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# Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second $q^b$ -derivatives

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## Abstract

In this paper, we obtain Hermite–Hadamard-type inequalities of convex functions by applying the notion of  $q^b$ -integral. We prove some new inequalities related with right-hand sides of  $q^b$ -Hermite–Hadamard inequalities for differentiable functions with convex absolute values of second derivatives. The results presented in this paper are a unification and generalization of the comparable results in the literature on Hermite–Hadamard inequalities.

**Keywords:** Hermite–Hadamard inequality;  $q$ -integral; Quantum calculus; Convex function

## 1 Introduction

The Hermite–Hadamard inequality introduced by Hermite and Hadamard (see also [1] and [2, p. 137]) is one of the most well-known inequalities in the theory of convex functional analysis. It has an interesting geometrical interpretation with several applications.

These inequalities state that if  $f : I \rightarrow \mathbb{R}$  is a convex function on an interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed manner if  $f$  is a concave function. Note that the Hermite–Hadamard inequalities may be viewed as a refinement of the concept of convexity and follows from Jensen's inequality. Hermite–Hadamard inequalities for convex functions have received much attention in the recent years, and, consequently, a remarkable variety of refinements and generalizations have been obtained.

Many well-known integral inequalities such as the Hölder, Hermite–Hadamard, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss–Chebyshev, and other integral inequalities have been studied in the setup of  $q$ -calculus using the concept of classical convexity. For more results in this direction, we refer to [3–18].

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The purpose of this paper is to study Hermite–Hadamard-like inequalities for convex functions by applying the new concept of  $q^b$ -integral. We also discuss the relation of our results with comparable results existing in the literature.

The organization of this paper is as follows. In Sect. 2, we give a brief description of the concepts of  $q$ -calculus and some related works in this direction. In Sect. 3, we present the Hermite-Hadamard-type inequalities for the  $q^b$ -integrals. We also study the relation between the results presented herein and comparable results in the literature. Section 4 contains some conclusions and further directions for the future research. We believe that the study initiated in this paper may inspire new research in this area.

## 2 Preliminaries of $q$ -calculus and some inequalities

In this section, we first present some known definitions and related inequalities in  $q$ -calculus. Set the following notation (see [19]):

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}, \quad q \in (0, 1).$$

Jackson [20] defined the  $q$ -Jackson integral of a given function  $f$  from 0 to  $b$  as follows:

$$\int_0^b f(x) d_q x = (1 - q)b \sum_{n=0}^{\infty} q^n f(bq^n), \quad \text{where } 0 < q < 1, \tag{2.1}$$

provided that the sum converges absolutely.

Jackson [20] defined the  $q$ -Jackson integral of a given function over the interval  $[a, b]$  as follows:

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

**Definition 1** ([21]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the  $q_a$ -derivative of  $f$  at  $x \in [a, b]$  is identified as

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, \quad x \neq a. \tag{2.2}$$

Since  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function, we can define

$${}_a D_q f(a) = \lim_{x \rightarrow a} {}_a D_q f(x).$$

The function  $f$  is said to be  $q_a$ -differentiable on  $[a, b]$  if  ${}_a D_q f(x)$  exists for all  $x \in [a, b]$ . If we take  $a = 0$  in (2.2), then we have  ${}_0 D_q f(x) = D_q f(x)$ , where  $D_q f(x)$  is the  $q$ -derivative of  $f$  at  $x \in [0, b]$  (see [19]) given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0.$$

**Definition 2** ([22]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the  $q^b$ -derivative of  $f$  at  $x \in [a, b]$  is given by

$${}^b D_q f(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

**Definition 3** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the second  $q^b$ -derivative of  $f$  at  $x \in [a, b]$  is given by

$$\begin{aligned} {}^bD_q^2 f(x) &= {}^bD_q({}^bD_q f(x)) \\ &= \frac{f(q^2ta + (1 - tq^2)b) - (1 + q)f(qta + (1 - qt)b) + qf(ta + (1 - t)b)}{(1 - q)^2q(b - a)^2t^2}. \end{aligned}$$

**Definition 4** ([21]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the  $q_a$ -definite integral on  $[a, b]$  is defined by

$$\begin{aligned} \int_a^b f(x) {}_a d_q x &= (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n b + (1 - q^n)a) \\ &= (b - a) \int_0^1 f((1 - t)a + tb) d_q t. \end{aligned}$$

Alp et al. [3] proved the following  $q_a$ -Hermite–Hadamard inequalities for convex functions in the setting of quantum calculus.

