mathematics

MDPI

Article

New Hermite–Hadamard and Ostrowski-Type Inequalities for Newly Introduced Co-Ordinated Convexity with Respect to a Pair of Functions

Muhammad Aamir Ali [1], Fongchan Wannalookkhee [2], Hüseyin Budak [3], Sina Etemad [4,*] and Shahram Rezapour [5,6,7,*]

[1] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
[2] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
[3] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey
[4] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz 3751-71379, Iran
[5] Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam
[6] Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam
[7] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: sina.etemad@azaruniv.ac.ir (S.E.); rezapourshahram@yahoo.ca or shahramrezapour@duytan.edu.vn or sh.rezapour@mail.cmuh.org.tw (S.R.)

check for updates

Citation: Ali, M.A.; Wannalookkhee, F.; Budak, H.; Etemad, S.; Rezapour, S. New Hermite–Hadamard- and Ostrowski-Type Inequalities for Newly Introduced Co-Ordinated Convexity with Respect to a Pair of Functions. *Mathematics* **2022**, *10*, 3469. https://doi.org/10.3390/math10193469

Academic Editor: Janusz Brzdęk

Received: 9 August 2022
Accepted: 21 September 2022
Published: 23 September 2022

Abstract: In both pure and applied mathematics, convex functions are used in many different problems. They are crucial to investigate both linear and non-linear programming issues. Since a convex function is one whose epigraph is a convex set, the theory of convex functions falls under the umbrella of convexity. However, it is a significant theory that affects practically all areas of mathematics. In this paper, we introduce the notions of (g, h)-convexity or convexity with respect to a pair of functions on co-ordinates and discuss its fundamental properties. Moreover, we establish some novel Hermite–Hadamard- and Ostrowski-type inequalities for newly introduced co-ordinated convexity. Additionally, it is presented that the newly introduced notion of the convexity and given inequalities are generalizations of existing studies in the literature. Lastly, we look at various mathematical examples and graphs to confirm the validity of the newly found inequalities.

Keywords: Hermite–Hadamard inequality; Ostrowski inequalities; convex functions; co-ordinated convex function

MSC: 26D10; 26D15; 26A51; 26B25

1. Introduction

In different branches of mathematical analysis, inequalities play a fundamental role in proving many famous theorems. Additionally, in recent years, such inequalities have been used in most papers discussing fractional mathematical models, fractional boundary value problems, etc. The application of some of them can be found in numerous articles, including [1–11]. These applications show the importance of study on different generalizations of inequalities.

Throughout this paper, we let $I = [\sigma, \rho] \subset \mathbb{R}$ and $\triangle := [\sigma, \rho] \times [\varsigma, d] \subseteq \mathbb{R}^2$. The Hermite–Hadamard inequality is a fudamental inequality which is presented as: if $\mathcal{F} : I \to \mathbb{R}$ is a convex function, then

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\varkappa)\, d\varkappa \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}. \quad (1)$$

The double inequality (1) was discussed by Hermite [12] in 1883, and ten years later in 1893, it was proved by Hadamard [13].

Another famous inequality is the Ostrowski inequality, which was established by Ostrowski [14] in 1938.

Theorem 1. *Let $\digamma : I \to \mathbb{R}$ be differentiable on I° (interior of I) with a bounded derivative, that is, $\|\digamma'\|_\infty := \sup_{\varkappa\in(\sigma,\rho)} |\digamma'(\varkappa)| < \infty$. Then, we have*

$$\left|\digamma(\varkappa) - \frac{1}{\rho-\sigma}\int_\sigma^\rho \digamma(\varkappa)\, d\varkappa\right| \le \left[\frac{1}{4} + \frac{(\varkappa - \frac{\sigma+\rho}{2})}{(\rho-\sigma)^2}\right](\rho-\sigma)\|\digamma'\|_\infty, \tag{2}$$

for all $\varkappa \in I$. The constant $\frac{1}{4}$ is the best possible.

The inequality (2) can be rewritten in the equivalent form

$$\left|\digamma(\varkappa) - \frac{1}{\rho-\sigma}\int_\sigma^\rho \digamma(\varkappa)\, d\varkappa\right| \le \left[\frac{(\varkappa-\sigma)^2 + (\rho-\varkappa)^2}{2(\rho-\sigma)}\right]\|\digamma'\|_\infty.$$

The Hermite–Hadamard inequality and Ostrowski inequality have been extensively studied by a number of researchers; see [15–20] and the references therein.

In 2001, S. S. Dragomir defined a new concept of a co-ordinated convex function.

Definition 1 ([21])**.** *$\digamma : \triangle \to \mathbb{R}$ is a co-ordinated convex map, if*

$$\digamma_\varkappa : [\varsigma, d] \ni v \mapsto \digamma(\varkappa, v) \in \mathbb{R} \quad \& \quad \digamma_\gamma : [\sigma,\rho] \ni u \mapsto \digamma(u,\gamma) \in \mathbb{R}, \tag{3}$$

are convex ($\forall\, \varkappa \in (\sigma,\rho), \forall\, \gamma \in (\varsigma, d)$).

A formal equivalent definition of such functions can be stated as the following:

Definition 2 ([21])**.** *$\digamma : \triangle \to \mathbb{R}$ is co-ordinated convex if*

$$\begin{aligned} \digamma(\xi\varkappa + (1-\xi)z, \lambda\gamma + (1-\lambda)w) \quad &\le \xi\lambda\digamma(\varkappa,\gamma) + \xi(1-\lambda)\digamma(\varkappa,w) \\ &+ (1-\xi)\lambda\digamma(z,\gamma) + (1-\xi)(1-\lambda)\digamma(z,w), \end{aligned} \tag{4}$$

$\forall\, \xi, \lambda \in [0,1]$ and $\forall\, (\varkappa,\gamma),(z,w) \in \triangle$.

