)b) - \Psi(p(ta + (1 - pt)b))}{(p - q)(b - a)t}. \end{aligned} \tag{11}$$

Applying an identical transformation, we obtain

$$\begin{aligned} & (b - a) \left[\int_0^\nu (qt + \gamma\nu - \gamma) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t + \int_\nu^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t \right] \\ &= (b - a) \int_0^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t \\ &+ (b - a) \int_0^\nu (1 - \gamma) {}^b D_{p,q}\Psi(ta + (1 - t)b) {}_d d_{p,q}t. \end{aligned} \tag{12}$$

Using (6) and (11), we obtain

$$\begin{aligned}
 & \int_0^1 t {}^b D_{p,q} \Psi(ta + (1-t)b) d_{p,q}t \\
 &= \int_0^1 \frac{\Psi(qta + (1-qt)b) - \Psi(pta + (1-pt)b)}{(p-q)(b-a)} d_{p,q}t \\
 &= \frac{1}{b-a} \left[\sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^{\lambda+1}} \Psi\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}a + \left(1 - \frac{q^{\lambda+1}}{p^{\lambda+1}}\right)b\right) - \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^{\lambda+1}} \Psi\left(\frac{q^\lambda}{p^\lambda}a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) \right] \\
 &= \frac{1}{b-a} \left[\frac{1}{q} \sum_{\lambda=0}^{\infty} \frac{q^{\lambda+1}}{p^{\lambda+1}} \Psi\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}a + \left(1 - \frac{q^{\lambda+1}}{p^{\lambda+1}}\right)b\right) - \frac{1}{p} \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^\lambda} \Psi\left(\frac{q^\lambda}{p^\lambda}a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) \right] \\
 &= \frac{1}{b-a} \left[\left(\frac{1}{q} - \frac{1}{p}\right) \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^\lambda} \Psi\left(\frac{q^\lambda}{p^\lambda}a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) - \frac{1}{q} \Psi(a) \right] \\
 &= \frac{1}{b-a} \left(\frac{p-q}{pq}\right) \sum_{\lambda=0}^{\infty} \frac{q^\lambda}{p^\lambda} \Psi\left(\frac{q^\lambda}{p^\lambda}a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) - \frac{1}{q(b-a)} \Psi(a) \\
 &= \frac{1}{pq(b-a)^2} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{1}{q(b-a)} \Psi(a). \tag{13}
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 & \int_0^1 {}^b D_{p,q} \Psi(ta + (1-t)b) d_{p,q}t \\
 &= \int_0^1 \frac{\Psi(qta + (1-qt)b) - \Psi(pta + (1-pt)b)}{(p-q)(b-a)t} d_{p,q}t \\
 &= \frac{1}{b-a} \left[\sum_{\lambda=0}^{\infty} \Psi\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}a + \left(1 - \frac{q^{\lambda+1}}{p^{\lambda+1}}\right)b\right) - \sum_{\lambda=0}^{\infty} \Psi\left(\frac{q^\lambda}{p^\lambda}a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) \right] \\
 &= \frac{1}{b-a} [\Psi(b) - \Psi(a)], \tag{14}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^\nu {}^b D_{p,q} \Psi(ta + (1-t)b) {}^\nu d_{p,q}t \\
 &= \int_0^\nu \frac{\Psi(qta + (1-qt)b) - \Psi(pta + (1-pt)b)}{(p-q)(b-a)t} {}^\nu d_{p,q}t \\
 &= \frac{1}{b-a} \left[\sum_{\lambda=0}^{\infty} \Psi\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}\nu a + \left(1 - \frac{q^{\lambda+1}}{p^{\lambda+1}}\right)b\right) - \sum_{\lambda=0}^{\infty} \Psi\left(\frac{q^\lambda}{p^\lambda}\nu a + \left(1 - \frac{q^\lambda}{p^\lambda}\right)b\right) \right] \\
 &= \frac{1}{b-a} [\Psi(b) - \Psi(\nu a + (1-\nu)b)]. \tag{15}
 \end{aligned}$$

Substituting (13) to (15) in (12), we obtain the required $(p, q)^b$ -integral identity. Therefore, the proof is completed. \square

Corollary 1. Under the assumptions of Theorem 4 with $\nu = 0, 1$ and $p / [2]_{p,q}$, the following new $(p, q)^b$ -integral identities hold:

(i)

$$(b-a) \int_0^1 (qt-1) {}^b D_{p,q} \Psi(ta + (1-t)b) d_{p,q}t = \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \Psi(b); \tag{16}$$

(ii)

$$(b - a) \left[\int_0^1 qt {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \Psi(a); \tag{17}$$

(iii)

$$\begin{aligned} (b - a) & \left[\int_0^{p/[2]_{p,q}} \left(qt - \frac{\gamma q}{[2]_{p,q}} \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right. \\ & \left. + \int_{p/[2]_{p,q}}^1 \left(qt + \frac{p\gamma}{[2]_{p,q}} - 1 \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] \\ & = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma \frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} - (1 - \gamma)\Psi \left(\frac{pa + qb}{[2]_{p,q}} \right). \end{aligned} \tag{18}$$

Corollary 2. Under the assumptions of Theorem 4 with $\gamma = 0, 1/3, 1/2$ and 1 , the following new $(p, q)^b$ -integral identities hold:

(i)

$$\begin{aligned} (b - a) & \left[\int_0^v qt {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t + \int_v^1 (qt - 1) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] \\ & = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \Psi(va + (1 - v)b); \end{aligned} \tag{19}$$

(ii)

$$\begin{aligned} (b - a) & \left[\int_0^v \left(qt + \frac{1}{3}v - \frac{1}{3} \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t + \int_v^1 \left(qt + \frac{1}{3}v - 1 \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] \\ & = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{1}{3} [v\Psi(a) + (1 - v)\Psi(b) + 2\Psi(va + (1 - v)b)]; \end{aligned} \tag{20}$$

(iii)

$$\begin{aligned} (b - a) & \left[\int_0^v \left(qt + \frac{1}{2}v - \frac{1}{2} \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t + \int_v^1 \left(qt + \frac{1}{2}v - 1 \right) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] \\ & = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{1}{2} [v\Psi(a) + (1 - v)\Psi(b) + \Psi(va + (1 - v)b)]; \end{aligned} \tag{21}$$

(iv)

$$\begin{aligned} (b - a) & \left[\int_0^v (qt + v - 1) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t + \int_v^1 (qt + v - 1) {}^b D_{p,q} \Psi (ta + (1 - t)b) d_{p,q}t \right] \\ & = \frac{1}{p(b - a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - [v\Psi(a) + (1 - v)\Psi(b)]. \end{aligned} \tag{22}$$

Remark 1. If $p = 1$, then (10) is reduced as follows:

$$\begin{aligned} (b - a) & \left[\int_0^v (qt + \gamma v - \gamma) {}^b D_q \Psi (ta + (1 - t)b) d_qt + \int_v^1 (qt + \gamma v - 1) {}^b D_q \Psi (ta + (1 - t)b) d_qt \right] \\ & = \frac{1}{b - a} \int_a^b \Psi(x) {}^b d_qx - \gamma [v\Psi(a) + (1 - v)\Psi(b)] - (1 - \gamma)\Psi(va + (1 - v)b), \end{aligned}$$

which appears in [46].