**Theorem 1** If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex differentiable function on  $[a, b]$  and  $0 < q < 1$ , then we have

$$f\left(\frac{qa + b}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a) + f(b)}{1 + q}. \tag{2.3}$$

In [3] and [23] authors established some bounds for the left- and right-hand sides of inequality (2.3).

On the other hand, Bermudo et al. [22] gave the following definition and obtained the related Hermite–Hadamard-type inequalities.

**Definition 5** ([22]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the  $q^b$ -definite integral on  $[a, b]$  is given by

$$\begin{aligned} \int_a^b f(x) {}^b d_q x &= (1 - q)(b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n)b) \\ &= (b - a) \int_0^1 f(ta + (1 - t)b) d_q t. \end{aligned}$$

**Theorem 2** ([22]) If  $f : [a, b] \rightarrow \mathbb{R}$  is a convex differentiable function on  $[a, b]$  and  $0 < q < 1$ , then we have the following  $q$ -Hermite–Hadamard inequalities:

$$f\left(\frac{a + qb}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a) + qf(b)}{1 + q}. \tag{2.4}$$

From Theorems 1 and 2 we obtain the following inequalities.

**Corollary 1** [22] For any convex function  $f : [a, b] \rightarrow \mathbb{R}$  and  $0 < q < 1$ , we have

$$f\left(\frac{qa + b}{1 + q}\right) + f\left(\frac{a + qb}{1 + q}\right) \leq \frac{1}{b - a} \left\{ \int_a^b f(x) {}_a d_q x + \int_a^b f(x) {}^b d_q x \right\} \leq f(a) + f(b) \tag{2.5}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \left\{ \int_a^b f(x)_a d_q x + \int_a^b f(x)_b d_q x \right\} \leq \frac{f(a)+f(b)}{2}. \tag{2.6}$$

**Theorem 3** (Hölder’s inequality, [24, p. 604]) *Suppose that  $x > 0, 0 < q < 1, p_1 > 1$ . If  $\frac{1}{p_1} + \frac{1}{r_1} = 1$ , then*

$$\int_0^x |f(x)g(x)| d_q x \leq \left( \int_0^x |f(x)|^{p_1} d_q x \right)^{\frac{1}{p_1}} \left( \int_0^x |g(x)|^{r_1} d_q x \right)^{\frac{1}{r_1}}.$$

In this paper, we will also find some bounds for right-hand side of inequality (2.4).

### 3 New Hermite–Hadamard-type inequalities for quantum integrals

We now give some new Hermite–Hadamard-type inequalities for functions whose second  $q^b$ -derivatives in absolute value are convex.

We start with the following useful lemma.

**Lemma 1** *If  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a twice  $q^b$ -differentiable function on  $(a, b)$  such that  ${}^b D_q^2 f$  is continuous and integrable on  $[a, b]$ , then we have:*

$$\begin{aligned} & \frac{f(a) + qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)_b d_q x \\ &= \frac{q^2(b-a)^2}{1+q} \int_0^1 t(1-qt) {}^b D_q^2 f (ta + (1-t)b) d_q t, \end{aligned} \tag{3.1}$$

where  $0 < q < 1$ .

*Proof* From Definition 2 it follows that

$$\begin{aligned} & {}^b D_q^2 f (ta + (1-t)b) \\ &= {}^b D_q ( {}^b D_q (f (ta + (1-t)b)) ) \\ &= {}^b D_q \left( \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t} \right) \\ &= \frac{1}{(1-q)(b-a)t} \left[ \frac{f(q^2ta + (1-tq^2)b) - f(qta + (1-qt)b)}{(1-q)q(b-a)t} \right. \\ & \quad \left. - \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t} \right] \\ &= \frac{f(q^2ta + (1-tq^2)b) - f(qta + (1-qt)b)}{(1-q)^2q(b-a)^2t^2} \\ & \quad - \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)^2(b-a)^2t^2} \\ &= \frac{f(q^2ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2q(b-a)^2t^2}. \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned}
 & \int_0^1 t(1-qt)^b {}^b D_q^2 f(ta + (1-t)b) d_q t \\
 &= \int_0^1 \frac{f(q^2 ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2 q(b-a)^2 t} d_q t \\
 & \quad - \int_0^1 q \left[ \frac{f(q^2 ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2 q(b-a)^2} \right] d_q t.
 \end{aligned} \tag{3.3}$$