He also presented the following version of Hermite–Hadamard-type inequalities for the aforesaid functions:

Theorem 2 ([21])**.** *If $\digamma : \triangle \to \mathbb{R}$ is a co-ordinated convex function, then*

$$\begin{aligned} \digamma\left(\frac{\sigma+\rho}{2}, \frac{\varsigma+d}{2}\right) \quad &\le \frac{1}{2}\left[\frac{1}{\rho-\sigma}\int_\sigma^\rho \digamma\left(\varkappa, \frac{\varsigma+d}{2}\right) d\varkappa + \frac{1}{d-\varsigma}\int_\varsigma^d \digamma\left(\frac{\sigma+\rho}{2}, \gamma\right) d\gamma\right] \\ &\le \frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d \digamma(\varkappa,\gamma)\, d\gamma\, d\varkappa \\ &\le \frac{1}{4}\left[\frac{1}{\rho-\sigma}\left(\int_\sigma^\rho \digamma(\varkappa,\varsigma)\, d\varkappa + \int_\sigma^\rho \digamma(\varkappa,d)\, d\varkappa\right)\right. \\ &\left. + \frac{1}{d-\varsigma}\left(\int_\varsigma^d \digamma(\sigma,\gamma)\, d\gamma + \int_\varsigma^d \digamma(\rho,\gamma)\, d\gamma\right)\right] \\ &\le \frac{1}{4}[\digamma(\sigma,\varsigma) + \digamma(\rho,\varsigma) + \digamma(\sigma,d) + \digamma(\rho,d)]. \end{aligned} \tag{5}$$

In relation to the convex functions on co-ordinates, Latif et al. [22] presented the following Ostrowski-type inequality in 2010:

Theorem 3 ([22]). *Let $\digamma : \Delta \to \mathbb{R}$ be a function with the twice-partial differentiability property on Δ° and $\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}$ be continuous and integrable on Δ. When $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}\right|$ is co-ordinated convex on Δ and $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}(\varkappa,\gamma)\right| \le M,\ (\varkappa,\gamma)\in\Delta$, then*

$$\left|\digamma(\varkappa,\gamma) + \frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d \digamma(u,v)\,d\eta\,d\xi - A\right| \le M\left[\frac{(\varkappa-\sigma)^2+(\rho-\varkappa)^2}{2(\rho-\sigma)}\right]\left[\frac{(\gamma-\varsigma)^2+(d-\gamma)^2}{2(d-\varsigma)}\right], \quad (6)$$

where

$$A(\varkappa,\gamma) = \frac{1}{d-\varsigma}\int_\varsigma^d \digamma(\varkappa,v)\,dv + \frac{1}{\rho-\sigma}\int_\sigma^\varsigma \digamma(u,\gamma)\,du. \quad (7)$$

Theorem 4 ([22]). *Let $\digamma : \Delta \to \mathbb{R}$ be a function with the twice-partial differentiability property on Δ°, and let $\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}$ be continuous and integrable on Δ. When $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}\right|^q$ is co-ordinated convex on Δ, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}(\varkappa,\gamma)\right| \le M,\ (\varkappa,\gamma)\in\Delta$, then*

$$\left|\digamma(\varkappa,\gamma) + \frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d \digamma(u,v)\,dv\,du - A\right| \le \frac{M}{(1+p)^{2/p}}\left[\frac{(\varkappa-\sigma)^2+(\rho-\varkappa)^2}{\rho-\sigma}\right]\left[\frac{(\gamma-\varsigma)^2+(d-\gamma)^2}{d-\varsigma}\right], \quad (8)$$

where A is defined as in (7).

Theorem 5 ([22]). *Let $\digamma : \Delta \to \mathbb{R}$ be a function with the twice-partial differentiability property on Δ°, and let $\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}$ be continuous and integrable on Δ. When $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}\right|^q$ is co-ordinated convex on Δ, where $q\ge 1$, and $\left|\frac{\partial^2 \digamma}{\partial\eta\,\partial\xi}(\varkappa,\gamma)\right| \le M,\ (\varkappa,\gamma)\in\Delta$, then*

$$\left|\digamma(\varkappa,\gamma) + \frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d \digamma(u,v)\,dv\,du - A\right| \le \frac{M}{4}\left[\frac{(\varkappa-\sigma)^2+(\rho-\varkappa)^2}{\rho-\sigma}\right]\left[\frac{(\gamma-\varsigma)^2+(d-\gamma)^2}{d-\varsigma}\right], \quad (9)$$

where A is defined as in (7).

In [23], B. Samet introduced a new class of convex functions with respect to a pair of functions, which is defined as:

Definition 3. *Let $g,h : I \to \mathbb{R}$ be two mappings. A mapping $\digamma : I \to \mathbb{R}$ is called (g,h) convex if the following inequality holds for $M(\varkappa,\gamma) = g(\varkappa)h(\gamma) + g(\gamma)h(\varkappa)$*

$$\digamma(\xi\varkappa + (1-\xi)\gamma) \le \xi^2\digamma(\varkappa) + (1-\xi)^2\digamma(\gamma) + \xi(1-\xi)M(\varkappa,\gamma), \quad (10)$$

for all $\xi\in[0,1]$ and $\varkappa,\gamma\in I$.

Remark 1. *If $g,h : I\subset\mathbb{R}\to[0,\infty)$ are two convex mappings, then $\digamma = gh$ is (g,h)-convex mapping.*

B. Samet also established a Hermite–Hadamard inequality (double inequality) for newly class of convex functions with respect to a pair of functions, presented as:

Theorem 6 ([23]). *Let $F : I \to \mathbb{R}$ be a (g,h)-convex function over I with $g,h : I \to \mathbb{R}$. If $F \in L^1(I)$ and $g,h \in L^2(I)$, then*

$$\begin{aligned} & 2F\left(\tfrac{\sigma+\rho}{2}\right) - \tfrac{1}{\rho-\sigma}\int_\sigma^\rho g(\sigma+\rho-\varkappa)h(\varkappa)d\varkappa \\ = \; & 2F\left(\tfrac{\sigma+\rho}{2}\right) - \tfrac{1}{\rho-\sigma}\int_\sigma^\rho g(\varkappa)h(\sigma+\rho-\varkappa)d\varkappa \\ \leq \; & \tfrac{1}{\rho-\sigma}\int_\sigma^\rho F(\varkappa)d\varkappa \\ \leq \; & \tfrac{F(\sigma)+F(\rho)}{3} + \tfrac{M(\sigma,\rho)}{6}. \end{aligned} \tag{11}$$

Remark 2. *In Theorem 6, if we set $g = F$ and $h = 1$ or $h = F$ and $g = 1$, then inequality (11) becomes the inequality (1).*

After that, Ali et al. [24] and Xie et al. [25] used the (g,h)-convexity and established the following fractional Hermite–Hadamard-type inequality.