Remark 2. If $p = 1$, then (16) to (18) are reduced as follows:

(i)

$$(b - a) \int_0^1 (qt - 1)^b {}^bD_q \Psi (ta + (1 - t)b) d_q t = \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \Psi (b);$$

(ii)

$$(b - a) \left[\int_0^1 qt {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] = \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \Psi (a);$$

(iii)

$$\begin{aligned} & (b - a) \left[\int_0^{1/[2]_q} \left(qt - \frac{\gamma q}{[2]_q} \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t + \int_{1/[2]_q}^1 \left(qt + \frac{\gamma}{[2]_q} - 1 \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \gamma \frac{\Psi (a) + q\Psi (b)}{[2]_q} - (1 - \gamma) \Psi \left(\frac{a + qb}{[2]_q} \right), \end{aligned}$$

respectively, which appears in [46].

Remark 3. If $p = 1$, then (19) to (22) are reduced as follows:

(i)

$$\begin{aligned} & (b - a) \left[\int_0^v qt {}^bD_q \Psi (ta + (1 - t)b) d_q t + \int_v^1 (qt - 1) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \Psi (va + (1 - v)b), \end{aligned} \tag{23}$$

which appears in [46]. In particular, if $v = 1/[2]_q$, then (23) leads to the midpoint-type integral identity as follows:

$$\begin{aligned} & (b - a) \left[\int_0^{1/[2]_q} qt {}^bD_q \Psi (ta + (1 - t)b) d_q t + \int_{1/[2]_q}^1 (qt - 1) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \Psi \left(\frac{a + qb}{[2]_q} \right), \end{aligned}$$

which appears in [50].

(ii)

$$\begin{aligned} & (b - a) \left[\int_0^v \left(qt + \frac{1}{3}v - \frac{1}{3} \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t + \int_v^1 \left(qt + \frac{1}{3}v - 1 \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \frac{1}{3} [v\Psi (a) + (1 - v)\Psi (b) + 2\Psi (va + (1 - v)b)], \end{aligned} \tag{24}$$

which appears in [46]. In particular, if $v = 1/[2]_q$, then (24) leads to the Simpson-like integral identity as follows:

$$\begin{aligned} & (b - a) \left[\int_0^{1/[2]_q} \left(qt - \frac{q}{3([2]_q)} \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right. \\ & \quad \left. + \int_{1/[2]_q}^1 \left(qt + \frac{1}{3([2]_q)} - 1 \right) {}^bD_q \Psi (ta + (1 - t)b) d_q t \right] \\ &= \frac{1}{b - a} \int_a^b \Psi (x) {}^b d_q x - \frac{1}{3} \left[\frac{\Psi (a) + q\Psi (b)}{[2]_q} + 2\Psi \left(\frac{a + qb}{[2]_q} \right) \right], \end{aligned}$$

which appears in [46].

(iii)

$$\begin{aligned}
 & (b-a) \left[\int_0^\nu \left(qt + \frac{1}{2}\nu - \frac{1}{2} \right) {}^b D_q \Psi (ta + (1-t)b) d_q t + \int_\nu^1 \left(qt + \frac{1}{2}\nu - 1 \right) {}^b D_q \Psi (ta + (1-t)b) d_q t \right] \\
 &= \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{2} [\nu \Psi(a) + (1-\nu) \Psi(b) + \Psi(\nu a + (1-\nu)b)],
 \end{aligned} \tag{25}$$

which appears in [46]. In particular, if $\nu = 1/[2]_q$, then (25) leads to the averaged midpoint-trapezoid-type integral identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{1/[2]_q} \left(qt - \frac{q}{2([2]_q)} \right) {}^b D_q \Psi (ta + (1-t)b) d_q t \right. \\
 & \quad \left. + \int_{1/[2]_q}^1 \left(qt + \frac{1}{2([2]_q)} - 1 \right) {}^b D_q \Psi (ta + (1-t)b) d_q t \right] \\
 &= \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{2} \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} + \Psi \left(\frac{a+qb}{[2]_q} \right) \right],
 \end{aligned}$$

which appears in [46].

(iv)

$$\begin{aligned}
 & (b-a) \left[\int_0^\nu (qt + \nu - 1) {}^b D_q \Psi (ta + (1-t)b) d_q t + \int_\nu^1 (qt + \nu - 1) {}^b D_q \Psi (ta + (1-t)b) d_q t \right] \\
 &= \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - [\nu \Psi(a) + (1-\nu) \Psi(b)],
 \end{aligned} \tag{26}$$

which appears in [46]. In particular, if $\nu = 1/[2]_q$, then (26) leads to the trapezoid-type integral identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{1/[2]_q} \left(qt - \frac{q}{[2]_q} - 1 \right) {}^b D_q \Psi (ta + (1-t)b) d_q t \right. \\
 & \quad \left. + \int_{1/[2]_q}^1 \left(qt + \frac{1}{[2]_q} - 1 \right) {}^b D_q \Psi (ta + (1-t)b) d_q t \right] \\
 &= \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{\Psi(a) + q\Psi(b)}{[2]_q},
 \end{aligned}$$

which appears in [50].

Remark 4. From Corollary 2, we have the new (p, q) -integral identities as follows:

(i) If we take $\nu = p/[2]_{p,q}$, then (19) leads to the midpoint-type identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{p/[2]_{p,q}} qt {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q} t \right. \\
 & \quad \left. + \int_{p/[2]_{p,q}}^1 (qt - 1) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q} t \right] \\
 &= \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \Psi \left(\frac{pa+qb}{[2]_{p,q}} \right),
 \end{aligned}$$

which was proposed by Aamir Ali et al. in [29].