By equality (2.1) we obtain that

$$\begin{aligned}
 & \int_0^1 \frac{f(q^2 ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2 q(b-a)^2 t} d_q t \\
 &= (1-q) \sum_{n=0}^{\infty} \frac{f(q^{n+2} a + (1-q^{n+2})b)}{(1-q)^2 q(b-a)^2} - (1-q)(1+q) \sum_{n=0}^{\infty} \frac{f(q^{n+1} a + (1-q^{n+1})b)}{(1-q)^2 q(b-a)^2} \\
 & \quad + q(1-q) \sum_{n=0}^{\infty} \frac{f(q^n a + (1-q^n)b)}{(1-q)^2 q(b-a)^2} \\
 &= \sum_{n=0}^{\infty} \frac{f(q^{n+2} a + (1-q^{n+2})b)}{(1-q)q(b-a)^2} - \sum_{n=0}^{\infty} \frac{f(q^{n+1} a + (1-q^{n+1})b)}{(1-q)q(b-a)^2} \\
 & \quad - q \left[ \sum_{n=0}^{\infty} \frac{f(q^{n+1} a + (1-q^{n+1})b)}{(1-q)q(b-a)^2} - \sum_{n=0}^{\infty} \frac{f(q^n a + (1-q^n)b)}{(1-q)q(b-a)^2} \right] \\
 &= \frac{f(b) - f(qa + (1-q)b)}{(1-q)q(b-a)^2} - q \left[ \frac{f(b) - f(a)}{(1-q)q(b-a)^2} \right].
 \end{aligned} \tag{3.4}$$

From (2.1) and Definition 5 it follows that

$$\begin{aligned}
 & \int_0^1 q \left[ \frac{f(q^2 ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2 q(b-a)^2} \right] d_q t \\
 &= q \left[ (1-q)(b-a) \sum_{n=0}^{\infty} \frac{q^{n+2} f(q^{n+2} a + (1-q^{n+2})b)}{(1-q)^2 q^3 (b-a)^3} \right. \\
 & \quad - (1-q)(1+q)(b-a) \sum_{n=0}^{\infty} \frac{q^{n+1} f(q^{n+1} a + (1-q^{n+1})b)}{(1-q)^2 q^2 (b-a)^3} \\
 & \quad \left. + q(1-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n f(q^n a + (1-q^n)b)}{(1-q)^2 q (b-a)^3} \right] \\
 &= q \left[ \frac{1}{(1-q)^2 q^3 (b-a)^3} \right. \\
 & \quad \times \left( \int_a^b f(x)^b d_q x - (1-q)(b-a)f(a) - (1-q)(b-a)qf(qa + (1-q)b) \right) \\
 & \quad \left. - \frac{1+q}{(1-q)^2 q^2 (b-a)^3} \left( \int_a^b f(x)^b d_q x - (1-q)(1+q)(b-a)f(a) \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{(1-q)^2(b-a)^3} \int_a^b f(x)^b d_q x \Big] \\
 & = \frac{1+q}{(b-a)^2 q^2} \int_a^b f(x)^b d_q x + \frac{q^2+q-1}{(1-q)q^2(b-a)^2} f(a) - \frac{f(qa+(1-q)b)}{(1-q)q(b-a)^2} \tag{3.5}
 \end{aligned}$$