Theorem 7 ([24,25]). *Consider $F : I \to \mathbb{R}$ as a (g,h)-convex function on I with $g,h : I \to \mathbb{R}$. If $F \in L^1(I)$ and $g,h \in L^2(I)$, then*

$$\begin{aligned} & 2F\left(\frac{\sigma+\rho}{2}\right) - \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[J^\alpha_{\sigma+}\Omega(\rho) + J^\alpha_{\rho-}\Omega(\sigma)\right] \\ \leq \; & \frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^\alpha}\left[J^\alpha_{\sigma+}F(\rho) + J^\alpha_{\rho-}F(\sigma)\right] \\ \leq \; & \frac{F(\sigma)+F(\rho)}{2}\left[\frac{\alpha}{\alpha+2} + \frac{2}{\alpha^2+3\alpha+2}\right] + \frac{\alpha}{\alpha^2+3\alpha+2}M(\sigma,\rho), \end{aligned} \tag{12}$$

$$\begin{aligned} & 2F\left(\frac{\sigma+\rho}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}\left[J^\alpha_{\sigma+}\Omega\left(\frac{\sigma+\rho}{2}\right) + J^\alpha_{\rho-}\Omega\left(\frac{\sigma+\rho}{2}\right)\right] \\ \leq \; & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}\left[J^\alpha_{\sigma+}F\left(\frac{\sigma+\rho}{2}\right) + J^\alpha_{\rho-}F\left(\frac{\sigma+\rho}{2}\right)\right] \\ \leq \; & \frac{[F(\sigma)+F(\rho)](\alpha+1)}{2(\alpha+2)} + \frac{M(\sigma,\rho)}{2(\alpha+2)}, \end{aligned}$$

and

$$\begin{aligned} & 2F\left(\tfrac{\sigma+\rho}{2}\right) - \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}\left[J^\alpha_{\frac{\sigma+\rho}{2}+}\Omega(\rho) + J^\alpha_{\frac{\sigma+\rho}{2}-}\Omega(\sigma)\right] \\ \leq \; & \tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^\alpha}\left[J^\alpha_{\frac{\sigma+\rho}{2}+}F(\rho) + J^\alpha_{\frac{\sigma+\rho}{2}-}F(\sigma)\right] \\ \leq \; & \tfrac{[F(\sigma)+F(\rho)]}{4}\left[2 - \tfrac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right] + \tfrac{M(\sigma,\rho)\alpha(\alpha+3)}{4(\alpha+1)(\alpha+2)}, \end{aligned} \tag{13}$$

where $\Omega(\varkappa) = g(\varkappa)h(\sigma+\rho-\varkappa)$.

For more results on Simpson- and Ostrowski-type inequalities via (g,h)-convexity, one can consult [23–25].

Motivated by the above literature, we conduct an anlaysis on some new generalizations of Hermite–Hadamard and Ostrowski type inequalities via the co-ordinated convex functions by defining a new notion of co-ordinated (g,h)-convexity or convexity with respect to a pair of functions. It is also shown that the newly established inequalities and class of convex functions are generalizations of the existing results in the literature.

A description of the paper is as follows: in Section 2, the notion of co-ordinated (g,h)-convexity is introduced. We also prove some of its important properties and give an example of a co-ordinated (g,h)-convex function. In Section 3, we establish some

Hermite–Hadamard-type inequalities for co-ordinated (g,h)-convex functions. In Section 4, we derive some new Ostrowski-type inequalities under the differentiable (g,h)-convexity. The examples showing the consistency of newly discussed inequalities are in Section 5. Section 6 concludes our work briefly.

2. Co-Ordinated (g,h)-Convex Functions

In this section, we introduce a new concept of the co-ordinated (g,h)-convex function, and then some of its properties are proved. We also give an example of a co-ordinated (g,h)-convex function at the end of the section.

Definition 4. *Let $g,h:\triangle\to\mathbb{R}$ be two given functions. A function $F:\triangle\to\mathbb{R}$ is called co-ordinated (g,h)-convex, if*

$$\begin{aligned}
F(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w) &\le \xi^2\eta^2F(\varkappa,\gamma)+\xi^2(1-\eta)^2F(\varkappa,w)\\
&+(1-\xi)^2\eta^2F(z,\gamma)+(1-\xi)^2(1-\eta)^2F(z,w)+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)\\
&+\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w)+\xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w)\\
&+\xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)+\eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w),
\end{aligned}$$

where

$$\begin{aligned}
M_1(\varkappa,\gamma,w)&=h(\varkappa,w)g(\varkappa,\gamma)+h(\varkappa,\gamma)g(\varkappa,w),\\
M_2(\varkappa,z,\gamma,w)&=h(\varkappa,w)g(z,\gamma)+h(z,\gamma)g(\varkappa,w),\\
M_3(\varkappa,z,\gamma)&=h(\varkappa,\gamma)g(z,\gamma)+h(z,\gamma)g(\varkappa,\gamma),\\
M_4(\varkappa,z,\gamma,w)&=h(\varkappa,\gamma)g(z,w)+h(z,w)g(\varkappa,\gamma),\\
M_5(\varkappa,z,w)&=h(\varkappa,w)g(z,w)+h(z,w)g(\varkappa,w),
\end{aligned}$$

and

$$M_6(z,\gamma,w)=h(z,\gamma)g(z,w)+h(z,w)g(z,\gamma),$$

for all $\xi,\eta\in[0,1]$, $\varkappa,z\in[\sigma,\rho]$ and $\gamma,w\in[\varsigma,d]$.

Some properties of co-ordinated (g,h)-convexity are proved as follows:

Proposition 1. *Let $g,h:\triangle\to\mathbb{R}$ be two given functions. Then, we have that*

(i) *F is co-ordinated (g,h)-convex if F is co-ordinated (h,g)-convex.*
(ii) *If F is co-ordinated (g,h)-convex, then F is co-ordinated $(\sigma^{-1}g,\sigma h)$-convex for all $\sigma\in\mathbb{R}$, $\sigma\neq 0$.*
(iii) *If $F,\overline{F}$ are co-ordinated (g,h)-convex, in this case, $F+\overline{F}$ is co-ordinated $(2g,h)$-convex and co-ordinated $(g,2h)$-convex.*
(iv) *If F is co-ordinated (g,h)-convex, then σF is co-ordinated $(\sigma g,h)$-convex and co-ordinated $(g,\sigma h)$-convex for all $\sigma>0$.*
(v) *Let F be co-ordinated (g,h)-convex, $F\ge 0$ and $(g\ge 0,h\le 0)$ or $(g\le 0,h\ge 0)$. Then F is co-ordinated convex.*
(vi) *If F is co-ordinated (g,h)-convex and $\overline{F}$ are co-ordinated $(g,\overline{h})$-convex, then $F+\overline{F}$ is co-ordinated $(g,h+\overline{h})$-convex.*
(vii) *If F is co-ordinated (g,h)-convex and $\overline{F}$ are co-ordinated $(\overline{g},h)$-convex, then $F+\overline{F}$ is co-ordinated $(g+\overline{g},h)$-convex.*
(viii) *If F is co-ordinated (g,h)-convex, then*

$$\mathcal{F}\left(\frac{\varkappa+z}{2},\frac{\gamma+w}{2}\right) \le \frac{\mathcal{F}(\varkappa,\gamma)+\mathcal{F}(\varkappa,w)+\mathcal{F}(z,\gamma)+\mathcal{F}(z,w)}{16}$$
$$+\frac{M_1(\varkappa,\gamma,w)+M_2(\varkappa,z,\gamma,w)+M_3(\varkappa,z,\gamma)+M_4(\varkappa,z,\gamma,w)+M_5(\varkappa,z,w)+M_6(z,\gamma,w)}{16},$$

for all $\varkappa, z \in [\sigma,\rho]$ *and* $\gamma, w \in [\varsigma, d]$.