(ii) Taking $\nu = p/[2]_{p,q}$, then (20) leads to the Simpson-like integral identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{p/[2]_{p,q}} \left(qt - \frac{q}{3([2]_{p,q})} \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right. \\
 & \quad \left. + \int_{p/[2]_{p,q}}^1 \left(qt + \frac{p}{3([2]_{p,q})} - 1 \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right] \\
 & = \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{1}{3} \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} + 2\Psi \left(\frac{pa + qb}{[2]_{p,q}} \right) \right].
 \end{aligned}$$

(iii) If we set $v = p/[2]_{p,q}$, then (21) leads to the averaged midpoint-trapezoid-type integral identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{p/[2]_{p,q}} \left(qt - \frac{q}{2([2]_{p,q})} \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right. \\
 & \quad \left. + \int_{p/[2]_{p,q}}^1 \left(qt + \frac{p}{2([2]_{p,q})} - 1 \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right] \\
 & = \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{1}{2} \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} + \Psi \left(\frac{pa + qb}{[2]_{p,q}} \right) \right].
 \end{aligned}$$

(iv) By setting $v = p/[2]_{p,q}$, then (22) leads to the trapezoid-type integral identity as follows:

$$\begin{aligned}
 & (b-a) \left[\int_0^{p/[2]_{p,q}} \left(qt - \frac{q}{[2]_{p,q}} - 1 \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right. \\
 & \quad \left. + \int_{p/[2]_{p,q}}^1 \left(qt + \frac{p}{[2]_{p,q}} - 1 \right) {}^b D_{p,q} \Psi (ta + (1-t)b) d_{p,q}t \right] \\
 & = \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}}.
 \end{aligned}$$

Theorem 3. Suppose that $\Psi : [a, b] \rightarrow \mathbb{R}$ is a $(p, q)^b$ -differentiable function on I_1 such that ${}^b D_{p,q} \Psi$ is continuous and integrable functions on I_2 with $\gamma, \nu \in [0, 1]$. If $|{}^b D_{p,q} \Psi|$ is convex function on $[a, b]$, then

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\
 & \leq (b-a) \left[[\Lambda_1(p, q, \gamma, \nu) + \Lambda_2(p, q, \gamma, \nu) - \Lambda_3(p, q, \gamma, \nu)] |{}^b D_{p,q} \Psi(a)| \right. \\
 & \quad \left. + [\Theta_1(p, q, \gamma, \nu) + \Theta_2(p, q, \gamma, \nu) - \Theta_3(p, q, \gamma, \nu)] |{}^b D_{p,q} \Psi(b)| \right], \tag{27}
 \end{aligned}$$

where $\Lambda_i(p, q, \gamma, \nu)$, $i = 1, 2, 3$, and $\Theta_j(p, q, \gamma, \nu)$, $j = 1, 2, 3$, are given by

$$\begin{aligned}
 \Lambda_1(p, q, \gamma, \nu) &= \int_0^\nu t|qt + \gamma\nu - \gamma| d_{p,q}t \\
 &= \begin{cases} \frac{\nu^2\gamma(1-\nu)}{[2]_{p,q}} - \frac{\nu^3q}{[3]_{p,q}}, & (\gamma + q)\nu \leq \gamma; \\ \frac{\nu^3q}{[3]_{p,q}} - \frac{\nu^2\gamma(1-\nu)}{[2]_{p,q}} + \frac{2\gamma^3(1-\nu)^3}{q^2} \left(\frac{1}{[2]_{p,q}} - \frac{1}{[3]_{p,q}} \right), & (\gamma + q)\nu > \gamma, \end{cases} \\
 \Theta_1(p, q, \gamma, \nu) &= \int_0^\nu (1-t)|qt + \gamma\nu - \gamma| d_{p,q}t \\
 &= \int_0^\nu |qt + \gamma\nu - \gamma| d_{p,q}t - \int_0^\nu t|qt + \gamma\nu - \gamma| d_{p,q}t \\
 &= \begin{cases} \nu\gamma(1-\nu) - \frac{\nu^2q + \nu^2\gamma(1-\nu)}{[2]_{p,q}} + \frac{\nu^3q}{[3]_{p,q}}, & (\gamma + q)\nu \leq \gamma; \\ \left[\frac{2\gamma^2(1-\nu)^2([2]_{p,q}-1) + \nu^2q^2 + \nu^2q\gamma(1-\nu)}{q[2]_{p,q}} - \frac{\nu^3q}{[3]_{p,q}} \right], & (\gamma + q)\nu > \gamma, \end{cases} \\
 \Lambda_2(p, q, \gamma, \nu) &= \int_0^1 t|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \begin{cases} \frac{1-\gamma\nu}{[2]_{p,q}} - \frac{q}{[3]_{p,q}}, & \gamma\nu + q \leq 1; \\ \frac{q}{[3]_{p,q}} - \frac{1-\gamma\nu}{[2]_{p,q}} + \frac{2(1-\gamma\nu)^3}{q^2} \left(\frac{1}{[2]_{p,q}} - \frac{1}{[3]_{p,q}} \right), & \gamma\nu + q > 1, \end{cases} \\
 \Theta_2(p, q, \gamma, \nu) &= \int_0^1 (1-t)|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \int_0^1 |qt + \gamma\nu - 1| d_{p,q}t - \int_0^1 t|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \begin{cases} (1-\gamma\nu) - \frac{q+1-\gamma\nu}{[2]_{p,q}} + \frac{q}{[3]_{p,q}}, & \gamma\nu + q \leq 1; \\ \left[\frac{2(1-\gamma\nu)^2([2]_{p,q}-1) + q^2 + q(1-\gamma\nu)}{q[2]_{p,q}} - \frac{q}{[3]_{p,q}} \right], & \gamma\nu + q > 1, \end{cases} \\
 \Lambda_3(p, q, \gamma, \nu) &= \int_0^\nu t|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \begin{cases} \frac{\nu^2(1-\gamma\nu)}{[2]_{p,q}} - \frac{\nu^3q}{[3]_{p,q}}, & \gamma\nu + q \leq 1; \\ \frac{\nu^3q}{[3]_{p,q}} - \frac{\nu^2(1-\gamma\nu)}{[2]_{p,q}} + \frac{2(1-\gamma\nu)^3}{q^2} \left(\frac{1}{[2]_{p,q}} - \frac{1}{[3]_{p,q}} \right), & \gamma\nu + q > 1, \end{cases} \\
 \Theta_3(p, q, \gamma, \nu) &= \int_0^\nu (1-t)|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \int_0^\nu |qt + \gamma\nu - 1| d_{p,q}t - \int_0^\nu t|qt + \gamma\nu - 1| d_{p,q}t \\
 &= \begin{cases} \nu(1-\gamma\nu) - \frac{\nu^2q + \nu^2(1-\gamma\nu)}{[2]_{p,q}} + \frac{\nu^3q}{[3]_{p,q}}, & \gamma\nu + q \leq 1; \\ \left[\frac{2(1-\gamma\nu)^2([2]_{p,q}-1) + \nu^2q^2 + \nu^2q(1-\gamma\nu)}{q[2]_{p,q}} - \frac{\nu^3q}{[3]_{p,q}} \right], & \gamma\nu + q > 1. \end{cases}
 \end{aligned}$$

