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Some New Post-Quantum Simpson's Type Inequalities for Coordinated Convex Functions

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Abstract: In this paper, we establish some new Simpson's type inequalities for coordinated convex functions by using post-quantum calculus. The results raised in this paper provide significant extensions and generalizations of other related results given in earlier works.

Keywords: Simpson's inequality; convex function; coordinated convex function; (p,q)-derivative; (p,q)-integral; (p,q)-calculus

MSC: 05A30; 26D10; 26D15; 26A51; 26B25; 81P68

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1. Introduction

Quantum calculus or q-calculus is known as the study of calculus without limits. It was first studied by the famous mathematician, Euler (1707–1783). In 1910, F. H. Jackson [1] determined the definite q-integral known as the q-Jackson integral. Quantum calculus has applications in several mathematical areas such as combinatorics, orthogonal polynomials, number theory, basic hypergeometric functions, quantum theory, mechanics, and theory of relativity, see for instance [2–7] and the references therein. The book by V. Kac and P. Cheung [8] covers the fundamental knowledge and also the basic theoretical concepts of quantum calculus.

In 2013, J. Tariboon and S. K. Ntouyas [9] defined the q-derivative and q-integral of a continuous function on finite intervals and proved some of its significant properties. They also proved Hölder, Hermite–Hadamard, trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss and Grüss–Čebyšev inequalities in the setup of q-calculus , see [10] for more details. Based on these results, there are many outcomes concerning q-calculus, see [11–16] and the references cited therein.

The further generalization of q-calculus is (p,q)-calculus (or post-quantum calculus), which was first considered by R. Chakrabati and R. Jagannathan [17]. In 2016, M. Tunç and E. Göv [18,19] introduced the (p,q)-derivative and (p,q)-integral on finite intervals, proved some of its properties and gave many integral inequalities via (p,q)-calculus. Recently, according to works of M. Tunç and E. Göv, many researchers started working in this direction, some more results on the study of (p,q)-calculus can be found in [20–25]. It is worth to note here that (p,q)-calculus cannot be derived directly by replacing q by q/p in q-calculus, but q-calculus can be retaken by setting $p=1$ in (p,q)-calculus.

Simpson's rules are a well-known technique for numerical estimations of integrals. Thomas Simpson (1710–1761) developed this method to estimate of definite integrals. Simpson's quadrature formula (sometimes is called Simpson 1/3 rule) is stated as

$$\int_a^b f(x)\,dx \approx \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1}$$

There are many estimations related to Simpson's quadrature rule, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Suppose that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable mapping on (a,b) and let $||f^{(4)}|| = \sup_{x\in(a,b)}|f^{(4)}| < \infty$. Then we have*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\,||f^{(4)}||\,(b-a)^4. \tag{2}$$

Recently, many researchers have played attention on Simpson's type inequalities for various classes of functions such as convex functions, s-convex functions, etc., see [26–28] for more details.

In 2020, H. Budak, S. Erden and M. A. Ali [29] presented new Simpson's type inequalities for convex functions via q-calculus. Recently, H. Kalsoom et al. [14] and M. A. Ali et al. [30] derived new Simpson's type inequalities for coordinated convex functions via q-calculus.

Previously, some generalizations of new Simpson's type inequalities for convex functions to q-calculus, but no one generalized (p,q)-calculus. H. Kalsoom et al. and M. A. Ali et al. provide some excellent and powerful techniques to get such extensions. Motivated by the above literature, we propose to prove new Simpson's type inequalities for coordinated convex functions by using (p,q)-calculus.

2. Preliminaries

Throughout this paper, we let $\triangle := [a,b]\times[c,d] \subseteq \mathbb{R}\times\mathbb{R}$, $0<q<p\le 1$ and $0<q_i<p_i\le 1$ for $i=1,2$. The definitions of (p,q)-calculus, coordinated functions, q-calculus and (p,q)-calculus for coordinates are given in [15,16,18,19,23–25,31,32]. Also, here and further, we use the following notation:

$$[n]_{p,q} := \frac{p^n-q^n}{p-q} = p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1}, \qquad \text{for} \quad n\in\mathbb{R}.$$

Definition 1 ([18])**.** *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,\frac{1}{p}(b-a)+a]$. Then the ${}_a(p,q)$-derivative of f at $x\in(a,\frac{1}{p}(b-a)+a]$ is defined by*

$${}_aD_{p,q}f(x) = \frac{f(px+(1-p)a)-f(qx+(1-q)a)}{(p-q)(x-a)}. \tag{3}$$

The ${}_a(p,q)$-integral of f on $[a,x]$ is defined by

$$\int_a^x f(t)\,{}_ad_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)a\right). \tag{4}$$

Definition 2 ([24])**.** *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[\frac{1}{p}(a-b)+b,b]$. Then the ${}^b(p,q)$-derivative of f at $x\in[\frac{1}{p}(a-b)+b,b)$ is defined by*

$${}^bD_{p,q}f(x) = \frac{f(qx+(1-q)b)-f(px+(1-p)b)}{(p-q)(b-x)}. \tag{5}$$

The $^b(p,q)$*-integral of* f *on* $[x,b]$ *is defined by*

$$\int_x^b f(t)\,{}^b d_{p,q}t = (p-q)(b-x)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1-\frac{q^n}{p^{n+1}}\right)b\right). \tag{6}$$

Definition 3 ([31]). *A function* $f : \triangle \to \mathbb{R}$ *is said to be convex on coordinates, if the partial mappings*

$$f_x : [c,d] \ni v \mapsto f(x,v) \in \mathbb{R} \quad \text{and} \quad f_y : [a,b] \ni u \mapsto f(u,y) \in \mathbb{R}$$

are convex for all $x \in (a,b)$ *and* $y \in (c,d)$.

A formal definition for coordinated convex functions may be stated as follows:

Definition 4. *A function* $f : \triangle \to \mathbb{R}$ *is said to be convex on coordinates, if*

$$\begin{aligned} f(tx+(1-t)z, \lambda y + (1-\lambda)w) &\le t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) \\ &\quad + (1-t)\lambda(1-])f(z,w) \end{aligned} \tag{7}$$

holds for all $t,\lambda \in [0,1]$ *and* $(x,y),(z,w) \in \triangle$.

Definition 5 ([15]). *Suppose that* $f : \triangle \to \mathbb{R}$ *is a continuous function of two variables. Then the* (q_1q_2)*-derivatives are given by*

$$\frac{{}_a\partial_{q_1} f(x,y)}{{}_a\partial_{q_1} x} = \frac{f(x,y) - f(q_1x + (1-q_1)a, y)}{(1-q_1)(x-a)}, \quad x \ne a, \tag{8}$$

$$\frac{{}_c\partial_{q_2} f(x,y)}{{}_c\partial_{q_2} y} = \frac{f(x,y) - f(x, q_2y + (1-q_2)c)}{(1-q_2)(y-c)}, \quad y \ne c, \tag{9}$$

and

$$\begin{aligned} \frac{{}_{a,c}\partial^2{}_{q_1,q_2} f(x,y)}{{}_a\partial_{q_1} x\, {}_c\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(x-a)(y-c)}[f(q_1x+(1-q_1)a, q_2y+(1-q_2)c) \\ &\quad - f(q_1x+(1-q_1)a, y) - f(x, q_2y+(1-q_2)c) + f(x,y)], \end{aligned} \tag{10}$$

for $x \ne a$ *and* $y \ne c$.

Definition 6 ([16]). *Suppose that* $f : \triangle \to \mathbb{R}$ *is a continuous function of two variables. Then the* (q_1q_2)*-derivatives are given by*

$$\frac{{}^b\partial_{q_1} f(x,y)}{{}^b\partial_{q_1} x} = \frac{f(q_1x+(1-q_1)b, y) - f(x,y)}{(1-q_1)(b-x)}, \quad x \ne b, \tag{11}$$

$$\frac{{}^d\partial_{q_2} f(x,y)}{{}^d\partial_{q_2} y} = \frac{f(x, q_2y+(1-q_2)d) - f(x,y)}{(1-q_2)(d-y)}, \quad y \ne d, \tag{12}$$

$$\begin{aligned} \frac{{}^b_c\partial^2{}_{q_1,q_2} f(x,y)}{{}^b\partial_{q_1} x\, {}_c\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(b-x)(y-c)}[f(q_1x+(1-q_1)b, q_2y+(1-q_2)c) \\ &\quad - f(q_1x+(1-q_1)b, y) - f(x, q_2y+(1-q_2)c) + f(x,y)], \quad x \ne b, y \ne c, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{{}^d_a\partial^2{}_{q_1,q_2} f(x,y)}{{}_a\partial_{q_1} x\, {}^d\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(x-a)(d-y)}[f(q_1x+(1-q_1)a, q_2y+(1-q_2)d) \\ &\quad - f(q_1x+(1-q_1)a, y) - f(x, q_2y+(1-q_2)d) + f(x,y)], \quad x \ne a, y \ne d, \end{aligned} \tag{14}$$

and

$$\frac{{}^{b,d}\partial^2_{q_1,q_2} f(x,y)}{{}^{b}\partial_{q_1} x\, {}^{d}\partial_{q_2} y} = \frac{1}{(1-q_1)(1-q_2)(b-x)(d-y)}[f(q_1x+(1-q_1)b, q_2y+(1-q_2)d) \\ -f(q_1x+(1-q_1)b,y) - f(x,q_2y+(1-q_2)d) + f(x,y)], \tag{15}$$

for $x \neq b$ and $y \neq d$.

Definition 7 ([23])**.** *Suppose that $f : \triangle \to \mathbb{R}$ is a continuous function of two variables. Then the (p_1, p_2, q_1, q_2)-derivatives are given by*

$$\frac{{}_{a}\partial_{p_1,q_1} f(x,y)}{{}_{a}\partial_{p_1,q_1} x} = \frac{f(p_1x+(1-p_1)a,y) - f(q_1x+(1-q_1)a,y)}{(p_1-q_1)(x-a)}, \quad x \neq a, \tag{16}$$

$$\frac{{}_{c}\partial_{p_2,q_2} f(x,y)}{{}_{c}\partial_{p_2,q_2} y} = \frac{f(x,p_2y+(1-p_2)c) - f(x,q_2y+(1-q_2)c)}{(p_2-q_2)(y-c)}, \quad y \neq c, \tag{17}$$

and

$$\frac{{}_{a,c}\partial^2_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\, {}_{c}\partial_{p_2,q_2} y} = \frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}[f(q_1x+(1-q_1)a, q_2y+(1-q_2)c) \\ -f(q_1x+(1-q_1)a, p_2y+(1-p_2)c) - f(p_1x+(1-p_1)a, q_2y+(1-q_2)c) \\ +f(p_1x+(1-p_1)a, p_2y+(1-p_2)c)], \tag{18}$$

for $x \neq a$ and $y \neq c$.