Proof. (i) It is immediately true by Definition 4.
(ii) We simply consider that

$$h(\varkappa,w)g(\varkappa,\gamma)+h(\varkappa,\gamma)g(\varkappa,w) = (\sigma h(\varkappa,w))\Big(\sigma^{-1}g(\varkappa,\gamma)\Big)+(\sigma h(\varkappa,\gamma))\Big(\sigma^{-1}g(\varkappa,w)\Big),$$
$$h(\varkappa,w)g(z,\gamma)+h(z,\gamma)g(\varkappa,w) = (\sigma h(\varkappa,w))\Big(\sigma^{-1}g(z,\gamma)\Big)+(\sigma h(z,\gamma))\Big(\sigma^{-1}g(\varkappa,w)\Big),$$
$$h(\varkappa,\gamma)g(z,\gamma)+h(z,\gamma)g(\varkappa,\gamma) = (\sigma h(\varkappa,\gamma))\Big(\sigma^{-1}g(z,\gamma)\Big)+(\sigma h(z,\gamma))\Big(\sigma^{-1}g(\varkappa,\gamma)\Big),$$
$$h(\varkappa,\gamma)g(z,w)+h(z,w)g(\varkappa,\gamma) = (\sigma h(\varkappa,\gamma))\Big(\sigma^{-1}g(z,w)\Big)+(\sigma h(z,w))\Big(\sigma^{-1}g(\varkappa,\gamma)\Big),$$
$$h(\varkappa,w)g(z,w)+h(z,w)g(\varkappa,w) = (\sigma h(\varkappa,w))\Big(\sigma^{-1}g(z,w)\Big)+(\sigma h(z,w))\Big(\sigma^{-1}g(\varkappa,w)\Big),$$

and

$$h(z,\gamma)g(z,w)+h(z,w)g(z,\gamma) = (\sigma h(z,\gamma))\Big(\sigma^{-1}g(z,w)\Big)+(\sigma h(z,w))\Big(\sigma^{-1}g(z,\gamma)\Big).$$

(iii) Since $\mathcal{F}, \overline{\mathcal{F}}$ are co-ordinated (g,h)-convex, we have

$$\begin{aligned}\mathcal{F}(\xi\varkappa &+(1-\xi)z,\eta\gamma+(1-\eta)w) \le \xi^2\eta^2\mathcal{F}(\varkappa,\gamma)+\xi^2(1-\eta)^2\mathcal{F}(\varkappa,w)\\ &+(1-\xi)^2\eta^2\mathcal{F}(z,\gamma)+(1-\xi)^2(1-\eta)^2\mathcal{F}(z,w)\\ &+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)+\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w)\\ &+\xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w)\\ &+\xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)+\eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\overline{\mathcal{F}}(\xi\varkappa &+(1-\xi)z,\eta\gamma+(1-\eta)w) \le \xi^2\eta^2\overline{\mathcal{F}}(\varkappa,\gamma)+\xi^2(1-\eta)^2\overline{\mathcal{F}}(\varkappa,w)\\ &+(1-\xi)^2\eta^2\overline{\mathcal{F}}(z,\gamma)+(1-\xi)^2(1-\eta)^2\overline{\mathcal{F}}(z,w)\\ &+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)+\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w)\\ &+\xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w)\\ &+\xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)+\eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w),\end{aligned} \tag{15}$$

for all $\xi,\eta \in [0,1]$, $\varkappa, z \in [\sigma,\rho]$ and $\gamma, w \in [\varsigma, d]$. Adding (14) and (15), we obtain

$$\begin{aligned}(\mathcal{F}+\overline{\mathcal{F}})(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w) &\le \xi^2\eta^2(\mathcal{F}+\overline{\mathcal{F}})(\varkappa,\gamma)+\xi^2(1-\eta)^2(\mathcal{F}+\overline{\mathcal{F}})(\varkappa,w)\\ &+(1-\xi)^2\eta^2(\mathcal{F}+\overline{\mathcal{F}})(z,\gamma)+(1-\xi)^2(1-\eta)^2(\mathcal{F}+\overline{\mathcal{F}})(z,w)\\ &+2\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)+2\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w)\\ &+2\xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)+2\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w)\\ &+2\xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)+2\eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w),\end{aligned}$$

which means that $\mathcal{F} + \overline{\mathcal{F}}$ is co-ordinated $(2g, h)$-convex and co-ordinated $(g, 2h)$-convex.

(iv) Let $\mathcal{F}$ be co-ordinated (g, h)-convex. In this case, multiplying (14) by $\sigma \geq 0$ yields

$$\begin{aligned}(\sigma\mathcal{F})(\xi\varkappa + (1-\xi)z, \eta\gamma + (1-\eta)w) &\leq \xi^2\eta^2(\sigma\mathcal{F})(\varkappa,\gamma) + \xi^2(1-\eta)^2(\sigma\mathcal{F})(\varkappa,w)\\ &+ (1-\xi)^2\eta^2(\sigma\mathcal{F})(z,\gamma) + (1-\xi)^2(1-\eta)^2(\sigma\mathcal{F})(z,w)\\ &+ \xi^2\eta(1-\eta)\sigma M_1(\varkappa,\gamma,w) + \xi\eta(1-\xi)(1-\eta)\sigma M_2(\varkappa,z,\gamma,w)\\ &+ \xi\eta^2(1-\xi)\sigma M_3(\varkappa,z,\gamma) + \xi\eta(1-\xi)(1-\eta)\sigma M_4(\varkappa,z,\gamma,w)\\ &+ \xi(1-\xi)(1-\eta)^2\sigma M_5(\varkappa,z,w) + \eta(1-\xi)^2(1-\eta)\sigma M_6(z,\gamma,w),\end{aligned}$$

which shows that $\sigma\mathcal{F}$ is co-ordinated $(\sigma g, h)$-convex and co-ordinated $(g, \sigma h)$-convex.