Proof. Taking the absolute value of both sides of (10), using Lemma 1 and applying the convexity of $|{}^bD_{p,q}\Psi|$, we obtain

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\
 &= (b-a) \left[\left| \int_0^\nu (qt + \gamma\nu - \gamma) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t + \int_\nu^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t \right| \right] \\
 &\leq (b-a) \left[\int_0^\nu |qt + \gamma\nu - \gamma| \left| {}^b D_{p,q}\Psi(ta + (1-t)b) \right| d_{p,q}t + \int_0^1 |qt + \gamma\nu - 1| \left| {}^b D_{p,q}\Psi(ta + (1-t)b) \right| d_{p,q}t \right. \\
 &\quad \left. - \int_0^\nu |qt + \gamma\nu - 1| \left| {}^b D_{p,q}\Psi(ta + (1-t)b) \right| d_{p,q}t \right] \\
 &\leq (b-a) \left[\left| {}^b D_{p,q}\Psi(a) \right| \int_0^\nu t|qt + \gamma\nu - \gamma| d_{p,q}t + \left| {}^b D_{p,q}\Psi(b) \right| \int_0^\nu (1-t)|qt + \gamma\nu - \gamma| d_{p,q}t \right. \\
 &\quad + \left| {}^b D_{p,q}\Psi(a) \right| \int_0^1 t|qt + \gamma\nu - 1| d_{p,q}t + \left| {}^b D_{p,q}\Psi(b) \right| \int_0^1 (1-t)|qt + \gamma\nu - 1| d_{p,q}t \\
 &\quad \left. - \left| {}^b D_{p,q}\Psi(a) \right| \int_0^\nu t|qt + \gamma\nu - 1| d_{p,q}t - \left| {}^b D_{p,q}\Psi(b) \right| \int_0^\nu (1-t)|qt + \gamma\nu - 1| d_{p,q}t \right] \\
 &= (b-a) \left[[\Lambda_1(p, q, \gamma, \nu) + \Lambda_2(p, q, \gamma, \nu) - \Lambda_3(p, q, \gamma, \nu)] \left| {}^b D_{p,q}\Psi(a) \right| \right. \\
 &\quad \left. + [\Theta_1(p, q, \gamma, \nu) + \Theta_2(p, q, \gamma, \nu) - \Theta_3(p, q, \gamma, \nu)] \left| {}^b D_{p,q}\Psi(b) \right| \right],
 \end{aligned}$$

which completes the proof. \square

Corollary 3. Under the assumptions of Theorem 3 with $\nu = p/[2]_{p,q}$, the following new $(p, q)^b$ -integral inequality holds:

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} \right] - (1-\gamma)\Psi\left(\frac{pa+qb}{[2]_{p,q}}\right) \right| \\
 &\leq (b-a) \left[\left[\Lambda_1\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) + \Lambda_2\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) - \Lambda_3\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) \right] \left| {}^b D_{p,q}\Psi(a) \right| \right. \\
 &\quad \left. + \left[\Theta_1\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) + \Theta_2\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) - \Theta_3\left(p, q, \gamma, \frac{1}{[2]_{p,q}}\right) \right] \left| {}^b D_{p,q}\Psi(b) \right| \right]. \tag{28}
 \end{aligned}$$

Remark 5. If $p = 1$, $\Lambda_i(p, q, \gamma, \nu) = \Lambda_i(q, \gamma, \nu)$ and $\Theta_j(p, q, \gamma, \nu) = \Theta_j(q, \gamma, \nu)$ for $i, j = 1, 2, 3$, then (27) is reduced as follows:

$$\begin{aligned}
 & \left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_qx - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\
 &\leq (b-a) \left[[\Lambda_1(q, \gamma, \nu) + \Lambda_2(q, \gamma, \nu) - \Lambda_3(q, \gamma, \nu)] \left| {}^b D_q\Psi(a) \right| \right. \\
 &\quad \left. + [\Theta_1(q, \gamma, \nu) + \Theta_2(q, \gamma, \nu) - \Theta_3(q, \gamma, \nu)] \left| {}^b D_q\Psi(b) \right| \right], \tag{29}
 \end{aligned}$$

where $\Lambda_i(q, \gamma, \nu)$, $i = 1, 2, 3$, and $\Theta_j(q, \gamma, \nu)$, $j = 1, 2, 3$, are defined by