Definition 8 ([32])**.** *Suppose that $f : \triangle \to \mathbb{R}$ is a continuous function of two variables. Then the (p_1, p_2, q_1, q_2)-derivatives are given by*

$$\frac{{}^{b}\partial_{p_1,q_1} f(x,y)}{{}^{b}\partial_{p_1,q_1} x} = \frac{f(q_1x+(1-q_1)b,y) - f(p_1x+(1-p_1)b,y)}{(p_1-q_1)(b-x)}, \quad x \neq b, \tag{19}$$

$$\frac{{}^{d}\partial_{p_2,q_2} f(x,y)}{{}^{d}\partial_{p_2,q_2} y} = \frac{f(x,q_2y+(1-q_2)d) - f(x,p_2y+(1-p_2)d)}{(p_2-q_2)(d-y)}, \quad y \neq d, \tag{20}$$

$$\frac{{}^{b}_{c}\partial^2_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\, {}_{c}\partial_{p_2,q_2} y} = \frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(y-c)}[f(q_1x+(1-q_1)b, p_2y+(1-p_2)c) \\ -f(p_1x+(1-p_1)b, p_2y+(1-p_2)c) - f(q_1x+(1-q_1)b, q_2y+(1-q_2)c) \\ +f(p_1x+(1-p_1)b, q_2y+(1-q_2)c)], \quad x \neq b,\ y \neq c, \tag{21}$$

$$\frac{{}^{d}_{a}\partial^2_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\, {}^{d}\partial_{p_2,q_2} y} = \frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(d-y)}[f(p_1x+(1-p_1)a, q_2y+(1-q_2)d) \\ -f(q_1x+(1-q_1)a, q_2y+(1-q_2)d) - f(p_1x+(1-p_1)a, p_2y+(1-p_2)d) \\ +f(q_1x+(1-q_1)a, p_2y+(1-p_2)d)], \quad x \neq a,\ y \neq d, \tag{22}$$

and

$$\frac{{}^{b,d}\partial^2_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\, {}^{d}\partial_{p_2,q_2} y} = \frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(d-y)}[f(q_1x+(1-q_1)b, q_2y+(1-q_2)d) \\ -f(p_1x+(1-p_1)b, q_2y+(1-q_2)d) - f(q_1x+(1-q_1)b, p_2y+(1-p_2)d) \\ +f(p_1x+(1-p_1)b, p_2y+(1-p_2)d)], \quad x \neq b,\ y \neq d. \tag{23}$$

Definition 9 ([23])**.** *Suppose that $f : \triangle \to \mathbb{R}$ is a continuous function of two variables. Then the definite (p_1, p_2, q_1, q_2)-integral is given by*

$$\int_a^x \int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(x-a)(y-c)$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)a, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right), \tag{24}$$

for $(x,y) \in [a, p_1 b + (1-p_1)a] \times [c, p_2 d + (1-p_2)c]$.

Definition 10 ([25])**.** *Suppose that* $f : \triangle \to \mathbb{R}$ *is a continuous function of two variables. Then the definite* (p_1, p_2, q_1, q_2)*-integrals are given by*

$$\int_a^x \int_y^d f(t,s)\ {}^d d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(x-a)(d-y)$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)a, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right), \tag{25}$$

for $(x,y) \in [a, p_1 b + (1-p_1)a] \times [c, p_2 d + (1-p_2)c]$,

$$\int_x^b \int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}^b d_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(b-x)(y-c)$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right), \tag{26}$$

for $(x,y) \in [a, p_1 b + (1-p_1)a] \times [c, p_2 d + (1-p_2)c]$ *and*

$$\int_x^b \int_y^d f(t,s)\ {}^d d_{p_2,q_2}s\ {}^b d_{p_1,q_1}t = (p_1-q_1)(p_2-q_2)(b-x)(d-y)$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}}x + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)b, \frac{q_2^m}{p_2^{m+1}}y + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right), \tag{27}$$

for $(x,y) \in [a, p_1 b + (1-p_1)a] \times [c, p_2 d + (1-p_2)c]$.

Lemma 1 ([32])**.** ((p_1, p_2, q_1, q_2)-Hölder's inequality for functions of two variables). *Let* f, g *be* (p_1, p_2, q_1, q_2)*-integrable functions on* $\triangle$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ *with* $\alpha, \beta > 1$*. Then the following inequality for functions of two variables holds:*

$$\int_a^x \int_c^y |f(t,s)g(t,s)|\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t \le \left(\int_a^x \int_c^y |f(t,s)|^\alpha\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t\right)^{1/\alpha}$$
$$\times \left(\int_a^x \int_c^y |g(t,s)|^\beta\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t\right)^{1/\beta}. \tag{28}$$

Lemma 2 ([32])**.** ((p_1, p_2, q_1, q_2)-power mean inequality for functions of two variables). *Let* f, g *be* (p_1, p_2, q_1, q_2)*-integrable functions on* $\triangle$ *and* $\alpha \ge 1$*. Then the following inequality for functions of two variables holds:*

$$\int_a^x \int_c^y |f(t,s)g(t,s)|\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t \le \left(\int_a^x \int_c^y |f(t,s)|\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t\right)^{1-1/\alpha}$$
$$\times \left(\int_a^x \int_c^y |f(t,s)||g(t,s)|^\alpha\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t\right)^{1/\alpha}. \tag{29}$$

Lemma 3 ([19]). *We have the equality*

$$\int_a^b (t-a)^{\alpha}\ {}_a d_{p,q}t = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_{p,q}}, \tag{30}$$

for $\alpha \in \mathbb{R} \setminus \{-1\}$.

3. Main Results

In this section, we prove some new (p,q)-Simpson's type inequalities for coordinated convex functions. We may start with Lemmas 4 and 5, which are useful in further considerations.

Lemma 4. *Let* $f : \triangle \to \mathbb{R}$ *be a continuous function and* $(x_1, y_1) \in (a,x) \times (b,y)$. *Then we have*

$$\begin{aligned}\int_{x_1}^{x}\int_{y_1}^{y} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t &= \int_a^x\int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t - \int_a^{x_1}\int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t \\ &\quad - \int_a^x\int_c^{y_1} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t + \int_a^{x_1}\int_b^{y_1} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t.\end{aligned} \tag{31}$$

Proof. By the definition of (p_1,p_2,q_1,q_2)-integral, we have

$$\begin{aligned}\int_{x_1}^{x}\int_{y_1}^{y} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t &= \int_{x_1}^x\int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t - \int_{x_1}^{x}\int_c^{y_1} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t \\ &= \int_a^x\int_0^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t - \int_a^{x_1}\int_c^y f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t \\ &\quad - \int_a^x\int_c^{y_1} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t + \int_a^{x_1}\int_b^{y_1} f(t,s)\ {}_c d_{p_2,q_2}s\ {}_a d_{p_1,q_1}t,\end{aligned}$$

which completes the proof. □

For convenience, we will use the following notations:

$$\Phi(t,s) := \frac{{}^{b,d}\partial^2{}_{p_1,p_2,q_1,q_2}f(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s}, \quad \Theta(t,s) := \frac{{}^{d}_{a}\partial^2{}_{p_1,p_2,q_1,q_2}f(t,s)}{{}_{a}\partial_{p_1,q_1}t\,{}^{d}\partial_{p_2,q_2}s},$$
$$\Psi(t,s) := \frac{{}^{b}_{c}\partial^2{}_{p_1,p_2,q_1,q_2}f(t,s)}{{}^{b}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s} \quad \text{and} \quad \Omega(t,s) := \frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2}f(t,s)}{{}_{a}\partial_{p_1,q_1}t\,{}_{c}\partial_{p_2,q_2}s}.$$

Lemma 5. *Let* $f : \triangle \to \mathbb{R}$ *be a twice partially* (p_1,p_2,q_1,q_2)*-differentiable function on* $\triangle^\circ$. *If the partial* (p_1,p_2,q_1,q_2)*-derivative* $\Phi(t,s)$ *is continuous and* (p_1,p_2,q_1,q_2)*-integrable on* $\triangle$, *then the following identity holds:*

$${}^{b,d}I_{p_1,p_2,q_1,q_2}(f) = (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b, sc+(1-s)d)\ {}_0 d_{p_2,q_2}s\ {}_0 d_{p_1,q_1}t, \tag{32}$$

where

$$
\begin{aligned}
&{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)\\
&= \frac{1}{9}\left\{(6p_2-5)f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+(6p_1-5)f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right.\\
&\left.+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}+\frac{1}{36}\{(36p_1p_2-30p_1-30p_2+25)f(a,c)\\
&+(30p_1-4(6p_1-5)-25)f(a,d)+(30p_2-4(6p_2-5)-25)f(b,c)+f(b,d)\}\\
&-\frac{p_1}{6(b-a)}\int_a^b (6p_2-5)f(p_1x+(1-p_1)b,c)+4f\left(p_1x+(1-p_1)b,\frac{c+d}{2}\right)\\
&+f(p_1x+(1-p_1)b,d)\ {}^{b}d_{p_1,q_1}x-\frac{p_2}{6(d-c)}\int_c^d f(a,p_2y+(1-p_2)d)\\
&+4f\left(\frac{a+b}{2},p_2y+(1-p_2)d\right)+(6p_1-5)f(b,p_2y+(1-p_2)d)\ {}^{d}d_{p_2,q_2}y\\
&+\frac{p_1p_2}{(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)b,p_2y+(1-p_2)d)\ {}^{d}d_{p_2,q_2}y\ {}^{b}d_{p_1,q_1}x,
\end{aligned}
$$

$$
\Lambda_{p_1,q_1}(t)=\begin{cases} p_1q_1t-\frac{1}{6}, & t\in[0,\frac{1}{2});\\ p_1q_1t-\frac{5}{6}, & t\in[\frac{1}{2},1]\end{cases}
$$

and

$$
\Lambda_{p_2,q_2}(s)=\begin{cases} p_2q_2s-\frac{1}{6}, & s\in[0,\frac{1}{2});\\ p_2q_2s-\frac{5}{6}, & s\in[\frac{1}{2},1].\end{cases}
$$