(v) Let $\mathcal{F} \geq 0$ be co-ordinated (g, h)-convex, where $g \geq 0$ and $h \leq 0$. Then, for all $\xi, \eta \in [0,1]$, $\varkappa, z \in [\sigma, \rho]$ and $\gamma, w \in [\varsigma, d]$, we have

$$\begin{aligned}\mathcal{F}(\xi\varkappa + (1-\xi)z, \eta\gamma + (1-\eta)w) &\leq \xi^2\eta^2\mathcal{F}(\varkappa,\gamma) + \xi^2(1-\eta)^2\mathcal{F}(\varkappa,w) + (1-\xi)^2\eta^2\mathcal{F}(z,\gamma)\\ &+ (1-\xi)^2(1-\eta)^2\mathcal{F}(z,w) + \xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)\\ &+ \xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w) + \xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)\\ &+ \xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w) + \xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)\\ &+ \eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w)\\ &\leq \xi\eta\mathcal{F}(\varkappa,\gamma) + \xi(1-\eta)\mathcal{F}(\varkappa,w) + (1-\xi)\eta\mathcal{F}(z,\gamma) + (1-\xi)(1-\eta)\mathcal{F}(z,w),\end{aligned}$$

which shows that $\mathcal{F}$ is co-ordinated convex. This holds similarly in the case of $g \leq 0$ and $h \geq 0$.

(vi) If $\mathcal{F}$ is co-ordinated (g, h)-convex and $\overline{\mathcal{F}}$ are co-ordinated $(g, \overline{h})$-convex, we may write

$$\begin{aligned}\mathcal{F}(\xi\varkappa &+ (1-\xi)z, \eta\gamma + (1-\eta)w) \leq \xi^2\eta^2\mathcal{F}(\varkappa,\gamma) + \xi^2(1-\eta)^2\mathcal{F}(\varkappa,w)\\ &+ (1-\xi)^2\eta^2\mathcal{F}(z,\gamma) + (1-\xi)^2(1-\eta)^2\mathcal{F}(z,w) + \xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w)\\ &+ \xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w) + \xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)\\ &+ \xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w) + \xi(1-\xi)(1-\eta)^2M_5(\varkappa,z,w)\\ &+ \eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w),\end{aligned} \tag{16}$$

and

$$\begin{aligned}\overline{\mathcal{F}}(\xi\varkappa &+ (1-\xi)z, \eta\gamma + (1-\eta)w) \leq \xi^2\eta^2\overline{\mathcal{F}}(\varkappa,\gamma) + \xi^2(1-\eta)^2\overline{\mathcal{F}}(\varkappa,w)\\ &+ (1-\xi)^2\eta^2\overline{\mathcal{F}}(z,\gamma) + (1-\xi)^2(1-\eta)^2\overline{\mathcal{F}}(z,w) + \xi^2\eta(1-\eta)M_7(\varkappa,\gamma,w)\\ &+ \xi\eta(1-\xi)(1-\eta)M_8(\varkappa,z,\gamma,w) + \xi\eta^2(1-\xi)M_9(\varkappa,z,\gamma)\\ &+ \xi\eta(1-\xi)(1-\eta)M_{10}(\varkappa,z,\gamma,w) + \xi(1-\xi)(1-\eta)^2M_{11}(\varkappa,z,w)\\ &+ \eta(1-\xi)^2(1-\eta)M_{12}(z,\gamma,w),\end{aligned} \tag{17}$$

where

$$M_7(\varkappa,\gamma,w) = \overline{h}(\varkappa,w)g(\varkappa,\gamma) + \overline{h}(\varkappa,\gamma)g(\varkappa,w),$$

$$M_8(\varkappa,z,\gamma,w) = \overline{h}(\varkappa,w)g(z,\gamma) + \overline{h}(z,\gamma)g(\varkappa,w),$$

$$M_9(\varkappa,z,\gamma) = \overline{h}(\varkappa,\gamma)g(z,\gamma) + \overline{h}(z,\gamma)g(\varkappa,\gamma),$$

$$M_{10}(\varkappa,z,\gamma,w)=\overline{h}(\varkappa,\gamma)g(z,w)+\overline{h}(z,w)g(\varkappa,\gamma),$$

$$M_{11}(\varkappa,z,w)=\overline{h}(\varkappa,w)g(z,w)+\overline{h}(z,w)g(\varkappa,w),$$

and

$$M_{12}(z,\gamma,w)=\overline{h}(z,\gamma)g(z,w)+\overline{h}(z,w)g(z,\gamma),$$

for all $\xi,\eta\in[0,1]$, $\varkappa,z\in[\sigma,\rho]$ and $\gamma,w\in[\varsigma,d]$. Adding (16) and (17), we obtain that $F+\overline{F}$ is co-ordinated $(g,h+\overline{h})$-convex.

(vii) The proof is similar to that of (vi).

(viii) Taking $\xi=\eta=\frac{1}{2}$ in (14), the desired inequality is obtained.
□

Proposition 2. *Let $F:\triangle\to\mathbb{R}$ be a co-ordinated convex function. Then, F is co-ordinated $(F,1_{\mathbb{R}\times\mathbb{R}})$-convex.*

Proof. Let $\xi,\eta\in[0,1]$, $\varkappa,z\in[\sigma,\rho]$ and $\gamma,w\in[\varsigma,d]$. By the co-ordinated convexity of F, we write