$$\begin{aligned}
 \Lambda_1(q, \gamma, \nu) &= \int_0^\nu t|qt + \gamma\nu - \gamma| d_q t \\
 &= \begin{cases} \frac{\nu^2\gamma(1-\nu)}{[2]_q} - \frac{\nu^3q}{[3]_q}, & (\gamma + q)\nu \leq \gamma; \\ \frac{\nu^3q}{[3]_q} - \frac{\nu^2\gamma(1-\nu)}{[2]_q} + \frac{2\gamma^3(1-\nu)^3}{[2]_q[3]_q}, & (\gamma + q)\nu > \gamma, \end{cases} \\
 \Theta_1(q, \gamma, \nu) &= \int_0^\nu (1-t)|qt + \gamma\nu - \gamma| d_q t \\
 &= \int_0^\nu |qt + \gamma\nu - \gamma| d_q t - \int_0^\nu t|qt + \gamma\nu - \gamma| d_q t \\
 &= \begin{cases} \nu\gamma(1-\nu) - \frac{\nu^2q + \nu^2\gamma(1-\nu)}{[2]_q} + \frac{\nu^3q}{[3]_q}, & (\gamma + q)\nu \leq \gamma; \\ \frac{2\gamma^2(1-\nu)^2 + \nu^2q + \nu^2\gamma(1-\nu)}{[2]_q} - \frac{\nu^3q}{[3]_q} - \nu\gamma(1-\nu) - \frac{2\gamma^3(1-\nu)^3}{[2]_q[3]_q}, & (\gamma + q)\nu > \gamma, \end{cases} \\
 \Lambda_2(q, \gamma, \nu) &= \int_0^1 t|qt + \gamma\nu - 1| d_q t \\
 &= \begin{cases} \frac{1-\gamma\nu}{[2]_q} - \frac{q}{[3]_q}, & \gamma\nu + q \leq 1; \\ \frac{q}{[3]_q} - \frac{1-\gamma\nu}{[2]_q} + \frac{2(1-\gamma\nu)^3}{[2]_q[3]_q}, & \gamma\nu + q > 1, \end{cases} \\
 \Theta_2(q, \gamma, \nu) &= \int_0^1 (1-t)|qt + \gamma\nu - 1| d_q t \\
 &= \int_0^1 |qt + \gamma\nu - 1| d_q t - \int_0^1 t|qt + \gamma\nu - 1| d_q t \\
 &= \begin{cases} (1-\gamma\nu) - \frac{q+1-\gamma\nu}{[2]_q} + \frac{q}{[3]_q}, & \gamma\nu + q \leq 1; \\ \frac{2(1-\gamma\nu)^2 + q + 1 - \gamma\nu}{[2]_q} - \frac{q}{[3]_q} - (1-\gamma\nu) - \frac{2(1-\gamma\nu)^3}{[2]_q[3]_q}, & \gamma\nu + q > 1, \end{cases} \\
 \Lambda_3(q, \gamma, \nu) &= \int_0^\nu t|qt + \gamma\nu - 1| d_q t \\
 &= \begin{cases} \frac{\nu^2(1-\gamma\nu)}{[2]_q} - \frac{\nu^3q}{[3]_q}, & \gamma\nu + q \leq 1; \\ \frac{\nu^3q}{[3]_q} - \frac{\nu^2(1-\gamma\nu)}{[2]_q} + \frac{2(1-\gamma\nu)^3}{[2]_q[3]_q}, & \gamma\nu + q > 1, \end{cases} \\
 \Theta_3(q, \gamma, \nu) &= \int_0^\nu (1-t)|qt + \gamma\nu - 1| d_q t \\
 &= \int_0^\nu |qt + \gamma\nu - 1| d_q t - \int_0^\nu t|qt + \gamma\nu - 1| d_q t \\
 &= \begin{cases} \nu(1-\gamma\nu) - \frac{\nu^2q + \nu^2(1-\gamma\nu)}{[2]_q} + \frac{\nu^3q}{[3]_q}, & \gamma\nu + q \leq 1; \\ \frac{2(1-\gamma\nu)^2 + \nu^2q + \nu^2(1-\gamma\nu)}{[2]_{p,q}} - \nu(1-\gamma\nu) - \frac{\nu^3q}{[3]_{p,q}} - \frac{2(1-\gamma\nu)^3}{[2]_q[3]_q}, & \gamma\nu + q > 1, \end{cases}
 \end{aligned}$$

which appears in [46].

Remark 6. If $p = 1$, then (28) is reduced as follows:

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \gamma \left[\frac{\Psi(a) + qf(b)}{[2]_q} \right] - (1-\gamma)\Psi\left(\frac{a+qb}{[2]_q}\right) \right| \\
 &\leq (b-a) \left[\left| \Lambda_1\left(q, \gamma, \frac{1}{[2]_q}\right) + \Lambda_2\left(q, \gamma, \frac{1}{[2]_q}\right) - \Lambda_3\left(q, \gamma, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(a) \right. \\
 &\quad \left. + \left| \Theta_1\left(q, \gamma, \frac{1}{[2]_q}\right) + \Theta_2\left(q, \gamma, \frac{1}{[2]_q}\right) - \Theta_3\left(q, \gamma, \frac{1}{[2]_q}\right) \right| {}^b D_q f(b) \right],
 \end{aligned}$$

which appears in [46].

Remark 7. From Corollary 3, we have the new $(p, q)^b$ -integral inequalities as follows:

(i) If we take $\gamma = 0$, then (28) leads to the midpoint-type integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \Psi\left(\frac{pa+qb}{[2]_{p,q}}\right) \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) + \Lambda_2\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) - \Lambda_3\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) + \Theta_2\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) - \Theta_3\left(p, q, 0, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(b) \right]. \end{aligned} \tag{30}$$

(ii) Taking $\gamma = 1/3$, then (28) leads to the Simpson-like integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \frac{1}{3} \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} + 2\Psi\left(\frac{pa+qb}{[2]_{p,q}}\right) \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) + \Lambda_2\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) - \Lambda_3\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) + \Theta_2\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) - \Theta_3\left(p, q, \frac{1}{3}, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(b) \right]. \end{aligned} \tag{31}$$

(iii) If we take $\gamma = 1/2$, then (28) leads to the averaged midpoint-trapezoid-type integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \frac{1}{2} \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} + 2\Psi\left(\frac{pa+qb}{[2]_{p,q}}\right) \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) + \Lambda_2\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) - \Lambda_2\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) + \Theta_2\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) - \Theta_3\left(p, q, \frac{1}{2}, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(b) \right]. \end{aligned} \tag{32}$$

(iv) By setting $\gamma = 1$, then (28) leads to the trapezoid-type integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \left[\frac{p\Psi(a) + q\Psi(b)}{[2]_{p,q}} \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) + \Lambda_2\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) - \Lambda_3\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) + \Theta_2\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) - \Theta_3\left(p, q, 1, \frac{p}{[2]_{p,q}}\right) \right| {}^b D_{p,q} \Psi(b) \right]. \end{aligned} \tag{33}$$

Remark 8. If $p = 1$, then (30) to (33) are reduced as follows:

(i) We obtain the midpoint-type integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \Psi\left(\frac{a+qb}{[2]_q}\right) \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(q, 0, \frac{1}{[2]_q}\right) + \Lambda_2\left(q, 0, \frac{1}{[2]_q}\right) - \Lambda_3\left(q, 0, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(q, 0, \frac{1}{[2]_q}\right) + \Theta_2\left(q, 0, \frac{1}{[2]_q}\right) - \Theta_3\left(q, 0, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(b) \right], \end{aligned}$$

which appears in [46].