Proof. By Lemma 4 and the definitions of $\Lambda_{p_1,q_1}(t)$ and $\Lambda_{p_2,q_2}(s)$, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&=\int_0^{1/2}\int_0^{1/2} \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_0^{1/2}\int_{1/2}^{1} \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_{1/2}^{1}\int_0^{1/2} \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_{1/2}^{1}\int_{1/2}^{1} \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&=\int_0^{1/2}\int_0^{1/2}\left(p_1q_1t-\frac{1}{6}\right)\left(p_2q_2s-\frac{1}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_0^{1/2}\int_{1/2}^{1}\left(p_1q_1t-\frac{1}{6}\right)\left(p_2q_2s-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_{1/2}^{1}\int_0^{1/2}\left(p_1q_1t-\frac{5}{6}\right)\left(p_2q_2s-\frac{1}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\int_{1/2}^{1}\int_{1/2}^{1}\left(p_1q_1t-\frac{5}{6}\right)\left(p_2q_2s-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&=\frac{4}{9}\int_0^{1/2}\int_0^{1/2}\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\frac{2}{3}\int_0^{1/2}\int_0^{1}\left(p_2q_2s-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+\frac{2}{3}\int_0^{1}\int_0^{1/2}\left(p_1q_1t-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1\int_0^1\left(p_1q_1t-\frac{5}{6}\right)\left(p_2q_2s-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&=: I_1+I_2+I_3+I_4. \qquad (33)
\end{aligned}
$$

By Definition 8, we have

$$
\begin{aligned}
I_1 &= \frac{4}{9}\int_0^{1/2}\int_0^{1/2}\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&= \frac{4}{9}\int_0^{1/2}\int_0^{1/2}\frac{1}{(p_1-q_1)(p_2-q_2)(b-a)(d-c)}[f(q_1ta+(1-q_1t)b,q_2sc+(1-q_2s)d)\\
&\quad -f(p_1ta+(1-p_1t)b,q_2sc+(1-q_2s)d)-f(q_1ta+(1-q_1t)b,p_2sc+(1-p_2s)d)\\
&\quad +f(p_1ta+(1-p_1t)b,p_2sc+(1-p_2s)d)]\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&= \frac{4}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{2p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{2p_2^{m+1}}\right)d\right)\right.\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m+1}}{2p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{2p_2^{m+1}}\right)d\right)\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m}}{2p_2^{m}}c+\left(1-\frac{q_2^{m}}{2p_2^{m}}\right)d\right)\\
&\quad \left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(\frac{q_1^{n+1}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m}}{2p_2^{m}}c+\left(1-\frac{q_2^{m}}{2p_2^{m}}\right)d\right)\right\}\\
&= \frac{4}{9(b-a)(d-c)}\left[f(b,d)-f\left(\frac{a+b}{2},d\right)-f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]. \qquad (34)
\end{aligned}
$$

Also, by Definition 8, we get the following equalities:

$$
\begin{aligned}
&\int_0^{1/2}\int_0^{1} s\,\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(b-a)(d-c)}[f(q_1ta+(1-q_1t)b,q_2sc+(1-q_2s)d)\\
&\quad -f(p_1ta+(1-p_1t)b,q_2sc+(1-q_2s)d)-f(q_1ta+(1-q_1t)b,p_2sc+(1-p_2s)d)\\
&\quad +f(p_1ta+(1-p_1t)b,p_2sc+(1-p_2s)d)]\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&= \frac{1}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\\
&\quad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&\quad \left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right\}\\
&= \frac{1}{(b-a)(d-c)}\left\{\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left[\sum_{n=0}^{\infty}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.\right.\\
&\quad \left.-\sum_{n=0}^{\infty}f\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right]
\end{aligned}
$$

$$+\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^{m+1}}\left[\sum_{n=0}^{\infty}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)\right.$$
$$\left.\left.-\sum_{n=0}^{\infty}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)\right]\right\}$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^{m+1}}{p_2^{m+1}}f\left(b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.$$
$$-\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)+\frac{1}{p_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(\frac{a+b}{2},\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)$$
$$\left.-\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^{m+1}}{p_2^{m+1}}f\left(b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right\}$$
$$=\frac{1}{(b-a)(d-c)}\left\{-\frac{1}{q_2}f(b,c)+\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)\right.$$
$$-\frac{1}{p_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)+\frac{1}{p_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(\frac{a+b}{2},\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)$$
$$\left.+\frac{1}{q_2}f\left(\frac{a+b}{2},c\right)-\frac{1}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(\frac{a+b}{2},\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)\right\}$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{p_2-q_2}{q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(b,\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)-\frac{1}{q_2}f(b,c)\right.$$
$$\left.-\frac{p_2-q_2}{p_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(\frac{a+b}{2},\frac{q_2^m}{p_2^m}c+\left(1-\frac{q_2^m}{p_2^m}\right)d\right)+\frac{1}{q_2}f\left(\frac{a+b}{2},c\right)\right\}$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_2(d-c)}\int_c^d f(b,p_2y+(1-p_2)d)\ {}^d d_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{q_2(d-c)}\int_c^d f\left(\frac{a+b}{2},p_2y+(1-p_2)d\right)\ {}^d d_{p_2,q_2}y-\frac{1}{q_2}f(b,c)+\frac{1}{q_2}f\left(\frac{a+b}{2},c\right)\right\} \tag{35}$$

and

$$\int_0^{1/2}\int_0^1\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$=\frac{1}{(b-a)(d-c)}\left[f(b,d)-f\left(\frac{a+b}{2},d\right)-f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]. \tag{36}$$

From (35) and (36), we obtain

$$I_2=\frac{2}{3}\int_0^{1/2}\int_0^1\left(p_2q_2s-\frac{5}{6}\right)\Phi(ta+(1-t)b,sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$=\frac{2}{3(b-a)(d-c)}\left[\frac{p_2}{d-c}\int_c^d f(b,p_2y+(1-p_2)d)\ {}^d d_{p_2,q_2}y\right.$$
$$\left.-\frac{p_2}{d-c}\int_c^d f\left(\frac{a+b}{2},p_2y+(1-p_2)d\right)\ {}^d d_{p_2,q_2}y-p_2f(b,c)+p_2f\left(\frac{a+b}{2},c\right)\right]$$
$$-\frac{10}{18(b-a)(d-c)}\left[f(b,d)-f\left(\frac{a+b}{2},d\right)-f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]$$
$$=\frac{1}{(b-a)(d-c)}\left[\frac{2p_2}{3(d-c)}\int_c^d f(b,p_2y+(1-p_2)d)\ {}^d d_{p_2,q_2}y\right.$$

$$-\frac{2p_2}{3(d-c)}\int_c^d f\left(\frac{a+b}{2}, p_2y+(1-p_2)d\right)\,{}^d d_{p_2,q_2}y-\left(\frac{6p_2-5}{9}\right)f(b,c)$$
$$-\frac{5}{9}f(b,d)+\frac{5}{9}f\left(\frac{a+b}{2},d\right)+\left(\frac{6p_2-5}{9}\right)f\left(\frac{a+b}{2},c\right)\Bigg]. \quad (37)$$

Similarly, we have

$$I_3=\frac{1}{(b-a)(d-c)}\left[\frac{2p_1}{3(b-a)}\int_a^b f(p_1x+(1-p_1)b,d)\,{}^b d_{p_1,q_1}x\right.$$
$$-\frac{2p_1}{3(b-a)}\int_a^b f\left(p_1x+(1-p_1)b,\frac{c+d}{2}\right)\,{}^b d_{p_1,q_1}x-\left(\frac{6p_1-5}{9}\right)f(a,d)$$
$$\left.-\frac{5}{9}f(b,d)+\frac{5}{9}f\left(b,\frac{c+d}{2}\right)+\left(\frac{6p_1-5}{9}\right)f\left(a,\frac{c+d}{2}\right)\right]. \quad (38)$$

Moreover, we get the following equalities:

$$\int_0^1\int_0^1 \Phi(ta+(1-t)b,sc+(1-s)d)\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t$$
$$=\frac{1}{(b-a)(d-c)}[f(b,d)-f(a,d)-f(b,c)+f(a,c)], \quad (39)$$

$$\int_0^1\int_0^1 s\,\Phi(ta+(1-t)b,sc+(1-s)d)\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_2(d-c)}\int_c^d f(b,p_2y+(1-p_2)d)\,{}^d d_{p_2,q_2}y\right.$$
$$\left.-\frac{1}{q_2(d-c)}\int_c^d f(a,p_2y+(1-p_2)d)\,{}^d d_{p_2,q_2}y-\frac{1}{q_2}f(b,c)+\frac{1}{q_2}f(a,c)\right\}, \quad (40)$$

$$\int_0^1\int_0^1 t\,\Phi(ta+(1-t)b,sc+(1-s)d)\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_1(b-a)}\int_a^b f(p_1x+(1-p_1)b,d)\,{}^b d_{p_1,q_1}x\right.$$
$$\left.-\frac{1}{q_1(b-a)}\int_a^b f(p_1x+(1-p_1)b,c)\,{}^b d_{p_1,q_1}x-\frac{1}{q_1}f(a,d)+\frac{1}{q_1}f(a,c)\right\} \quad (41)$$

and

$$\int_0^1\int_0^1 ts\,\Phi(ta+(1-t)b,sc+(1-s)d)\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t$$
$$=\frac{1}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^{n+1}p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^{n+1}p_2^{m+1}}f\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^{n+1}p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^{n+1}p_2^{m+1}}f\left(\frac{q_1^{n}}{2p_1^{n}}a+\left(1-\frac{q_1^{n}}{2p_1^{n}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\right\}$$
$$=\frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^{n+1}q_2^{m+1}}{p_1^{n+1}p_2^{m+1}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\right.$$

$$
\begin{aligned}
&-\frac{1}{p_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m+1}}{p_1^np_2^{m+1}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m+1}}{p_2^{m+1}}c+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)d\right)\\
&-\frac{1}{q_1p_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^{n+1}q_2^{m}}{p_1^{n+1}p_2^{m}}f\left(\frac{q_1^{n+1}}{2p_1^{n+1}}a+\left(1-\frac{q_1^{n+1}}{2p_1^{n+1}}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&+\frac{1}{p_1p_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\Bigg\}\\
=&\frac{1}{(b-a)(d-c)}\Bigg\{\frac{1}{q_1q_2}\Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,c\right)+f(a,c)\Bigg]\\
&-\frac{1}{p_1q_2}\Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&-\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,c\right)\Bigg]\\
&-\frac{1}{q_1p_2}\Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\Bigg]\\
&+\frac{1}{p_1p_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\Bigg\}\\
=&\frac{1}{(b-a)(d-c)}\Bigg\{\frac{(p_1-q_1)(p_2-q_2)}{p_1p_2q_1q_2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^{m}}{p_1^np_2^{m}}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)\\
&-\frac{p_2-q_2}{q_1p_2q_2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}f\left(a,\frac{q_2^{m}}{p_2^{m}}c+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)d\right)-\frac{p_1-q_1}{q_1p_1q_2}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}f\left(\frac{q_1^n}{2p_1^n}a+\left(1-\frac{q_1^n}{2p_1^n}\right)b,c\right)-\frac{1}{q_1q_2}f(a,c)\Bigg\}\\
=&\frac{1}{(b-a)(d-c)}\Bigg\{\frac{1}{q_1q_2(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)b,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y\ {}^bd_{p_1,q_1}x\\
&-\frac{1}{q_1q_2(b-a)}\int_a^b f(p_1x+(1-p_1)b,c)\ {}^bd_{p_1,q_1}x-\frac{1}{q_1q_2(d-c)}\int_c^d f(a,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y\\
&+\frac{1}{q_1q_2}f(a,c)\Bigg\}.
\end{aligned}\tag{42}
$$