$$\begin{aligned}
&F(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w)\\
&\leq \xi\eta F(\varkappa,\gamma)+\xi(1-\eta)F(\varkappa,v)+(1-\xi)\eta F(z,\gamma)+(1-\xi)(1-\eta)F(z,w)\\
&=\xi^2\eta^2F(\varkappa,\gamma)+\xi^2(1-\eta)^2F(\varkappa,w)+(1-\xi)^2\eta^2F(z,\gamma)+(1-\xi)^2(1-\eta)^2F(z,w)\\
&\quad+\xi^2\eta(1-\eta)(F(\varkappa,\gamma)+F(\varkappa,w))+\xi\eta(1-\xi)(1-\eta)(F(z,\gamma)+F(\varkappa,w))\\
&\quad+\xi\eta^2(1-\xi)(F(z,\gamma)+F(\varkappa,\gamma))+\xi\eta(1-\xi)(1-\eta)(F(z,w)+F(\varkappa,\gamma))\\
&\quad+\xi(1-\xi)(1-\eta)^2(F(z,w)+F(\varkappa,w))+\eta(1-\xi)^2(1-\eta)(F(z,w)+F(z,\gamma))\\
&=\xi^2\eta^2F(\varkappa,\gamma)+\xi^2(1-\eta)^2F(\varkappa,w)+(1-\xi)^2\eta^2F(z,\gamma)+(1-\xi)^2(1-\eta)^2F(z,w)\\
&\quad+\xi^2\eta(1-\eta)(1_{\mathbb{R}\times\mathbb{R}}(\varkappa,w)F(\varkappa,\gamma)+1_{\mathbb{R}\times\mathbb{R}}(\varkappa,\gamma)F(\varkappa,w))\\
&\quad+\xi\eta(1-\xi)(1-\eta)(1_{\mathbb{R}\times\mathbb{R}}(\varkappa,w)F(z,\gamma)+1_{\mathbb{R}\times\mathbb{R}}(z,\gamma)F(\varkappa,w))\\
&\quad+\xi\eta^2(1-\xi)(1_{\mathbb{R}\times\mathbb{R}}(\varkappa,\gamma)F(z,\gamma)+1_{\mathbb{R}\times\mathbb{R}}(z,\gamma)F(\varkappa,\gamma))\\
&\quad+\xi\eta(1-\xi)(1-\eta)(1_{\mathbb{R}\times\mathbb{R}}(\varkappa,\gamma)F(z,w)+1_{\mathbb{R}\times\mathbb{R}}(z,w)F(\varkappa,\gamma))\\
&\quad+\xi(1-\xi)(1-\eta)^2(1_{\mathbb{R}\times\mathbb{R}}(\varkappa,w)F(z,w)+1_{\mathbb{R}\times\mathbb{R}}(z,w)F(\varkappa,w))\\
&\quad+\eta(1-\xi)^2(1-\eta)(1_{\mathbb{R}\times\mathbb{R}}(z,\gamma)F(z,w)+1_{\mathbb{R}\times\mathbb{R}}(z,w)F(z,\gamma)),
\end{aligned}$$

which proves that F is co-ordinated $(F,1_{\mathbb{R}\times\mathbb{R}})$-convex. □

Proposition 3. *Let $g,h:\triangle\to\mathbb{R}$ be two co-ordinated convex functions. Then, $F=gh$ is co-ordinated (g,h)-convex.*

Proof. Let $\xi,\eta\in[0,1]$, $\varkappa,z\in[\sigma,\rho]$ and $\gamma,w\in[\varsigma,d]$. Since $g,h:\triangle\to\mathbb{R}$ are two functions with the co-ordinated convexity property, we have

$$\begin{aligned}
&F(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w)\\
&\quad=g(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w)h(\xi\varkappa+(1-\xi)z,\eta\gamma+(1-\eta)w)\\
&\quad\leq[\xi\eta g(\varkappa,\gamma)+\xi(1-\eta)g(\varkappa,w)+(1-\xi)\eta g(z,\gamma)+(1-\xi)(1-\eta)g(z,w)]
\end{aligned}$$

$$\times [\xi\eta h(\varkappa,\gamma) + \xi(1-\eta)h(\varkappa,w) + (1-\xi)\eta h(z,\gamma) + (1-\xi)(1-\eta)h(z,w)]$$
$$= \xi^2\eta^2 F(\varkappa,\gamma) + \xi^2(1-\eta)^2 F(\varkappa,w) + (1-\xi)^2\eta^2 F(z,\gamma) + (1-\xi)^2(1-\eta)^2 F(z,w)$$
$$+ \xi^2\eta(1-\eta)M_1(\varkappa,\gamma,w) + \xi\eta(1-\xi)(1-\eta)M_2(\varkappa,z,\gamma,w) + \xi\eta^2(1-\xi)M_3(\varkappa,z,\gamma)$$
$$+ \xi\eta(1-\xi)(1-\eta)M_4(\varkappa,z,\gamma,w) + \xi(1-\xi)(1-\eta)^2 M_5(\varkappa,z,w)$$
$$+ \eta(1-\xi)^2(1-\eta)M_6(z,\gamma,w).$$

This gurantees that F is co-ordinated (g,h)-convex. □

Example 1. *Let* $F : [0,1]^2 \to \mathbb{R}$ *be a function as*

$$F(\varkappa,\gamma) = \varkappa\gamma\, e^{-\varkappa-\gamma}.$$

Consider the functions $g,h : [0,1]\times[0,1] \to \mathbb{R}$ *defined by*

$$g(\varkappa,\gamma) = \varkappa\gamma,$$

and

$$h(\varkappa,\gamma) = e^{-\varkappa-\gamma}.$$

Since g,h *are non-negative co-ordinated convex functions, by Proposition 3,* $F = gh$ *is a co-ordinated* (g,h)*-convex function.*

3. Hermite–Hadamard-Type Inequalities

Now, we here derive some new Hermite–Hadamard-type inequalities for co-ordinated (g,h)-convex functions. We also show that our results can be reduced to previous work.

Theorem 8. *Let* $F : \triangle \to \mathbb{R}$ *be a function with the co-ordinated* (g,h)*-convexity on* $\triangle$*. If* $F \in L^1(\triangle)$ *and* $g,h \in L^2(\triangle)$*, then the following Hermite–Hadamard-type inequalities are formulated as*

$$\begin{aligned}
&4F\left(\tfrac{\sigma+\rho}{2},\tfrac{\varsigma+d}{2}\right) - \tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)[g(\varkappa,\varsigma+d-\gamma) + g(\sigma+\rho-\varkappa,\gamma)\\
&\quad + g(\sigma+\rho-\varkappa,\varsigma+d-\gamma)]\,d\gamma\,d\varkappa\\
&\le \tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(\varkappa,\gamma)\,d\gamma\,d\varkappa\\
&\le \tfrac{1}{9}\Big[F(\sigma,\varsigma) + F(\sigma,d) + F(\rho,\varsigma) + F(\rho,d) + \tfrac{K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)+M(\sigma,\rho,\varsigma)+O(\sigma,\rho,d)}{2}\\
&\quad + \tfrac{L(\sigma,\rho,\varsigma,d)+N(\sigma,\rho,\varsigma,d)}{4}\Big],
\end{aligned} \tag{18}$$

where

$$K(\sigma,\varsigma,d) = h(\sigma,d)g(\sigma,\varsigma) + h(\sigma,\varsigma)g(\sigma,d),$$
$$L(\sigma,\rho,\varsigma,d) = h(\sigma,d)g(\rho,\varsigma) + h(\rho,\varsigma)g(\sigma,d),$$
$$M(\sigma,\rho,\varsigma) = h(\sigma,\varsigma)g(\rho,\varsigma) + h(\rho,\varsigma)g(\sigma,\varsigma),$$
$$N(\sigma,\rho,\varsigma,d) = h(\sigma,\varsigma)g(\rho,d) + h(\rho,d)g(\sigma,\varsigma),$$
$$O(\sigma,\rho,d) = h(\sigma,d)g(\rho,d) + h(\rho,d)g(\sigma,d),$$

and

$$P(\rho,\varsigma,d) = h(\rho,\varsigma)g(\rho,d) + h(\rho,d)g(\rho,\varsigma).$$