(ii) We obtain the Simpson-like integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{3} \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} + 2\Psi\left(\frac{a+qb}{[2]_q}\right) \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) + \Lambda_2\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) - \Lambda_3\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) + \Theta_2\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) - \Theta_3\left(q, \frac{1}{3}, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(b) \right], \end{aligned} \tag{34}$$

which appears in [46]. Moreover, if $q \rightarrow 1$, then (34) is reduced as follows:

$$\left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{3} \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} + 2\Psi\left(\frac{a+qb}{[2]_q}\right) \right] \right| \leq \frac{5(b-a)}{72} [|\Psi'(b)| + |\Psi'(a)|],$$

which appears in [51].

(iii) We obtain the averaged midpoint-trapezoid-like integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{2} \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} + 2\Psi\left(\frac{a+qb}{[2]_q}\right) \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) + \Lambda_2\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) - \Lambda_2\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) + \Theta_2\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) - \Theta_3\left(q, \frac{1}{2}, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(b) \right], \end{aligned} \tag{35}$$

which appears in [46]. Moreover, if $q \rightarrow 1$, then (35) is reduced as follows:

$$\left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \frac{1}{2} \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} + 2\Psi\left(\frac{a+qb}{[2]_q}\right) \right] \right| \leq \frac{b-a}{16} [|\Psi'(b)| + |\Psi'(a)|],$$

which appears in [52].

(iv) We obtain the trapezoid-type integral inequality as follows:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_q x - \left[\frac{\Psi(a) + q\Psi(b)}{[2]_q} \right] \right| \\ & \leq (b-a) \left[\left| \Lambda_1\left(q, 1, \frac{1}{[2]_q}\right) + \Lambda_2\left(q, 1, \frac{1}{[2]_q}\right) - \Lambda_3\left(q, 1, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(a) \right. \\ & \quad \left. + \left| \Theta_1\left(q, 1, \frac{1}{[2]_q}\right) + \Theta_2\left(q, 1, \frac{1}{[2]_q}\right) - \Theta_3\left(q, 1, \frac{1}{[2]_q}\right) \right| {}^b D_q \Psi(b) \right], \end{aligned} \tag{36}$$

which appears in [52].

Theorem 4. Suppose that $\Psi : [a, b] \rightarrow \mathbb{R}$ is a $(p, q)^b$ -differentiable function on I_1 such that ${}^b D_{p,q} \Psi$ is continuous and integrable functions on I_2 with $\gamma, \nu \in [0, 1]$. If $|{}^b D_{p,q} \Psi|^r$ for $r > 1$ is a convex function on $[a, b]$, then

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ & \leq (b-a)(\Delta_1(p, q, \gamma, \nu))^{1-1/r} \left(\Lambda_2(p, q, \gamma, \nu) \left| {}^b D_{p,q} \Psi(a) \right|^r + \Theta_2(p, q, \gamma, \nu) \left| {}^b D_{p,q} \Psi(b) \right|^r \right)^{1/r} \\ & \quad + (b-a)(1-\gamma)\nu^{1-1/r} \left(\frac{\nu^2}{[2]_{p,q}} \left| {}^b D_{p,q} \Psi(a) \right|^r + \frac{\nu([2]_{p,q} - \nu)}{[2]_{p,q}} \left| {}^b D_{p,q} \Psi(b) \right|^r \right)^{1/r}, \end{aligned} \tag{37}$$

where $\Theta_2(p, q, \gamma, \nu)$ is given in Theorem 3 and $\Delta_1(p, q, \gamma, \nu)$ is defined by

$$\begin{aligned} \Delta_1(p, q, \gamma, \nu) &= \int_0^\nu |qt + \gamma\nu - 1| d_{p,q}t \\ &= \begin{cases} (1 - \gamma\nu) - \frac{q}{[2]_{p,q}}, & \gamma\nu + q \leq 1; \\ \frac{2(1 - \gamma\nu)^2([2]_{p,q} - 1) + q^2}{q[2]_{p,q}} - (1 - \gamma\nu), & \gamma\nu + q > 1. \end{cases} \end{aligned}$$

Proof. Taking the absolute value of both sides of (10) and using the power-mean inequality for (p, q) -integrals, we obtain

$$\begin{aligned} &\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ &= (b-a) \left| \left[\int_0^\nu (qt + \gamma\nu - \gamma) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t + \int_\nu^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t \right] \right| \\ &= (b-a) \left| \left[\int_0^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t + \int_0^\nu (1-\gamma) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t \right] \right| \\ &\leq (b-a) \left[\int_0^1 |qt + \gamma\nu - 1| {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q}t + (1-\gamma) \int_0^\nu |{}^b D_{p,q}\Psi(ta + (1-t)b)| d_{p,q}t \right] \\ &\leq (b-a) \left(\int_0^1 |qt + \gamma\nu - 1| d_{p,q}t \right)^{1-1/r} \left(\int_0^1 |qt + \gamma\nu - 1| {}^b D_{p,q}\Psi(ta + (1-t)b)^r d_{p,q}t \right)^{1/r} \\ &\quad + (b-a)(1-\gamma) \left(\int_0^\nu 1 d_{p,q}t \right)^{1-1/r} \left(\int_0^\nu |{}^b D_{p,q}\Psi(ta + (1-t)b)|^r d_{p,q}t \right)^{1/r}. \end{aligned}$$