From (39)–(42), we obtain

$$
\begin{aligned}
I_4=&\frac{p_1p_2}{(b-a)(d-c)}\Bigg\{\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)b,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y\ {}^bd_{p_1,q_1}x\\
&-\frac{1}{(b-a)}\int_a^b f(p_1x+(1-p_1)b,c)\ {}^bd_{p_1,q_1}x-\frac{1}{(d-c)}\int_c^d f(a,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y\\
&+f(a,c)\}-\frac{5p_1}{6(b-a)(d-c)}\Bigg\{\frac{1}{(b-a)}\int_a^b f(p_1x+(1-p_1)b,d)\ {}^bd_{p_1,q_1}x\\
&-\frac{1}{(b-a)}\int_a^b f(p_1x+(1-p_1)b,c)\ {}^bd_{p_1,q_1}x-f(a,d)+f(a,c)\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{5p_2}{6(b-a)(d-c)}\left\{ \frac{1}{(d-c)} \int_c^d f(b, p_2 y + (1-p_2)d)\ {}^d d_{p_2,q_2} y \right. \\
&- \frac{1}{(d-c)} \int_c^d f(a, p_2 y + (1-p_2)d)\ {}^d d_{p_2,q_2} y - f(b,c) + f(a,c) \Big\} \\
&+ \frac{25}{36(b-a)(d-c)} \{ f(b,d) - f(a,d) - f(b,c) + f(a,c) \}.
\end{aligned} \tag{43}
$$

Replacing (34), (37), (38) and (43) in (33) and multiplying the resulting one with $(b-a)(d-c)$, the proof is completed. □

Remark 1. *Under the condition of Lemma 5 with $p_1 = p_2 = 1$, we have Lemma 3 in [30].*

Remark 2. *Under the condition of Lemma 5 with $p_1 = p_2 = 1$ and q_1, q_2 tend to 1, we have the following identity:*

$$
\begin{aligned}
&\frac{f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right)}{9} \\
&\quad + \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{36} \\
&\quad - \frac{1}{6(b-a)} \int_a^b f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d)\, dx \\
&\quad - \frac{1}{6(d-c)} \int_c^d f(a,y) + 4f\left(\frac{a+b}{2}, y\right) + f(b,y)\, dy \\
&\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dy\, dx \\
&= (b-a)(d-c) \int_0^1 \int_0^1 \Lambda(t)\Lambda(s) \frac{\partial^2 f(ta + (1-t)b), sc + (1-s)d)}{\partial t \partial s}\, ds\, dt,
\end{aligned}
$$

which was appeared in [33].

For brevity, we give some notations of integrals before giving the main theorems as follows:

$$
A_1(p,q) = \int_0^{1/2} t \left| pqt - \frac{1}{6} \right| {}_0 d_{p,q} t, \tag{44}
$$

$$
A_2(p,q) = \int_0^{1/2} (1-t) \left| pqt - \frac{1}{6} \right| {}_0 d_{p,q} t, \tag{45}
$$

$$
A_3(p,q) = \int_{1/2}^1 t \left| pqt - \frac{5}{6} \right| {}_0 d_{p,q} t, \tag{46}
$$

$$
A_4(p,q) = \int_{1/2}^1 (1-t) \left| pqt - \frac{5}{6} \right| {}_0 d_{p,q} t, \tag{47}
$$

$$
A_5(p,q) = \int_0^{1/2} \left| pqt - \frac{1}{6} \right| {}_0 d_{p,q} t, \tag{48}
$$

$$
A_6(p,q) = \int_{1/2}^1 \left| pqt - \frac{5}{6} \right| {}_0 d_{p,q} t. \tag{49}
$$

Now we give some new quantum estimates by using the lemmas given previously.

Theorem 2. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Phi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Phi(t,s)|$ is convex on $\triangle$ then the following identity holds:*

$$\begin{aligned}|^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)[&(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Phi(a,c)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Phi(b,c)| \\ &+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Phi(a,d)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Phi(b,d)|]. \end{aligned} \tag{50}$$

Proof. By taking modulus of the identity in Lemma 5, we have

$$\begin{aligned}&|^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\leq (b-a)(d-c)\int_0^1\int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|\ |\Phi(ta+(1-t)b, sc+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t. \end{aligned} \tag{51}$$

Now, using the convexity of $|\Phi(t,s)|$, then we get

$$\begin{aligned}&|^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\leq (b-a)(d-c)\int_0^1 \Lambda_{p_1,q_1}(t)\bigg\{\int_0^1 \Lambda_{p_2,q_2}(s)[\ s\ |\Phi(ta+(1-t)b,c)| \\ &\quad + (1-s)\ |\Phi(ta+(1-t)b,d)|]\ {}_0d_{p_2,q_2}s\bigg\}\ {}_0d_{p_1,q_1}t. \end{aligned} \tag{52}$$

Computing the integrals appearing in the right hand side of (52), we have

$$\begin{aligned}&\int_0^1 \Lambda_{p_2,q_2}(s)[\ s\ |\Phi(ta+(1-t)b,c)| + (1-s)\ |\Phi(ta+(1-t)b,d)|]\ {}_0d_{p_2,q_2}s \\ &= \int_0^{1/2} s\left|p_2q_2s-\frac{1}{6}\right|\ |\Phi(ta+(1-t)b,c)|\ {}_0d_{p_2,q_2}s \\ &\quad + \int_0^{1/2}(1-s)\left|p_2q_2s-\frac{1}{6}\right|\ |\Phi(ta+(1-t)b,d)|\ {}_0d_{p_2,q_2}s \\ &\quad + \int_{1/2}^1 s\left|p_2q_2s-\frac{5}{6}\right|\ |\Phi(ta+(1-t)b,c)|\ {}_0d_{p_2,q_2}s \\ &\quad + \int_{1/2}^1 (1-s)\left|p_2q_2s-\frac{5}{6}\right|\ |\Phi(ta+(1-t)b,d)|\ {}_0d_{p_2,q_2}s. \end{aligned}$$

From (44)–(47), we obtain

$$\begin{aligned}&\int_0^1 \Lambda_{p_2,q_2}(s)[\ s\ |\Phi(ta+(1-t)b,c)| + (1-s)\ |\Phi(ta+(1-t)b,d)|]\ {}_0d_{p_2,q_2}s \\ &= |\Phi(ta+(1-t)b,c)|[A_1(p_2,q_2)+A_3(p_2,q_2)] \\ &\quad + |\Phi(ta+(1-t)b,d)|[A_2(p_2,q_2)+A_4(p_2,q_2)]. \end{aligned}$$

Thus, we have

$$\begin{aligned}&|^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\leq (b-a)(d-c)\int_0^1 \Lambda_{p_1,q_1}(t)\{|\Phi(ta+(1-t)b,c)|[A_1(p_2,q_2)+A_3(p_2,q_2)] \\ &\quad + |\Phi(ta+(1-t)b,d)|[A_2(p_2,q_2)+A_4(p_2,q_2)]\}\ {}_0d_{p_1,q_1}t \\ &\leq (b-a)(d-c)\int_0^1 \Lambda_{p_1,q_1}(t)\{[t\ |\Phi(a,c)| + (1-t)\ |\Phi(b,c)|][A_1(p_2,q_2)+A_3(p_2,q_2)] \\ &\quad + [t\ |\Phi(a,d)| + (1-t)\ |\Phi(b,d)|][A_2(p_2,q_2)+A_4(p_2,q_2)]\}\ {}_0d_{p_1,q_1}t \\ &\leq (b-a)(d-c)\bigg\{[A_1(p_2,q_2)+A_3(p_2,q_2)]\bigg[|\Phi(a,c)|\int_0^{1/2} t\left|p_1q_1t-\frac{1}{6}\right|\ {}_0d_{p_1,q_1}t \end{aligned}$$

$$+ |\Phi(a,c)| \int_{1/2}^{1} t \left| p_1q_1t - \frac{5}{6} \right| {}_0d_{p_1,q_1}t + |\Phi(b,c)| \int_0^{1/2} (1-t) \left| p_1q_1t - \frac{1}{6} \right| {}_0d_{p_1,q_1}t$$
$$+ |\Phi(b,c)| \int_{1/2}^{1} (1-t) \left| p_1q_1t - \frac{5}{6} \right| {}_0d_{p_1,q_1}t \Big]$$
$$+ [A_2(p_2,q_2) + A_4(p_2,q_2)] \Big[|\Phi(a,d)| \int_0^{1/2} t \left| p_1q_1t - \frac{1}{6} \right| {}_0d_{p_1,q_1}t$$
$$+ |\Phi(a,d)| \int_{1/2}^{1} t \left| p_1q_1t - \frac{5}{6} \right| {}_0d_{p_1,q_1}t + |\Phi(b,d)| \int_0^{1/2} (1-t) \left| p_1q_1t - \frac{1}{6} \right| {}_0d_{p_1,q_1}t$$
$$+ |\Phi(b,d)| \int_{1/2}^{1} (1-t) \left| p_1q_1t - \frac{5}{6} \right| {}_0d_{p_1,q_1}t \Big] \Big\}.$$