Using (3.4) and (3.5) in (3.3), we have

$$\begin{aligned}
 & \int_0^1 t(1-qt)^b {}^b D_q^2 f (ta+(1-t)b) d_q t \\
 & = \frac{f(b)-f(qa+(1-q)b)}{(1-q)q(b-a)^2} - q \left[ \frac{f(b)-f(a)}{(1-q)q(b-a)^2} \right] \\
 & \quad - \frac{1+q}{(b-a)^2 q^2} \int_a^b f(x)^b d_q x - \frac{q^2+q-1}{(1-q)q^2(b-a)^2} f(a) + \frac{f(qa+(1-q)b)}{(1-q)q(b-a)^2} \\
 & = \frac{f(a)+qf(b)}{(b-a)^2 q^2} - \frac{1+q}{(b-a)^2 q^2} \int_a^b f(x)^b d_q x. \tag{3.6}
 \end{aligned}$$

Multiplying both sides of (3.6) by  $\frac{(b-a)^2 q^2}{1+q}$ , we obtain the required identity (3.1) and hence we complete the proof of Lemma 1. □

*Remark 1* If we take the limit as  $q \rightarrow 1^-$  in Lemma 1, then we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta+(1-t)b) dt,$$

as given in [25].

**Theorem 4** *If  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a twice  $q^b$ -differentiable function on  $(a, b)$  such that  ${}^b D_q^2 f$  is continuous and integrable on  $[a, b]$ , then we have the following inequality, provided that  $|{}^b D_q^2 f|$  is convex on  $[a, b]$ :*

$$\begin{aligned}
 & \left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b d_q x \right| \\
 & \leq \frac{q^2(b-a)^2}{(1+q)(q^2+q+1)(q^3+q^2+q+1)} \left[ |{}^b D_q^2 f(a)| + q^2 |{}^b D_q^2 f(b)| \right],
 \end{aligned}$$

where  $0 < q < 1$ .

*Proof* Taking the modulus in Lemma 1 and applying the convexity of  $|{}^b D_q^2 f|$ , we obtain

$$\begin{aligned}
 & \left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b d_q x \right| \\
 & \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) |{}^b D_q^2 f (ta+(1-t)b)| d_q t \\
 & \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) [t |{}^b D_q^2 f(a)| + (1-t) |{}^b D_q^2 f(b)|] d_q t
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{q^2(b-a)^2}{1+q} \left[ |{}^bD_q^2 f(a)| \int_0^1 t(t(1-qt)) d_q t + |{}^bD_q^2 f(b)| \int_0^1 (1-t)(t(1-qt)) d_q t \right] \\
 &= \frac{q^2(b-a)^2}{1+q} \left[ \frac{|{}^bD_q^2 f(a)|}{(q^2+q+1)(q^3+q^2+q+1)} + \frac{q^2|{}^bD_q^2 f(b)|}{(q^2+q+1)(q^3+q^2+q+1)} \right],
 \end{aligned}$$

which completes the proof. □

*Remark 2* Under the assumptions of Theorem 4 with the limit as  $q \rightarrow 1^-$ , we have the following trapezoidal inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^2}{12} \left[ \frac{|f''(a)|+|f''(b)|}{2} \right],$$

as given by Sarikaya and Aktan [26, Proposition 2].

**Theorem 5** Suppose that  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a twice  $q^b$ -differentiable function on  $(a, b)$  and  ${}^bD_q^2 f$  is continuous and integrable on  $[a, b]$ . If  $|{}^bD_q^2 f|^{p_1}, p_1 > 1$ , is convex on  $[a, b]$ , then we have the following inequality:

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{p_1}}(1+q+q^2)} \left( \frac{1}{q^3+q^2+q+1} \right)^{\frac{1}{p_1}} (|{}^bD_q^2 f(a)|^{p_1} + q^2|{}^bD_q^2 f(b)|^{p_1})^{\frac{1}{p_1}},
 \end{aligned}$$

where  $0 < q < 1$ .