Applying the convexity of $|{}^b D_{p,q}\Psi|^r$, we have

$$\begin{aligned} &\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q}x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ &\leq (b-a) \left(\int_0^1 |qt + \gamma\nu - 1| d_{p,q}t \right)^{1-1/r} \\ &\quad \times \left(|{}^b D_{p,q}\Psi(a)|^r \int_0^1 t |qt + \gamma\nu - 1| d_{p,q}t + |{}^b D_{p,q}\Psi(b)|^r \int_0^1 (1-t) |qt + \gamma\nu - 1| d_{p,q}t \right)^{1/r} \\ &\quad + (b-a)(1-\gamma) \left(\int_0^\nu 1 d_{p,q}t \right)^{1-1/r} \left(|{}^b D_{p,q}\Psi(a)|^r \int_0^\nu t d_{p,q}t + |{}^b D_{p,q}\Psi(b)|^r \int_0^\nu (1-t) d_{p,q}t \right)^{1/r} \\ &\leq (b-a)(\Delta_1(p, q, \gamma, \nu))^{1-1/r} \left(\Lambda_2(p, q, \gamma, \nu) |{}^b D_{p,q}\Psi(a)|^r + \Theta_2(p, q, \gamma, \nu) |{}^b D_{p,q}\Psi(b)|^r \right)^{1/r} \\ &\quad + (b-a)(1-\gamma)v^{1-1/r} \left(\frac{v^2}{[2]_{p,q}} |{}^b D_{p,q}\Psi(a)|^r + \frac{v([2]_{p,q} - v)}{[2]_{p,q}} |{}^b D_{p,q}\Psi(b)|^r \right)^{1/r}, \end{aligned}$$

which completes the proof. \square

Remark 9. If $p = 1$, then (37) is reduced as follows:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b \Psi(x) {}^b d_qx - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ &\leq (b-a)(\Delta_1(q, \gamma, \nu))^{1-1/r} \left(\Lambda_2(q, \gamma, \nu) |{}^b D_q\Psi(a)|^r + \Theta_2(q, \gamma, \nu) |{}^b D_q\Psi(b)|^r \right)^{1/r} \\ &\quad + (b-a)(1-\gamma)v^{1-1/r} \left(\frac{v^2}{[2]_q} |{}^b D_q\Psi(a)|^r + \frac{v([2]_q - v)}{[2]_q} |{}^b D_q\Psi(b)|^r \right)^{1/r}, \end{aligned}$$

where $\Theta_2(q, \gamma, \nu)$ is given in Remark 5 and $\Delta_1(q, \gamma, \nu)$ is defined by

$$\begin{aligned} \Delta_1(q, \gamma, \nu) &= \int_0^\nu |qt + \gamma\nu - 1| d_q t \\ &= \begin{cases} (1 - \gamma\nu) - \frac{q}{[2]_q}, & \gamma\nu + q \leq 1; \\ \frac{2(1 - \gamma\nu)^2 + q}{[2]_q} - (1 - \gamma\nu), & \gamma\nu + q > 1, \end{cases} \end{aligned}$$

which appears in [46].

Theorem 5. Suppose that $\Psi : [a, b] \rightarrow \mathbb{R}$ is a $(p, q)^b$ -differentiable function on I_1 such that ${}^bD_{p,q}\Psi$ is a continuous and integrable function on I_2 with $\gamma, \nu \in [0, 1]$. If $|{}^bD_{p,q}\Psi|^r$ for $r > 1$ with $1/s + 1/r = 1$ is a convex function on $[a, b]$, then

$$\begin{aligned} &\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ &\leq (b-a)(\Delta_2(p, q, \gamma, \nu))^{1/s} \left(\frac{|{}^bD_{p,q}\Psi(a)|^r + ([2]_{p,q} - 1) |{}^bD_{p,q}\Psi(b)|^r}{[2]_{p,q}} \right)^{1/r} \\ &\quad + (b-a)(1-\gamma)\nu^{1/s} \left(\frac{\nu^2}{[2]_{p,q}} |{}^bD_{p,q}\Psi(a)|^r + \frac{\nu([2]_{p,q} - \nu)}{[2]_{p,q}} |{}^bD_{p,q}\Psi(b)|^r \right)^{1/r}, \end{aligned} \tag{38}$$

where

$$\Delta_2(p, q, \gamma, \nu) = \int_0^1 |qt + \gamma\nu - 1|^s d_{p,q} t.$$

Proof. Taking the absolute value of both sides of (10) and using Theorem 1, we obtain

$$\begin{aligned} &\left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \gamma[\nu\Psi(a) + (1-\nu)\Psi(b)] - (1-\gamma)\Psi(\nu a + (1-\nu)b) \right| \\ &= (b-a) \left| \int_0^\nu (qt + \gamma\nu - \gamma) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q} t + \int_\nu^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q} t \right| \\ &= (b-a) \left| \int_0^1 (qt + \gamma\nu - 1) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q} t + \int_0^\nu (1-\gamma) {}^b D_{p,q}\Psi(ta + (1-t)b) d_{p,q} t \right| \\ &\leq (b-a) \left[\int_0^1 |qt + \gamma\nu - 1| |{}^b D_{p,q}\Psi(ta + (1-t)b)| d_{p,q} t + (1-\gamma) \int_0^\nu |{}^b D_{p,q}\Psi(ta + (1-t)b)| d_{p,q} t \right] \\ &\leq (b-a) \left(\int_0^1 |qt + \gamma\nu - 1|^s d_{p,q} t \right)^{1/s} \left(\int_0^1 |{}^b D_{p,q}\Psi(ta + (1-t)b)|^r d_{p,q} t \right)^{1/r} \\ &\quad + (b-a)(1-\gamma) \left(\int_0^\nu 1^s d_{p,q} t \right)^{1/s} \left(\int_0^\nu |{}^b D_{p,q}\Psi(ta + (1-t)b)|^r d_{p,q} t \right)^{1/r}. \end{aligned}$$

Applying the convexity of $|{}^bD_{p,q}\Psi|^r$, we have

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \gamma[v\Psi(a) + (1-v)\Psi(b)] - (1-\gamma)\Psi(va + (1-v)b) \right| \\
 & \leq (b-a) \left(\int_0^1 |qt + \gamma v - 1|^s d_{p,q} t \right)^{1/s} \left(|{}^b D_{p,q} \Psi(a)|^r \int_0^1 t d_{p,q} t + |{}^b D_{p,q} \Psi(b)|^r \int_0^1 (1-t) d_{p,q} t \right)^{1/r} \\
 & \quad + (b-a)(1-\gamma) \left(\int_0^v 1^s d_{p,q} t \right)^{1/s} \left(|{}^b D_{p,q} \Psi(a)|^r \int_0^v t d_{p,q} t + |{}^b D_{p,q} \Psi(b)|^r \int_0^v (1-t) d_{p,q} t \right)^{1/r} \\
 & = (b-a)(\Delta_2(p, q, \gamma, v))^{1/s} \left(\frac{|{}^b D_{p,q} \Psi(a)|^r + ([2]_{p,q} - 1) |{}^b D_{p,q} \Psi(b)|^r}{[2]_{p,q}} \right)^{1/r} \\
 & \quad + (b-a)(1-\gamma)v^{1/s} \left(\frac{v^2}{[2]_{p,q}} |{}^b D_{p,q} \Psi(a)|^r + \frac{v([2]_{p,q} - v)}{[2]_{p,q}} |{}^b D_{p,q} \Psi(b)|^r \right)^{1/r},
 \end{aligned}$$

which completes the proof. \square

Remark 10. If $p = 1$, then (38) is reduced as follows:

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b \Psi(x) {}^b d_{p,q} x - \gamma[v\Psi(a) + (1-v)\Psi(b)] - (1-\gamma)\Psi(va + (1-v)b) \right| \\
 & \leq (b-a)(\Delta_2(q, \gamma, v))^{1/s} \left(\frac{|{}^b D_q \Psi(a)|^r + q |{}^b D_q \Psi(b)|^r}{[2]_q} \right)^{1/r} \\
 & \quad + (b-a)(1-\gamma)v^{1/s} \left(\frac{v^2}{[2]_q} |{}^b D_q \Psi(a)|^r + \frac{v([2]_q - v)}{[2]_q} |{}^b D_q \Psi(b)|^r \right)^{1/r}, \tag{39}
 \end{aligned}$$

where

$$\Delta_2(q, \gamma, v) = \int_0^1 |qt + \gamma v - 1|^s d_q t,$$

which appears in [46].

Corollary 4. Under the assumptions of Theorems 4 and 5 with $v = p/[2]_{p,q}$, if we choose $\gamma = 0, \gamma = 1/3, \gamma = 1/2$ and $\gamma = 1$, then we obtain the midpoint-type integral inequality, the Simpson-like integral inequality, the averaged midpoint-trapezoid-type integral inequality and the trapezoid-type integral inequality, respectively.

Remark 11. From Corollary 4, if $p = 1$, then we have some q -integral inequalities, which appears in [46].

4. Examples

In this section, we show two examples to investigate our main theorems.

Example 1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 3 with $p = 3/4, q = 1/2, v = 3/5$ and $\gamma = 1/3$, the left side of (27) becomes

$$\begin{aligned}
 & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b f(x) {}^b d_{p,q} x - \gamma[vf(a) + (1-v)f(b)] - (1-\gamma)f(va + (1-v)b) \right| \\
 & = \left| \frac{4}{3} \int_{\frac{1}{4}}^1 x^2 {}^1 d_{\frac{3}{4}, \frac{1}{2}} x - \frac{1}{3} \cdot \left[\frac{3}{5} f(0) + \frac{2}{3} f(1) \right] - \frac{2}{3} f\left(\frac{2}{5}\right) \right| \\
 & \approx |0.2736 - 0.1333 - 0.1066| \approx 0.0337,
 \end{aligned}$$

and the right side of (27) becomes

$$\begin{aligned} & (b - a) \left[|\Lambda_1(p, q, \gamma, \nu) + \Lambda_2(p, q, \gamma, \nu) - \Lambda_3(p, q, \gamma, \nu)| \left| {}^b D_{p,q} f(a) \right| \right. \\ & \left. + |\Theta_1(p, q, \gamma, \nu) + \Theta_2(p, q, \gamma, \nu) - \Theta_3(p, q, \gamma, \nu)| \left| {}^b D_{p,q} f(b) \right| \right] \\ & \approx (1 - 0) [|0.0517 + 0.2189 - 0.1394| \cdot |0.75| + |0.0264 + 0.1810 - 0.1605| \cdot |2|] \\ & \approx 0.1922. \end{aligned}$$

It is clear that

$$0.0337 \leq 0.1922,$$

which demonstrates the result described in Theorem 3.

Example 2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 3 with $p = 3/4$, $q = 1/2$, $\nu = 3/5$, $\gamma = 1/3$ and $r = 5$, the left side of (37) becomes

$$\begin{aligned} & \left| \frac{1}{p(b-a)} \int_{pa+(1-p)b}^b f(x) {}^b d_{p,q} x - \gamma [\nu f(a) + (1-\nu)f(b)] - (1-\gamma) f(\nu a + (1-\nu)b) \right| \\ & = \left| \frac{4}{3} \int_{\frac{1}{4}}^1 x^2 {}^1 d_{\frac{3}{4}, \frac{1}{2}} x - \frac{1}{3} \cdot \left[\frac{3}{5} f(0) + \frac{2}{3} f(1) \right] - \frac{2}{3} f\left(\frac{2}{5}\right) \right| \\ & \approx |0.2736 - 0.1333 - 0.1066| \approx 0.0337, \end{aligned}$$

and the right side of (37) becomes

$$\begin{aligned} & (b - a) (\Delta_1(p, q, \gamma, \nu))^{1-1/r} \left(|\Lambda_2(p, q, \gamma, \nu)| \left| {}^b D_{p,q} \Psi(a) \right|^r + |\Theta_2(p, q, \gamma, \nu)| \left| {}^b D_{p,q} \Psi(b) \right|^r \right)^{1/r} \\ & + (b - a) (1 - \gamma) \nu^{1-1/r} \left(\frac{\nu^2}{[2]_{p,q}} \left| {}^b D_{p,q} \Psi(a) \right|^r + \frac{\nu([2]_{p,q} - \nu)}{[2]_{p,q}} \left| {}^b D_{p,q} \Psi(b) \right|^r \right)^{1/r} \\ & \approx (1 - 0) (0.4)^{1-1/5} \left(0.2189 \cdot |0.75|^5 + 0.1810 \cdot |2|^5 \right)^{1/5} \\ & + (1 - 0) (1 - 0.3333) (0.6)^{1-1/5} \left(0.2880 \cdot |0.75|^5 + 0.3120 \cdot |2|^5 \right)^{1/5} \\ & \approx 1.3868. \end{aligned}$$

It is clear that

$$0.0337 \leq 1.3868,$$

which demonstrates the result described in Theorem 4.

5. Conclusions

In this work, we established some new estimates of $(p, q)^b$ -integral inequalities related to Hermite–Hadamard inequalities for which the first-order $(p, q)^b$ -derivatives in absolute value are convex functions. The main results in this study were proven to be generalizations of some previously proved results of q^b -integral inequalities related to Hermite–Hadamard inequalities for q^b -differentiable convex functions. Furthermore, the obtained results were used to study some special cases, namely the midpoint-type integral inequality, Simpson-like integral inequality, averaged midpoint-trapezoid-type integral inequality, and trapezoid-type integral inequality. Examples were given to illustrate the investigated results.

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