Proof. Let

$$u_1(\xi) = \xi\sigma + (1-\xi)\rho, \quad u_2(\xi) = (1-\xi)\sigma + \xi\rho,$$
$$v_1(\eta) = \eta\varsigma + (1-\eta)d, \quad v_2(\eta) = (1-\eta)\varsigma + \eta d,$$

for $\xi, \eta \in [0,1]$.

By the co-ordinated (g,h)-convexity of $\digamma$ on $[\sigma,\rho]\times[\varsigma,d]$, we have

$$\begin{aligned}
&16\digamma\left(\tfrac{\sigma+\rho}{2}, \tfrac{\varsigma+d}{2}\right) = 16\digamma\left(\tfrac{u_1(\xi)+u_2(\xi)}{2}, \tfrac{v_1(\eta)+v_2(\eta)}{2}\right)\\
&\le \digamma(u_1(\xi),v_1(\eta)) + \digamma(u_1(\xi),v_2(\eta)) + \digamma(u_2(\xi),v_1(\eta)) + \digamma(u_2(\xi),v_2(\eta))\\
&\quad + M_1(u_1(\xi),v_1(\eta),v_2(\eta)) + M_2(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta)) + M_3(u_1(\xi),u_2(\xi),v_1(\eta))\\
&\quad + M_4(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta)) + M_5(u_1(\xi),u_2(\xi),v_2(\eta)) + M_6(u_2(\xi),v_1(\eta),v_2(\eta)).
\end{aligned} \tag{19}$$

Integrating both sides of (19) over $[0,1]\times[0,1]$ with respect to ξ and η, we obtain

$$\begin{aligned}
16\digamma\left(\tfrac{\sigma+\rho}{2}, \tfrac{\varsigma+d}{2}\right) &\le \int_0^1\int_0^1 \digamma(u_1(\xi),v_1(\eta)) + \digamma(u_1(\xi),v_2(\eta)) + \digamma(u_2(\xi),v_1(\eta))\\
&\quad + \digamma(u_2(\xi),v_2(\eta)) + M_1(u_1(\xi),v_1(\eta),v_2(\eta)) + M_2(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta))\\
&\quad + M_3(u_1(\xi),u_2(\xi),v_1(\eta)) + M_4(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta))\\
&\quad + M_5(u_1(\xi),u_2(\xi),v_2(\eta)) + M_6(u_2(\xi),v_1(\eta),v_2(\eta))\,d\eta\,d\xi.
\end{aligned} \tag{20}$$

We consider

$$\begin{aligned}
\int_0^1\int_0^1 \digamma(u_1(\xi),v_1(\eta))\,d\eta\,d\xi &= \int_0^1\int_0^1 \digamma(\xi\sigma+(1-\xi)\rho, \eta\varsigma+(1-\eta)d)\,d\eta\,d\xi\\
&= \frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d \digamma(\varkappa,\gamma)\,d\gamma\,d\varkappa\\
&= \int_0^1\int_0^1 \digamma(u_1(\xi),v_2(\eta))\,d\eta\,d\xi\\
&= \int_0^1\int_0^1 \digamma(u_2(\xi),v_1(\eta))\,d\eta\,d\xi\\
&= \int_0^1\int_0^1 \digamma(u_2(\xi),v_2(\eta))\,d\eta\,d\xi.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int_0^1\int_0^1 M_1(u_1(\xi),v_1(\eta),v_2(\eta))\,d\eta\,d\xi &= \int_0^1\int_0^1 h(u_1(\xi),v_2(\eta))g(u_1(\xi),v_1(\eta))\,d\eta\,d\xi\\
&\quad + \int_0^1\int_0^1 h(u_1(\xi),v_1(\eta))g(u_1(\xi),v_2(\eta))\,d\eta\,d\xi\\
&= \int_0^1\int_0^1 h(\xi\sigma+(1-\xi)\rho,(1-\eta)\varsigma+\eta d)g(\xi\sigma+(1-\xi)\rho,\eta\varsigma+(1-\eta)d)\,d\eta\,d\xi\\
&\quad + \int_0^1\int_0^1 h(\xi\sigma+(1-\xi)\rho,\eta\varsigma+(1-\eta)d)g(\xi\sigma+(1-\xi)\rho,(1-\eta)\varsigma+\eta d)\,d\eta\,d\xi\\
&= \frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\varkappa,\varsigma+d-\gamma)\,d\gamma\,d\varkappa.
\end{aligned}$$

Similarly, we get

$$\int_0^1\int_0^1 M_2(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta))\,d\eta\,d\xi=\frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\sigma+\rho-\varkappa,\varsigma+d-\gamma)\,d\gamma\,d\varkappa,$$

$$\int_0^1\int_0^1 M_3(u_1(\xi),u_2(\xi),v_1(\eta))\,d\eta\,d\xi=\frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\sigma+\rho-\varkappa,\varsigma+d-\gamma)\,d\gamma\,d\varkappa,$$

$$\int_0^1\int_0^1 M_4(u_1(\xi),u_2(\xi),v_1(\eta),v_2(\eta))\,d\eta\,d\xi=\frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\sigma+\rho-\varkappa,\varsigma+d-\gamma)\,d\gamma\,d\varkappa,$$

$$\int_0^1\int_0^1 M_5(u_1(\xi),u_2(\xi),v_2(\eta))\,d\eta\,d\xi=\frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\sigma+\rho-\varkappa,\gamma)\,d\gamma\,d\varkappa,$$

and

$$\int_0^1\int_0^1 M_5(u_2(\xi),v_1(\eta),v_2(\eta))\,d\eta\,d\xi=\frac{2}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)g(\varkappa,\varsigma+d-\gamma)\,d\gamma\,d\varkappa.$$