From (44)–(47), we obtain

$$\begin{aligned} &|{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\quad \leq (b-a)(d-c)\{(A_1(p_1,q_1) + A_3(p_1,q_1))(A_1(p_2,q_2) + A_3(p_2,q_2))|\Phi(a,c)| \\ &\qquad + (A_2(p_1,q_1) + A_4(p_1,q_1))(A_1(p_2,q_2) + A_3(p_2,q_2))|\Phi(b,c)| \\ &\qquad + (A_1(p_1,q_1) + A_3(p_1,q_1))(A_2(p_2,q_2) + A_4(p_2,q_2))|\Phi(a,d)| \\ &\qquad + (A_2(p_1,q_1) + A_4(p_1,q_1))(A_2(p_2,q_2) + A_4(p_2,q_2))|\Phi(b,d)|\}. \end{aligned}$$

The proof is completed. □

Remark 3. *If $p_1 = p_2 = 1$ then Theorem 2 reduces to Theorem 7 in [30].*

Remark 4. *If $p_1 = p_2 = 1$ and q_1, q_2 tend to 1 then Theorem 2 reduces to Theorem 7 in [33].*

Theorem 3. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1,p_2,q_1,q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1,p_2,q_1,q_2)-derivative $\Phi(t,s)$ be continuous and (p_1,p_2,q_1,q_2)-integrable on $\triangle$. If $|\Phi(t,s)|^\alpha$ is convex on $\triangle$ for some $p > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then the following identity holds:*

$$\begin{aligned} |{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)&\left(\int_0^1 \int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t \right)^{1/\beta} \\ &\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}} |\Phi(a,c)|^\alpha + \frac{p_2+q_2-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}} |\Phi(a,d)|^\alpha \right. \\ &\left. + \frac{p_1+q_1-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}} |\Phi(b,c)|^\alpha + \frac{(p_1+q_1-1)(p_2+q_2-1)}{[2]_{p_1,q_1}[2]_{p_2,q_2}} |\Phi(b,d)|^\alpha \right)^{1/\alpha}. \end{aligned} \tag{53}$$

Proof. Applying Lemma 1 to the right hand side of (51), we have

$$\begin{aligned} &|{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\quad \leq (b-a)(d-c)\left(\int_0^1 \int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t \right)^{1/\beta} \\ &\qquad \left(\int_0^1 \int_0^1 |\Phi(ta+(1-t)b, sc+(1-s)d)|^\alpha \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t \right)^{1/\alpha}. \end{aligned} \tag{54}$$

By convexity of $|\Phi(t,s)|^\alpha$, (54) becomes

$$\begin{aligned} &|{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \\ &\quad \leq (b-a)(d-c)\left(\int_0^1 \int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t \right)^{1/\beta} \end{aligned}$$

$$\left(\int_0^1\int_0^1 ts\ |\Phi(a,c)|^\alpha + t(1-s)\ |\Phi(a,d)|^\alpha \right.$$
$$\left.(1-t)s\ |\Phi(b,c)|^\alpha + (1-t)(1-s)\ |\Phi(b,d)|^\alpha\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}. \quad (55)$$

Applying Lemma 3 for $a = 0$ and $\alpha = 1$, we obtain

$$\int_0^1\int_0^1 ts\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t = \left(\int_0^1 t\ {}_0d_{p_1,q_1}t\right)\left(\int_0^1 s\ {}_0d_{p_2,q_2}s\right) = \frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \quad (56)$$

$$\int_0^1\int_0^1 t(1-s)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t = \frac{p_2+q_2-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \quad (57)$$

$$\int_0^1\int_0^1 (1-t)s\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t = \frac{p_1+q_1-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}, \quad (58)$$

and

$$\int_0^1\int_0^1 (1-t)(1-s)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t = \frac{(p_2+q_2-1)(p_2+q_2-1)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}. \quad (59)$$

Substituting (56)–(59) in (55), the proof is completed. □

Remark 5. *If $p_1 = p_2 = 1$ then Theorem 3 reduces to Theorem 8 in [30].*

Theorem 4. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Phi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Phi(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha \geq 1$, then the following identity holds:*

$$\begin{aligned}|{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)\Big\{&[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}\\ &[A_1(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,c)|^\alpha + A_2(p_1,q_1)|\Phi(b,c)|^\alpha)\\ &+A_2(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,d)|^\alpha + A_2(p_1,q_1)|\Phi(b,d)|^\alpha)]^{1/\alpha}\\ &+[A_5(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha}\\ &[A_3(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,c)|^\alpha + A_2(p_1,q_1)|\Phi(b,c)|^\alpha)\\ &+A_4(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,d)|^\alpha + A_2(p_1,q_1)|\Phi(b,d)|^\alpha)]^{1/\alpha}\\ &+[A_6(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}\\ &[A_1(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,c)|^\alpha + A_4(p_1,q_1)|\Phi(b,c)|^\alpha)\\ &+A_2(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,d)|^\alpha + A_4(p_1,q_1)|\Phi(b,d)|^\alpha)]^{1/\alpha}\\ &+[A_6(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha}\\ &[A_3(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,c)|^\alpha + A_4(p_1,q_1)|\Phi(b,c)|^\alpha)\\ &+A_4(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,d)|^\alpha + A_4(p_1,q_1)|\Phi(b,d)|^\alpha)]^{1/\alpha}\Big\}. \quad (60)\end{aligned}$$

Proof. Applying Lemma 2 to the right hand side of of (51), we have

$$\begin{aligned}&|{}^{b,d}I_{p_1,p_2,q_1,q_2}(f)|\\ &\leq (b-a)(d-c)\left\{\left(\int_0^{1/2}\int_0^{1/2}\left|p_1q_1t - \frac{1}{6}\right|\left|p_2q_2s - \frac{1}{6}\right|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\right.\end{aligned}$$

$$\begin{aligned}
&\left(\int_0^{1/2}\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{1}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{\alpha}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}\\
&+\left(\int_0^{1/2}\int_{1/2}^{1}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{5}{6}\right|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\\
&\left(\int_0^{1/2}\int_{1/2}^{1}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{5}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{p}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}\\
&+\left(\int_{1/2}^{1}\int_0^{1/2}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{1}{6}\right|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\\
&\left(\int_{1/2}^{1}\int_0^{1/2}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{1}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{p}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}\\
&+\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{5}{6}\right|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\\
&\left.\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{5}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{p}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}\right\}. \qquad (61)
\end{aligned}$$

By convexity of $|\Phi(t,s)|^{\alpha}$, we have

$$\begin{aligned}
&\left(\int_0^{1/2}\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{1}{6}\right|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\\
&\left(\int_0^{1/2}\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{1}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{\alpha}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha}\\
&\leq\left[\left(\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|\ {}_0d_{p_1,q_1}t\right)\left(\int_0^{1/2}\left|p_2q_2s-\frac{1}{6}\right|\ {}_0d_{p_2,q_2}s\right)\right]^{1-1/\alpha}\\
&\quad\left[\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|\left\{\int_0^{1/2}\left|p_2q_2s-\frac{1}{6}\right|(s|\Phi(ta+(1-t)b,c)|^{\alpha}\right.\right.\\
&\quad\left.\left.+(1-s)|\Phi(ta+(1-t)b,d)|^{\alpha})\ {}_0d_{p_2,q_2}s\right\}\ {}_0d_{p_1,q_1}t\right]^{1/\alpha}\\
&=[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}\\
&\quad\left[A_1(p_2,q_2)\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right||\Phi(ta+(1-t)b,c)|^{\alpha}\ {}_0d_{p_1,q_1}t\right.\\
&\quad\left.+A_2(p_2,q_2)\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right||\Phi(ta+(1-t)b,d)|^{\alpha}\ {}_0d_{p_1,q_1}t\right]^{1/\alpha}\\
&\leq[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}\\
&\quad\left[A_1(p_2,q_2)\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|(t|\Phi(a,c)|^{\alpha}+(1-t)|\Phi(b,c)|^{\alpha})\ {}_0d_{p_1,q_1}t\right.\\
&\quad\left.+A_2(p_2,q_2)\int_0^{1/2}\left|p_1q_1t-\frac{1}{6}\right|(t|\Phi(a,d)|^{\alpha}+(1-t)|\Phi(b,d)|^{\alpha})\ {}_0d_{p_1,q_1}t\right]^{1/\alpha}\\
&=[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}\\
&\quad[A_1(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,c)|^{\alpha}+A_2(p_1,q_1)|\Phi(b,c)|^{\alpha})\\
&\quad+A_2(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,d)|^{\alpha}+A_2(p_1,q_1)|\Phi(b,d)|^{\alpha})]^{1/\alpha}, \qquad (62)
\end{aligned}$$

$$\left(\int_0^{1/2}\int_{1/2}^{1}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{5}{6}\right|\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1-1/\alpha}$$
$$\left(\int_0^{1/2}\int_{1/2}^{1}\left|p_1q_1t-\frac{1}{6}\right|\left|p_2q_2s-\frac{5}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{\alpha}\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\alpha}$$
$$\leq [A_5(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha}$$
$$[A_4(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,c)|^{\alpha}+A_2(p_1,q_1)|\Phi(b,c)|^{\alpha})$$
$$+A_4(p_2,q_2)(A_1(p_1,q_1)|\Phi(a,d)|^{\alpha}+A_2(p_1,q_1)|\Phi(b,d)|^{\alpha})]^{1/\alpha}, \tag{63}$$

$$\left(\int_{1/2}^{1}\int_0^{1/2}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{1}{6}\right|\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1-1/\alpha}$$
$$\left(\int_{1/2}^{1}\int_0^{1/2}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{1}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{\alpha}\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\alpha}$$
$$\leq [A_6(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha}$$
$$[A_1(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,c)|^{\alpha}+A_4(p_1,q_1)|\Phi(b,c)|^{\alpha})$$
$$+A_2(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,d)|^{\alpha}+A_4(p_1,q_1)|\Phi(b,d)|^{\alpha})]^{1/\alpha} \tag{64}$$

and

$$\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{5}{6}\right|\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1-1/\alpha}$$
$$\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|p_1q_1t-\frac{5}{6}\right|\left|p_2q_2s-\frac{5}{6}\right||\Phi(ta+(1-t)b,sc+(1-s)d)|^{\alpha}\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\alpha}$$
$$\leq [A_6(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha}$$
$$[A_3(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,c)|^{\alpha}+A_4(p_1,q_1)|\Phi(b,c)|^{\alpha})$$
$$+A_4(p_2,q_2)(A_3(p_1,q_1)|\Phi(a,d)|^{\alpha}+A_4(p_1,q_1)|\Phi(b,d)|^{\alpha})]^{1/\alpha}. \tag{65}$$

Substituting (62)–(65) in (61), the proof is completed. □

Remark 6. *If $p_1=p_2=1$ then Theorem 4 reduces to Theorem 9 in [30].*

4. Examples

In this section, we give some examples to show how the application of our theorems. Our theorems can be used to find the boundaries of (p_1,p_2,q_1,q_2)-integral of very complicated functions in the following examples.