*Proof* Taking the modulus in Lemma 1 and applying the well-known power mean inequality, we have

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) |{}^bD_q^2 f(ta+(1-t)b)| d_q t \\
 &\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt)) d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( \int_0^1 (t(1-qt)) |{}^bD_q^2 f(ta+(1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}}.
 \end{aligned}$$

By the convexity of  $|{}^bD_q^2 f|^{p_1}$  we have

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt)) d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( \int_0^1 (t(1-qt)) [t|{}^bD_q^2 f(a)|^{p_1} + (1-t)|{}^bD_q^2 f(b)|^{p_1}] d_q t \right)^{\frac{1}{p_1}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt)) d_q t \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( |{}^b D_q^2 f(a)|^{p_1} \int_0^1 t(t(1-qt)) d_q t + |{}^b D_q^2 f(b)|^{p_1} \int_0^1 (1-t)(t(1-qt)) d_q t \right)^{\frac{1}{p_1}} \\
 &= \frac{q^2(b-a)^2}{1+q} \left( \frac{1}{(1+q)(1+q+q^2)} \right)^{1-\frac{1}{p_1}} \\
 &\quad \times \left( \frac{|{}^b D_q^2 f(a)|^{p_1}}{(q^2+q+1)(q^3+q^2+q+1)} + \frac{q^2 |{}^b D_q^2 f(b)|^{p_1}}{(q^2+q+1)(q^3+q^2+q+1)} \right)^{\frac{1}{p_1}},
 \end{aligned}$$

which completes the proof. □

*Remark 3* If we take the limit as  $q \rightarrow 1^-$  in Theorem 5, then we have

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12.2^{\frac{1}{p_1}}} (|f''(a)|^{p_1} + |f''(b)|^{p_1})^{\frac{1}{p_1}}.$$

**Theorem 6** Suppose that  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$  is a twice  $q^b$ -differentiable function on  $(a, b)$  and  ${}^b D_q^2 f$  is continuous and integrable on  $[a, b]$ . If  $|{}^b D_q^2 f|^{p_1}$  is convex on  $[a, b]$  for some  $p_1 > 1$  and  $\frac{1}{r_1} + \frac{1}{p_1} = 1$ , then we have

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{1+q} (u_1)^{\frac{1}{r_1}} \left( \frac{|{}^b D_q^2 f(a)|^{p_1} + q |{}^b D_q^2 f(b)|^{p_1}}{q+1} \right)^{\frac{1}{p_1}}, \tag{3.7}
 \end{aligned}$$

where  $u_1 = (1-q) \sum_{n=0}^{\infty} (q^n)^{r_1+1} (1-q^{n+1})^{r_1}$  and  $0 < q < 1$ .

*Proof* Taking the modulus in Lemma 1 and applying well-known Hölder’s inequality, we obtain

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) |{}^b D_q^2 f(ta+(1-t)b)| d_q t \\
 &\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt))^{r_1} d_q t \right)^{\frac{1}{r_1}} \left( \int_0^1 |{}^b D_q^2 f(ta+(1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}}.
 \end{aligned}$$

Since  $|{}^b D_q^2 f|^{p_1}$  is convex, we have

$$\begin{aligned}
 &\left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b d_q x \right| \\
 &\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt))^{r_1} d_q t \right)^{\frac{1}{r_1}}
 \end{aligned}$$

$$\begin{aligned} & \times \left( |{}^b D_q^2 f(a)|^{p_1} \int_0^1 t d_q t + |{}^b D_q^2 f(b)|^{p_1} \int_0^1 (1-t) d_q t \right)^{\frac{1}{p_1}} \\ & = \frac{q^2(b-a)^2}{1+q} (u_1)^{\frac{1}{r_1}} \left( \frac{|{}^b D_q^2 f(a)|^{p_1} + q|{}^b D_q^2 f(b)|^{p_1}}{q+1} \right)^{\frac{1}{p_1}}. \end{aligned}$$

Thus

$$u_1 = \int_0^1 (t(1-qt))^{r_1} d_q t = (1-q) \sum_{n=0}^{\infty} (q^n)^{r_1+1} (1-q^{n+1})^{r_1},$$

which completes the proof. □

*Remark 4* If we take the limit as  $q \rightarrow 1^-$  in Theorem 6, then we have

$$u_1 = \int_0^1 (t(1-t))^{r_1} dt = B(r_1 + 1, r_1 + 1),$$

where  $B(x, y)$  is the Euler beta function. Moreover, inequality (3.7) reduces to

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} (B(r_1 + 1, r_1 + 1))^{\frac{1}{r_1}} \left( \frac{|f''(a)|^{p_1} + |f''(b)|^{p_1}}{2} \right)^{\frac{1}{p_1}}. \end{aligned}$$

We obtain another Hermite–Hadamard-type inequality for powers in terms of the second quantum derivatives.