Example 1. *Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be defined by $f(x,y)=\tan xy$, $p_1=p_2=\frac{3}{4}$ and $q_1=q_2=\frac{1}{4}$. Then f is twice partially $\left(\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)$-differentiable on $(0,1)\times(0,1)$, $\Phi(t,s)$ is continuous and $\left(\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)$-integrable on $[0,1]\times[0,1]$, and $|\Phi(t,s)|$ is convex on $[0,1]\times[0,1]$. Note that, it is very difficult to find $\left(\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)$-integral of f by using the definition directly. But we can find its boundaries. By applying Theorem 2, we have*

$$\begin{aligned}
|{}^{1,1}I_{\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}}(f)| \leq (1-0)(1-0)\Bigg[&\left(A_1\left(\frac{3}{4},\frac{1}{4}\right)+A_3\left(\frac{3}{4},\frac{1}{4}\right)\right)\left(A_1\left(\frac{3}{4},\frac{1}{4}\right)+A_3\left(\frac{3}{4},\frac{1}{4}\right)\right)|\Phi(0,0)| \\
&+\left(A_2\left(\frac{3}{4},\frac{1}{4}\right)+A_4\left(\frac{3}{4},\frac{1}{4}\right)\right)\left(A_1\left(\frac{3}{4},\frac{1}{4}\right)+A_3\left(\frac{3}{4},\frac{1}{4}\right)\right)|\Phi(1,0)| \\
&+\left(A_1\left(\frac{3}{4},\frac{1}{4}\right)+A_3\left(\frac{3}{4},\frac{1}{4}\right)\right)\left(A_2\left(\frac{3}{4},\frac{1}{4}\right)+A_4\left(\frac{3}{4},\frac{1}{4}\right)\right)|\Phi(0,1)| \\
&+\left(A_2\left(\frac{3}{4},\frac{1}{4}\right)+A_4\left(\frac{3}{4},\frac{1}{4}\right)\right)\left(A_2\left(\frac{3}{4},\frac{1}{4}\right)+A_4\left(\frac{3}{4},\frac{1}{4}\right)\right)|\Phi(1,1)|\Bigg].
\end{aligned}$$

Since

$$\begin{aligned}
A_1\left(\frac{3}{4},\frac{1}{4}\right) &= \int_0^{1/2} t\left(\frac{1}{6}-\frac{3t}{6}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{1}{6}\cdot\frac{1}{8}-\frac{3}{16}\cdot\frac{2}{13} = -\frac{5}{624}, \\
A_2\left(\frac{3}{4},\frac{1}{4}\right) &= \int_0^{1/2} (1-t)\left(\frac{1}{6}-\frac{3t}{6}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{1}{6}\cdot\frac{1}{2}-\frac{34}{96}\cdot\frac{1}{8}+\frac{3}{16}\cdot\frac{2}{13} = -\frac{113}{1664}, \\
A_3\left(\frac{3}{4},\frac{1}{4}\right) &= \int_{1/2}^{1} t\left(\frac{5}{6}-\frac{3t}{6}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{5}{6}\cdot\frac{7}{8}-\frac{3}{16}\cdot\frac{14}{13} = \frac{329}{624}, \\
A_4\left(\frac{3}{4},\frac{1}{4}\right) &= \int_{1/2}^{1} (1-t)\left(\frac{5}{6}-\frac{3t}{6}\right)\,{}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{5}{6}\cdot\frac{1}{2}-\frac{98}{96}\cdot\frac{7}{8}+\frac{3}{16}\cdot\frac{14}{13} = -\frac{457}{1664}, \\
\Phi(0,0) &= 4\left(\tan\frac{3}{4}\cdot\frac{3}{4}-\tan\frac{1}{4}\cdot\frac{3}{4}-\tan\frac{3}{4}\cdot\frac{1}{4}+\tan\frac{1}{4}\cdot\frac{1}{4}\right)\approx 1.25425, \\
\Phi(1,0) &= \Phi(0,1)=\Phi(1,1)=0,
\end{aligned}$$

we derive

$$|{}^{1,1}I_{\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}}(f)| \leq \left(-\frac{5}{624}+\frac{329}{624}\right)\left(-\frac{5}{624}+\frac{329}{624}\right)1.25425 = 0.338147.$$

Example 2. *From Example 1, we try to apply Theorem 3 with $\alpha=\beta=2$. We get*

$$\begin{aligned}
|{}^{1,1}I_{\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}}(f)| \leq (1-0)(1-0)&\left(\int_0^1\int_0^1 |\Lambda_{\frac{3}{4},\frac{1}{4}}(t)\Lambda_{\frac{3}{4},\frac{1}{4}}(s)|^2\,{}_0d_{\frac{3}{4},\frac{1}{4}}s\,{}_0d_{\frac{3}{4},\frac{1}{4}}t\right)^{1/2} \\
&\left(\frac{1}{[2]_{\frac{3}{4},\frac{1}{4}}[2]_{\frac{3}{4},\frac{1}{4}}}|\Phi(0,0)|^2+\frac{\frac{3}{4}+\frac{1}{4}-1}{[2]_{\frac{3}{4},\frac{1}{4}}[2]_{\frac{3}{4},\frac{1}{4}}}|\Phi(0,1)|^2\right. \\
&\left.+\frac{\frac{3}{4}+\frac{1}{4}-1}{[2]_{\frac{3}{4},\frac{1}{4}}[2]_{\frac{3}{4},\frac{1}{4}}}|\Phi(1,0)|^2+\frac{(\frac{3}{4}+\frac{1}{4}-1)(\frac{3}{4}+\frac{1}{4}-1)}{[2]_{\frac{3}{4},\frac{1}{4}}[2]_{\frac{3}{4},\frac{1}{4}}}|\Phi(1,1)|^2\right)^{1/2}.
\end{aligned} \tag{66}$$

Since

$$\begin{aligned}
&\int_0^1\int_0^1 |\Lambda_{\frac{3}{4},\frac{1}{4}}(t)\Lambda_{\frac{3}{4},\frac{1}{4}}(s)|^2\,{}_0d_{\frac{3}{4},\frac{1}{4}}s\,{}_0d_{\frac{3}{4},\frac{1}{4}}t = -0.05942297\ldots, \\
&[2]_{\frac{3}{4},\frac{1}{4}} = \frac{3}{4}+\frac{1}{4} = 1, \\
&\Phi(0,0) = 4\left(\tan\frac{3}{4}\cdot\frac{3}{4}-\tan\frac{1}{4}\cdot\frac{3}{4}-\tan\frac{3}{4}\cdot\frac{1}{4}+\tan\frac{1}{4}\cdot\frac{1}{4}\right)\approx 1.25425, \\
&\Phi(1,0) = \Phi(0,1)=\Phi(1,1)=0,
\end{aligned}$$

the right hand-side of (66) is

$$(-0.05942297\ldots)^{1/2}(1.25425)^{1/2}\notin\mathbb{R}.$$

Therefore, Theorem 3 with $\alpha = \beta = 2$ is not applicable for this $\left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right)$-integral of f.

Example 3. *From Example 1, we try to apply Theorem 4 with $\alpha = \frac{3}{2}$. We get*

$$
\begin{aligned}
|^{1,1}I_{\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}}(f)| \leq (1-0)(1-0)\Bigg\{ & \left[A_5\left(\frac{3}{4},\frac{1}{4}\right)A_5\left(\frac{3}{4},\frac{1}{4}\right)\right]^{1-2/3} \\
& \left[A_1\left(\frac{3}{4},\frac{1}{4}\right)A_1\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,0)|^{3/2} + A_2\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,0)|^{3/2}\right. \\
& \left.+A_2\left(\frac{3}{4},\frac{1}{4}\right)A_1\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,1)|^{3/2} + A_2\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,1)|^{3/2}\right]^{2/3} \\
& + \left[A_5\left(\frac{3}{4},\frac{1}{4}\right)A_6\left(\frac{3}{4},\frac{1}{4}\right)\right]^{1-2/3} \\
& \left[A_3\left(\frac{3}{4},\frac{1}{4}\right)A_1\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,0)|^{3/2} + A_2\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,0)|^{3/2}\right. \\
& \left.+A_4\left(\frac{3}{4},\frac{1}{4}\right)A_1\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,1)|^{3/2} + A_2\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,1)|^{3/2}\right]^{2/3} \\
& + \left[A_6\left(\frac{3}{4},\frac{1}{4}\right)A_5\left(\frac{3}{4},\frac{1}{4}\right)\right]^{1-2/3} \\
& \left[A_1\left(\frac{3}{4},\frac{1}{4}\right)A_3\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,0)|^{3/2} + A_4\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,0)|^{3/2}\right. \\
& \left.+A_2\left(\frac{3}{4},\frac{1}{4}\right)A_3\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,1)|^{3/2} + A_4\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,1)|^{3/2}\right]^{1/p} \\
& + [A_6(p_1,q_1)A_6(p_2,q_2)]^{1-2/3} \\
& \left[A_3\left(\frac{3}{4},\frac{1}{4}\right)A_3\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,0)|^{3/2} + A_4\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,0)|^{3/2})\right. \\
& \left.+A_4\left(\frac{3}{4},\frac{1}{4}\right)A_3\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(0,1)|^{3/2} + A_4\left(\frac{3}{4},\frac{1}{4}\right)|\Phi(1,1)|^{3/2}\right]^{2/3}\Bigg\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
A_1\left(\frac{3}{4},\frac{1}{4}\right) &= \int_0^{1/2} t\left(\frac{1}{6}-\frac{3t}{6}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{1}{6}\cdot\frac{1}{8} - \frac{3}{16}\cdot\frac{2}{13} = -\frac{5}{624}, \\
A_2\left(\frac{3}{4},\frac{1}{4}\right) &= \int_0^{1/2} (1-t)\left(\frac{1}{6}-\frac{3t}{6}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{1}{6}\cdot\frac{1}{2} - \frac{34}{96}\cdot\frac{1}{8} + \frac{3}{16}\cdot\frac{2}{13} = -\frac{113}{1664}, \\
A_3\left(\frac{3}{4},\frac{1}{4}\right) &= \int_{1/2}^{1} t\left(\frac{5}{6}-\frac{3t}{6}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{5}{6}\cdot\frac{7}{8} - \frac{3}{16}\cdot\frac{14}{13} = \frac{329}{624}, \\
A_4\left(\frac{3}{4},\frac{1}{4}\right) &= \int_{1/2}^{1} (1-t)\left(\frac{5}{6}-\frac{3t}{6}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{5}{6}\cdot\frac{1}{2} - \frac{98}{96}\cdot\frac{7}{8} + \frac{3}{16}\cdot\frac{14}{13} = -\frac{457}{1664}, \\
A_5\left(\frac{3}{4},\frac{1}{4}\right) &= \int_0^{1/2} \left(\frac{1}{6}-\frac{3t}{16}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{1}{6}\cdot\frac{1}{2} - \frac{3}{16}\cdot\frac{1}{8} = \frac{23}{384}, \\
A_6\left(\frac{3}{4},\frac{1}{4}\right) &= \int_{1/2}^{1} (1-t)\left(\frac{5}{6}-\frac{3t}{16}\right)\, {}_0d_{\frac{3}{4},\frac{1}{4}}t = \frac{5}{6}\cdot 1 - \frac{13}{16}\cdot\frac{1}{2} - \frac{5}{6}\cdot\frac{1}{2} + \frac{3}{16}\cdot\frac{1}{8} = \frac{133}{384}, \\
\Phi(0,0) &= 4\left(\tan\frac{3}{4}\cdot\frac{3}{4} - \tan\frac{1}{4}\cdot\frac{3}{4} - \tan\frac{3}{4}\cdot\frac{1}{4} + \tan\frac{1}{4}\cdot\frac{1}{4}\right) \approx 1.25425, \\
\Phi(1,0) &= \Phi(0,1) = \Phi(1,1) = 0,
\end{aligned}
$$

we derive

$$
\begin{aligned}
|^{1,1}I_{\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}}(f)| \leq & \left[\left(\frac{23}{384}\right)\left(\frac{23}{384}\right)\right]^{1/3}\left[\left(\frac{-5}{624}\right)\left(\frac{-5}{624}\right)1.25425^{3/2}\right]^{2/3} \\
& + 2\left[\left(\frac{23}{384}\right)\left(\frac{133}{384}\right)\right]^{1/3}\left[\left(\frac{-5}{624}\right)\left(\frac{329}{624}\right)1.25425^{3/2}\right]^{2/3} \\
& + \left[\left(\frac{133}{384}\right)\left(\frac{133}{384}\right)\right]^{1/3}\left[\left(\frac{329}{624}\right)\left(\frac{329}{624}\right)1.25425^{3/2}\right]^{2/3} \\
& = 0.208945.
\end{aligned}
$$

5. Additional Results

In this section, we present some results without proof because the proofs are similar to the ones given in last section.

Lemma 6. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$. If the partial (p_1, p_2, q_1, q_2)-derivative $\Theta(t,s)$ is continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$, then the following identity holds:*

$$
\begin{aligned}
&{}_a^d I_{p_1,p_2,q_1,q_2}(f) \\
&= (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Theta(tb+(1-t)a, sc+(1-s)d)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t,
\end{aligned}
\tag{67}
$$

where

$$
\begin{aligned}
&{}_a^d I_{p_1,p_2,q_1,q_2}(f) \\
&= \frac{1}{9}\left\{(6p_2-5)f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)+(6p_1-5)f\left(b,\frac{c+d}{2}\right)\right. \\
&\left.+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}+\frac{1}{36}\{(36p_1p_2-30p_1-30p_2+25)f(b,c) \\
&+(30p_1-4(6p_1-5)-25)f(b,d)+(30p_2-4(6p_2-5)-25)f(a,c)+f(a,d)\} \\
&-\frac{p_1}{6(b-a)}\int_a^b (6p_2-5)f(p_1x+(1-p_1)a,c)+4f\left(p_1x+(1-p_1)a,\frac{c+d}{2}\right) \\
&+f(p_1x+(1-p_1)a,d)\ {}_ad_{p_1,q_1}x-\frac{p_2}{6(d-c)}\int_c^d f(b,p_2y+(1-p_2)d) \\
&+4f\left(\frac{a+b}{2},p_2y+(1-p_2)d\right)+(6p_1-5)f(a,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y \\
&+\frac{p_1p_2}{(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)a,p_2y+(1-p_2)d)\ {}^dd_{p_2,q_2}y\ {}_ad_{p_1,q_1}x.
\end{aligned}
$$

Remark 7. *If $p_1 = p_2 = 1$ then Lemma 6 reduces to Lemma 4 in [30].*

We use Lemma 6 to derive Theorems 5–7 as follows:

Theorem 5. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Theta(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Theta(t,s)|$ is convex on $\triangle$ then the following identity holds:*

$$\begin{aligned}|{}_a^d I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)[&(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Theta(b,c)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Theta(a,c)| \\ &+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Theta(b,d)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Theta(a,d)|].\end{aligned} \tag{68}$$

Remark 8. *If $p_1 = p_2 = 1$ then Theorem 5 reduces to Theorem 10 in [30].*

Theorem 6. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Theta(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Theta(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then the following identity holds:*

$$\begin{aligned}|{}_a^d I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)&\left(\int_0^1\int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t\right)^{1/\beta} \\ &\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Theta(b,c)|^\alpha + \frac{p_2+q_2-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Theta(b,d)|^\alpha\right. \\ &\left.+\frac{p_1+q_1-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Theta(a,c)|^\alpha + \frac{(p_1+q_1-1)(p_2+q_2-1)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Theta(a,d)|^\alpha\right)^{1/\alpha}.\end{aligned} \tag{69}$$

Remark 9. *If $p_1 = p_2 = 1$ then Theorem 6 reduces to Theorem 11 in [30].*

Theorem 7. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Theta(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Theta(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha \geq 1$, then the following identity holds:*

$$\begin{aligned}|{}_a^d I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)\Big\{&[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\ &[A_1(p_2,q_2)(A_1(p_1,q_1)|\Theta(b,c)|^\alpha + A_2(p_1,q_1)|\Theta(a,c)|^\alpha) \\ &+A_2(p_2,q_2)(A_1(p_1,q_1)|\Theta(b,d)|^\alpha + A_2(p_1,q_1)|\Theta(a,d)|^\alpha)]^{1/\alpha} \\ &+[A_5(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\ &[A_3(p_2,q_2)(A_1(p_1,q_1)|\Theta(b,c)|^\alpha + A_2(p_1,q_1)|\Theta(a,c)|^\alpha) \\ &+A_4(p_2,q_2)(A_1(p_1,q_1)|\Theta(b,d)|^\alpha + A_2(p_1,q_1)|\Theta(a,d)|^\alpha)]^{1/\alpha} \\ &+[A_6(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\ &[A_1(p_2,q_2)(A_3(p_1,q_1)|\Theta(b,c)|^\alpha + A_4(p_1,q_1)|\Theta(a,c)|^\alpha) \\ &+A_2(p_2,q_2)(A_3(p_1,q_1)|\Theta(b,d)|^\alpha + A_4(p_1,q_1)|\Theta(a,d)|^\alpha)]^{1/\alpha} \\ &+[A_6(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\ &[A_3(p_2,q_2)(A_3(p_1,q_1)|\Theta(b,c)|^\alpha + A_4(p_1,q_1)|\Theta(a,c)|^\alpha) \\ &+A_4(p_2,q_2)(A_3(p_1,q_1)|\Theta(b,d)|^\alpha + A_4(p_1,q_1)|\Theta(a,d)|^\alpha)]^{1/\alpha}\Big\}.\end{aligned} \tag{70}$$

Remark 10. *If $p_1 = p_2 = 1$ then Theorem 7 reduces to Theorem 12 in [30].*

Lemma 7. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\psi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. Then the following identity holds:*

$$\begin{aligned}&{}_c^b I_{p_1,p_2,q_1,q_2}(f) \\ &= (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\psi(ta+(1-t)b, sd+(1-s)c) \, {}_0d_{p_2,q_2}s \, {}_0d_{p_1,q_1}t,\end{aligned} \tag{71}$$

where

$$
\begin{aligned}
&{}_c^b I_{p_1,p_2,q_1,q_2}(f) \\
&= \frac{1}{9}\left\{ f\left(\frac{a+b}{2},c\right) + (6p_2-5)f\left(\frac{a+b}{2},d\right) + (6p_1-5)f\left(a,\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right)\right. \\
&\left. +4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\} + \frac{1}{36}\{(36p_1p_2 - 30p_1 - 30p_2 + 25)f(a,d) \\
&+(30p_1 - 4(6p_1-5) - 25)f(a,c) + (30p_2 - 4(6p_2-5) - 25)f(b,d) + f(b,c)\} \\
&- \frac{p_1}{6(b-a)}\int_a^b (6p_2-5)f(p_1x+(1-p_1)b,d) + 4f\left(p_1x+(1-p_1)b,\frac{c+d}{2}\right) \\
&+f(p_1x+(1-p_1)b,c)\ {}^b d_{p_1,q_1}x - \frac{p_2}{6(d-c)}\int_c^d f(a,p_2y+(1-p_2)c) \\
&+4f\left(\frac{a+b}{2},p_2y+(1-p_2)c\right) + (6p_1-5)f(b,p_2y+(1-p_2)c)\ {}_c d_{p_2,q_2}y \\
&+ \frac{p_1p_2}{(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)b,p_2y+(1-p_2)c)\ {}_c d_{p_2,q_2}y\ {}^b d_{p_1,q_1}x.
\end{aligned}
$$

Remark 11. *If $p_1 = p_2 = 1$ then Lemma 7 reduces to Lemma 5 in [30].*

We use Lemma 7 to derive Theorems 8–10 as follows:

Theorem 8. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\psi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\psi(t,s)|$ is convex on $\triangle$ then the following identity holds:*

$$
\begin{aligned}
|{}_c^b I_{p_1,p_2,q_1,q_2}(f)| \le (b-a)(d-c)[&(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\psi(a,d)| \\
&+ (A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\psi(b,d)| \\
&+ (A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\psi(a,c)| \\
&+ (A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\psi(b,c)|]. \qquad (72)
\end{aligned}
$$

Remark 12. *If $p_1 = p_2 = 1$ then Theorem 8 reduces to Theorem 13 in [30].*

Theorem 9. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\psi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\psi(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then the following identity holds:*

$$
\begin{aligned}
|{}_c^b I_{p_1,p_2,q_1,q_2}(f)| \le (b-a)(d-c)&\left(\int_0^1\int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\beta} \\
&\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\psi(a,d)|^\alpha + \frac{p_2+q_2-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\psi(a,c)|^\alpha\right. \\
&\left.+\frac{p_1+q_1-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\psi(b,d)|^\alpha + \frac{(p_1+q_1-1)(p_2+q_2-1)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\psi(b,c)|^\alpha\right)^{1/\alpha}. \qquad (73)
\end{aligned}
$$

Remark 13. *If $p_1 = p_2 = 1$ then Theorem 9 reduces to Theorem 14 in [30].*

Theorem 10. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\psi(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\psi(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha \ge 1$, then the following identity holds:*

$$
\begin{aligned}
|{}_c^b I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)\Big\{ & [A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\
& [A_1(p_2,q_2)(A_1(p_1,q_1)|\psi(a,d)|^\alpha + A_2(p_1,q_1)|\psi(b,d)|^\alpha) \\
& + A_2(p_2,q_2)(A_1(p_1,q_1)|\psi(a,c)|^\alpha + A_2(p_1,q_1)|\psi(b,c)|^\alpha)]^{1/\alpha} \\
& + [A_5(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\
& [A_3(p_2,q_2)(A_1(p_1,q_1)|\psi(a,d)|^\alpha + A_2(p_1,q_1)|\psi(b,d)|^\alpha) \\
& + A_4(p_2,q_2)(A_1(p_1,q_1)|\psi(a,c)|^\alpha + A_2(p_1,q_1)|\psi(b,c)|^\alpha)]^{1/\alpha} \\
& + [A_6(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\
& [A_1(p_2,q_2)(A_3(p_1,q_1)|\psi(a,d)|^\alpha + A_4(p_1,q_1)|\psi(b,d)|^\alpha) \\
& + A_2(p_2,q_2)(A_3(p_1,q_1)|\psi(a,c)|^\alpha + A_4(p_1,q_1)|\psi(b,c)|^\alpha)]^{1/\alpha} \\
& + [A_6(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\
& [A_3(p_2,q_2)(A_3(p_1,q_1)|\psi(a,d)|^\alpha + A_4(p_1,q_1)|\psi(b,d)|^\alpha) \\
& + A_4(p_2,q_2)(A_3(p_1,q_1)|\psi(a,c)|^\alpha + A_4(p_1,q_1)|\psi(b,c)|^\alpha)]^{1/\alpha}\Big\}. \qquad (74)
\end{aligned}
$$

Remark 14. *If $p_1 = p_2 = 1$ then Theorem 10 reduces to Theorem 15 in [30].*

Lemma 8. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$. If the partial (p_1, p_2, q_1, q_2)-derivative $\Omega(t,s)$ is continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$ then the following identity holds:*

$$
\begin{aligned}
& {}_{a,c}I_{p_1,p_2,q_1,q_2}(f) \\
&= (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)\Omega(tb+(1-t)a, sd+(1-s)c)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t, \qquad (75)
\end{aligned}
$$

where

$$
\begin{aligned}
& {}_{a,c}I_{p_1,p_2,q_1,q_2}(f) \\
&= \frac{1}{9}\Big\{f\Big(\frac{a+b}{2},c\Big) + (6p_2-5)f\Big(\frac{a+b}{2},d\Big) + f\Big(a,\frac{c+d}{2}\Big) + (6p_1-5)f\Big(b,\frac{c+d}{2}\Big) \\
&\quad + 4f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)\Big\} + \frac{1}{36}\{(36p_1p_2 - 30p_1 - 30p_2 + 25)f(b,d) \\
&\quad + (30p_1 - 4(6p_1-5) - 25)f(b,c) + (30p_2 - 4(6p_2-5) - 25)f(a,d) + f(a,c)\} \\
&\quad - \frac{p_1}{6(b-a)}\int_a^b (6p_2-5)f(p_1x+(1-p_1)a,d) + 4f\Big(p_1x+(1-p_1)a,\frac{c+d}{2}\Big) \\
&\quad + f(p_1x+(1-p_1)a,c)\ {}_ad_{p_1,q_1}x - \frac{p_2}{6(d-c)}\int_c^d f(b,p_2y+(1-p_2)c) \\
&\quad + 4f\Big(\frac{a+b}{2},p_2y+(1-p_2)c\Big) + (6p_1-5)f(a,p_2y+(1-p_2)c)\ {}_cd_{p_2,q_2}y \\
&\quad + \frac{p_1p_2}{(b-a)(d-c)}\int_a^b\int_c^d f(p_1x+(1-p_1)a, p_2y+(1-p_2)c)\ {}_cd_{p_2,q_2}y\ {}_ad_{p_1,q_1}x.
\end{aligned}
$$

Remark 15. *If $p_1 = p_2 = 1$ then Lemma 8 reduces to Lemma 4 in [29].*

We use Lemma 8 to derive Theorems 11–13 as follows:

Theorem 11. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Omega(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Omega(t,s)|$ is convex on $\triangle$ then the following identity holds:*

$$\begin{aligned}|{}_{a,c}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)[&(A_1(p_1,q_1)+A_3(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Omega(b,d)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_1(p_2,q_2)+A_3(p_2,q_2))|\Omega(a,d)| \\ &+(A_1(p_1,q_1)+A_3(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Omega(b,c)| \\ &+(A_2(p_1,q_1)+A_4(p_1,q_1))(A_2(p_2,q_2)+A_4(p_2,q_2))|\Omega(a,c)|]. \end{aligned} \tag{76}$$

Remark 16. *If $p_1 = p_2 = 1$ then Theorem 11 reduces to Theorem 8 in [29].*

Theorem 12. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Omega(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Omega(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then the following identity holds:*

$$\begin{aligned}|{}_{a,c}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)&\left(\int_0^1\int_0^1 |\Lambda_{p_1,q_1}(t)\Lambda_{p_2,q_2}(s)|^\beta \,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\beta} \\ &\left(\frac{1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Omega(b,d)|^\alpha + \frac{p_2+q_2-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Omega(b,c)|^\alpha\right. \\ &\left.+\frac{p_1+q_1-1}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Omega(a,d)|^\alpha + \frac{(p_1+q_1-1)(p_2+q_2-1)}{[2]_{p_1,q_1}[2]_{p_2,q_2}}|\Omega(a,c)|^\alpha\right)^{1/\alpha}. \end{aligned} \tag{77}$$

Remark 17. *If $p_1 = p_2 = 1$ then Theorem 12 reduces to Theorem 9 in [29].*

Theorem 13. *Let $f : \triangle \to \mathbb{R}$ be a twice partially (p_1, p_2, q_1, q_2)-differentiable function on $\triangle^\circ$ and the partial (p_1, p_2, q_1, q_2)-derivative $\Omega(t,s)$ be continuous and (p_1, p_2, q_1, q_2)-integrable on $\triangle$. If $|\Omega(t,s)|^\alpha$ is convex on $\triangle$ for some $\alpha \geq 1$, then the following identity holds:*

$$\begin{aligned}|{}_{a,c}I_{p_1,p_2,q_1,q_2}(f)| \leq (b-a)(d-c)\Big\{&[A_5(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\ &[A_1(p_2,q_2)(A_1(p_1,q_1)|\Omega(b,d)|^\alpha + A_2(p_1,q_1)|\Omega(a,d)|^\alpha) \\ &+A_2(p_2,q_2)(A_1(p_1,q_1)|\Omega(b,c)|^\alpha + A_2(p_1,q_1)|\Omega(a,c)|^\alpha)]^{1/\alpha} \\ &+[A_5(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\ &[A_3(p_2,q_2)(A_1(p_1,q_1)|\Omega(b,d)|^\alpha + A_2(p_1,q_1)|\Omega(a,d)|^\alpha) \\ &+A_4(p_2,q_2)(A_1(p_1,q_1)|\Omega(b,c)|^\alpha + A_2(p_1,q_1)|\Omega(a,c)|^\alpha)]^{1/\alpha} \\ &+[A_6(p_1,q_1)A_5(p_2,q_2)]^{1-1/\alpha} \\ &[A_1(p_2,q_2)(A_3(p_1,q_1)|\Omega(b,d)|^\alpha + A_4(p_1,q_1)|\Omega(a,d)|^\alpha) \\ &+A_2(p_2,q_2)(A_3(p_1,q_1)|\Omega(b,c)|^\alpha + A_4(p_1,q_1)|\Omega(a,c)|^\alpha)]^{1/\alpha} \\ &+[A_6(p_1,q_1)A_6(p_2,q_2)]^{1-1/\alpha} \\ &[A_3(p_2,q_2)(A_3(p_1,q_1)|\Omega(b,d)|^\alpha + A_4(p_1,q_1)|\Omega(a,d)|^\alpha) \\ &+A_4(p_2,q_2)(A_3(p_1,q_1)|\Omega(b,c)|^\alpha + A_4(p_1,q_1)|\Omega(a,c)|^\alpha)]^{1/\alpha}\Big\}. \end{aligned} \tag{78}$$

Remark 18. *If $p_1 = p_2 = 1$ then Theorem 13 reduces to Theorem 10 in [29].*

6. Conclusions

We proved some new Simpson's type inequalities for coordinated convex functions by using (p,q)-calculus. We first proved the key lemma and then we used such the lemma and property of convexity, Hölder's inequality, power mean inequality to prove our main theorems. Some previously published results of other researchers are deduced as special cases of our results for p_1, p_2 are unity and q_1, q_2 tend to one. This means that if $p_1 = p_2 = 1$ then our theorems are reduced to the work of M. A. Ali et al. [30]; moreover, if

$p_1 = p_2 = 1,\ q_1, q_2 \to 1$ then our work is reduced to the work of M. E. Ozdemir and A. O. Akdemir [33].

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