**Theorem 7** *With assumptions of Theorem 6, we have the inequality*

$$\begin{aligned} & \left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \frac{1}{[r_1+1]_q} \right)^{\frac{1}{r_1}} (u_2 |{}^b D_q^2 f(a)|^{p_1} + u_3 |{}^b D_q^2 f(b)|^{p_1})^{\frac{1}{p_1}}, \end{aligned} \tag{3.8}$$

where

$$u_2 = (1-q) \sum_{n=0}^{\infty} q^{2n} (1-q^{n+1})^{p_1} \quad \text{and} \quad u_3 = (1-q) \sum_{n=0}^{\infty} q^n (1-q^n) (1-q^{n+1})^{p_1}.$$

*Proof* Taking the modulus of the right-hand side of the equality in Lemma 1 and applying well-known Hölder’s inequality, we obtain

$$\begin{aligned} & \left| \frac{f(a)+qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \right| \\ & \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) |{}^b D_q^2 f(ta+(1-t)b)| d_q t \\ & \leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 t^{r_1} d_q t \right)^{\frac{1}{r_1}} \left( \int_0^1 (1-qt)^{p_1} |{}^b D_q^2 f(ta+(1-t)b)|^{p_1} d_q t \right)^{\frac{1}{p_1}}. \end{aligned}$$

Since  $|{}^bD_q^2 f|^{p_1}$  is convex, we have

$$\begin{aligned} & \left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) {}^b d_q x \right| \\ & \leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 t^{r_1} d_q t \right)^{\frac{1}{r_1}} \\ & \quad \times \left( |{}^bD_q^2 f(a)|^{p_1} \int_0^1 (1 - qt)^{p_1} t d_q t + |{}^bD_q^2 f(b)|^{p_1} \int_0^1 (1 - qt)^{p_1} (1 - t) d_q t \right)^{\frac{1}{p_1}} \\ & = \frac{q^2(b - a)^2}{1 + q} \left( \frac{1}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} (u_2 |{}^bD_q^2 f(a)|^{p_1} + u_3 |{}^bD_q^2 f(b)|^{p_1})^{\frac{1}{p_1}}. \end{aligned}$$

We can easily see that

$$u_2 = \int_0^1 (1 - qt)^{p_1} t d_q t = (1 - q) \sum_{n=0}^{\infty} q^{2n} (1 - q^{n+1})^{p_1}$$

and

$$u_3 = \int_0^1 (1 - qt)^{p_1} (1 - t) d_q t = (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^n) (1 - q^{n+1})^{p_1}.$$

This completes the proof. □

*Remark 5* If we take the limit as  $q \rightarrow 1^-$  in Theorem 7, then we have

$$u_2 = \int_0^1 (1 - t)^{p_1} t dt = \frac{1}{(p_1 + 1)(p_1 + 2)}$$

and

$$u_3 = \int_0^1 (1 - t)^{p_1} (1 - t) dt = \frac{1}{p_1 + 2}.$$

Moreover, inequality (3.8) reduces to

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b - a)^2}{2} \left( \frac{1}{r_1 + 1} \right)^{\frac{1}{r_1}} \left( \frac{1}{(p_1 + 1)(p_1 + 2)} \right)^{\frac{1}{p_1}} ((p_1 + 2) |f''(a)|^{p_1} + |f''(b)|^{p_1})^{\frac{1}{p_1}}. \end{aligned}$$

#### 4 Conclusions

In this paper, we obtained Hermite–Hadamard-type inequalities for convex functions by applying the newly defined  $q^b$ -integral. The results proved in this paper are a potential generalization of the existing comparable results in the literature. As future directions, we can find similar inequalities through different types of convexities.

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### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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### References

1. Dragomir, S.S., Pearce, C.: Selected topics on Hermite–Hadamard inequalities and applications. *Mathematics Preprint Archive* **2003**(3), 463–817 (2003)
2. Pečarić, J.E., Tong, Y.L.: *Convex Functions, Partial Orderings, and Statistical Applications*. Academic Press, Boston (1992)
3. Alp, N., Sankaya, M.Z., Kunt, M., İşcan, İ.:  $q$ -Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ., Sci.* **30**(2), 193–203 (2018)
4. Yang, X.-Z., Farid, G., Nazeer, W., Yussouf, M., Chu, M.-C., Dong, C.-F.: Fractional generalized Hadamard and Fejér–Hadamard inequalities for  $m$ -convex functions. *AIMS Math.* **5**(6), 6325–6340 (2020)
5. Budak, H., Ali, M.A., Tarhanaci, M.: Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* **186**(3), 899–910 (2020)
6. Guo, S.-Y., Chu, Y.-M., Farid, G., Mehmood, S., Nazeer, W.: Fractional Hadamard and Fejér–Hadamard inequalities associated with exponentially  $(s, m)$ -convex functions. *J. Funct. Spaces* **2020**, Article ID 2410385 (2020)
7. Ernst, T.: *A Comprehensive Treatment of  $Q$ -calculus*. Springer, Berlin (2012)
8. Ernst, T.: *The History of  $Q$ -calculus and a New Method*. Department of Mathematics, Uppsala University, Sweden (2000)
9. Nwaeze, E.R., Tameru, A.M.: New parameterized quantum integral inequalities via  $\eta$ -quasiconvexity. *Adv. Differ. Equ.* **2019**(1), 425 (2019)
10. Gauchman, H.: Integral inequalities in  $q$ -calculus. *Comput. Math. Appl.* **47**(2–3), 281–300 (2004)
11. Jhanthanam, S., Tariboon, J., Ntouyas, S.K., Nonlaopon, K.: On  $q$ -Hermite–Hadamard inequalities for differentiable convex functions. *Mathematics* **7**(7), 632 (2019)
12. Khan, M.A., Mohammad, N., Nwaeze, E.R., Chu, Y.-M.: Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**(1), 1 (2020)
13. Liu, W.-J., Zhuang, H.-F.: Some quantum estimates of Hermite–Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **7**(2), 501–522 (2017)
14. Noor, M.A., Noor, K.I., Awan, M.U.: Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **269**, 242–251 (2015)
15. Noor, M.A., Awan, M.U., Noor, K.I.: Quantum Ostrowski inequalities for  $q$ -differentiable convex functions. *J. Math. Inequal.* **10**(4), 1013–1018 (2016)
16. Sudsutad, W., Ntouyas, S.K., Tariboon, J.: Quantum integral inequalities for convex functions. *J. Math. Inequal.* **9**(3), 781–793 (2015)
17. Vivas-Cortez, M., Aamir Ali, M., Kashuri, A., Bashir Sial, I., Zhang, Z.: Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus. *Symmetry* **12**(9), 1476 (2020)
18. Zhuang, H., Liu, W., Park, J.: Some quantum estimates of Hermite–Hadamard inequalities for quasi-convex functions. *Mathematics* **7**(2), 152 (2019)
19. Kac, V., Cheung, P.: *Quantum Calculus*. Springer, Berlin (2001)
20. Jackson, F.H.: On  $q$ -definite integrals. *Q. J. Pure Appl. Math.* **41**, 193–203 (1910)
21. Tariboon, J., Ntouyas, S.K.: Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**(1), 282 (2013)
22. Bermudo, S., Kórus, P., Valdés, J.N.: On  $q$ -Hermite–Hadamard inequalities for general convex functions. *Acta Math. Hung.* 1–11 (2020)
23. Noor, M.A., Noor, K.I., Awan, M.U.: Some quantum estimates for Hermite–Hadamard inequalities. *Appl. Math. Comput.* **251**, 675–679 (2015)
24. Anastassiou, G.A.: *Intelligent Mathematics: Computational Analysis*. Springer, New York (2011)

25. Alomari, M.W., Darus, M., Dragomir, S.S.: New inequalities of Hermite–Hadamard type for functions whose second derivatives absolute values are quasi-convex. *Tamkang J. Math.* **41**(4), 353–359 (2010)
26. Sarikaya, M.Z., Aktan, N.: On the generalization of some integral inequalities and their applications. *Math. Comput. Model.* **54**(9–10), 2175–2182 (2011)

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