Thus, (20) becomes

$$\begin{aligned}4F\left(\frac{\sigma+\rho}{2},\frac{\varsigma+d}{2}\right)\leq{}&\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(\varkappa,\gamma)\,d\gamma\,d\varkappa\\&+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d h(\varkappa,\gamma)[g(\varkappa,\varsigma+d-\gamma)\\&+g(\sigma+\rho-\varkappa,\gamma)+g(\sigma+\rho-\varkappa,\varsigma+d-\gamma)]\,d\gamma\,d\varkappa.\end{aligned}$$

Therefore, the first inequality of (18) is proven.
Since F is co-ordinated (g,h)-convex on $[\sigma,\rho]\times[\varsigma,d]$, we have

$$\begin{aligned}&F(\xi\sigma+(1-\xi)\rho,\eta\varsigma+(1-\eta)d)\\&\leq\xi^2\eta^2F(\sigma,\varsigma)+\xi^2(1-\eta)^2F(\sigma,d)+(1-\xi)^2\eta^2F(\rho,\varsigma)+(1-\xi)^2(1-\eta)^2F(\rho,d)\\&\quad+\xi^2\eta(1-\eta)M_1(\sigma,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)M_2(\sigma,\rho,\varsigma,d)+\xi\eta^2(1-\xi)M_3(\sigma,\rho,\varsigma)\\&\quad+\xi\eta(1-\xi)(1-\eta)M_4(\sigma,\rho,\varsigma,d)+\xi(1-\xi)(1-\eta)^2M_5(\sigma,\rho,d)+\eta(1-\xi)^2(1-\eta)M_6(\rho,\varsigma,d)\\&=\xi^2\eta^2F(\sigma,\varsigma)+\xi^2(1-\eta)^2F(\sigma,d)+(1-\xi)^2\eta^2F(\rho,\varsigma)+(1-\xi)^2(1-\eta)^2F(\rho,d)\\&\quad+\xi^2\eta(1-\eta)K(\sigma,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)L(\sigma,\rho,\varsigma,d)+\xi\eta^2(1-\xi)M(\sigma,\rho,\varsigma)\\&\quad+\xi\eta(1-\xi)(1-\eta)N(\sigma,\rho,\varsigma,d)+\xi(1-\xi)(1-\eta)^2O(\sigma,\rho,d)+\eta(1-\xi)^2(1-\eta)P(\rho,\varsigma,d),\end{aligned}\tag{21}$$

$$\begin{aligned}&F(\xi\sigma+(1-\xi)\rho,(1-\eta)\varsigma+\eta d)\\&\leq\xi^2\eta^2F(\sigma,d)+\xi^2(1-\eta)^2F(\sigma,\varsigma)+(1-\xi)^2\eta^2F(\rho,d)+(1-\xi)^2(1-\eta)^2F(\rho,\varsigma)\\&\quad+\xi^2\eta(1-\eta)M_1(\sigma,d,\varsigma)+\xi\eta(1-\xi)(1-\eta)M_2(\sigma,\rho,d,\varsigma)+\xi\eta^2(1-\xi)M_3(\sigma,\rho,d)\\&\quad+\xi\eta(1-\xi)(1-\eta)M_4(\sigma,\rho,d,\varsigma)+\xi(1-\xi)(1-\eta)^2M_5(\sigma,\rho,\varsigma)+\eta(1-\xi)^2(1-\eta)M_6(\rho,d,\varsigma)\\&=\xi^2\eta^2F(\sigma,d)+\xi^2(1-\eta)^2F(\sigma,\varsigma)+(1-\xi)^2\eta^2F(\rho,d)+(1-\xi)^2(1-\eta)^2F(\rho,\varsigma)\\&\quad+\xi^2\eta(1-\eta)K(\sigma,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)N(\sigma,\rho,\varsigma,d)+\xi\eta^2(1-\xi)O(\sigma,\rho,d)\\&\quad+\xi\eta(1-\xi)(1-\eta)L(\sigma,\rho,\varsigma,d)+\xi(1-\xi)(1-\eta)^2M(\sigma,\rho,\varsigma)+\eta(1-\xi)^2(1-\eta)P(\rho,\varsigma,d),\end{aligned}\tag{22}$$

$$\begin{aligned}
&F((1-\xi)\sigma+\xi\rho,\eta\varsigma+(1-\eta)d)\\
&\leq \xi^2\eta^2F(\rho,\varsigma)+\xi^2(1-\eta)^2F(\rho,d)+(1-\xi)^2\eta^2F(\sigma,\varsigma)+(1-\xi)^2(1-\eta)^2F(\sigma,d)\\
&\quad+\xi^2\eta(1-\eta)M_1(\rho,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)M_2(\rho,\sigma,\varsigma,d)+\xi\eta^2(1-\xi)M_3(\rho,\sigma,\varsigma)\\
&\quad+\xi\eta(1-\xi)(1-\eta)M_4(\rho,\sigma,\varsigma,d)+\xi(1-\xi)(1-\eta)^2M_5(\rho,\sigma,d)+\eta(1-\xi)^2(1-\eta)M_6(\sigma,\varsigma,d)\\
&=\xi^2\eta^2F(\rho,\varsigma)+\xi^2(1-\eta)^2F(\rho,d)+(1-\xi)^2\eta^2F(\sigma,\varsigma)+(1-\xi)^2(1-\eta)^2F(\sigma,d)\\
&\quad+\xi^2\eta(1-\eta)P(\rho,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)N(\sigma,\rho,\varsigma,d)+\xi\eta^2(1-\xi)M(\sigma,\rho,\varsigma)\\
&\quad+\xi\eta(1-\xi)(1-\eta)L(\sigma,\rho,\varsigma,d)+\xi(1-\xi)(1-\eta)^2O(\sigma,\rho,d)+\eta(1-\xi)^2(1-\eta)K(\sigma,\varsigma,d),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&F((1-\xi)\sigma+\xi\rho,(1-\eta)\varsigma+\eta d)\\
&\leq \xi^2\eta^2F(\rho,d)+\xi^2(1-\eta)^2F(\rho,\varsigma)+(1-\xi)^2\eta^2F(\sigma,d)+(1-\xi)^2(1-\eta)^2F(\sigma,\varsigma)\\
&\quad+\xi^2\eta(1-\eta)M_1(\rho,d,\varsigma)+\xi\eta(1-\xi)(1-\eta)M_2(\rho,\sigma,d,\varsigma)+\xi\eta^2(1-\xi)M_3(\rho,\sigma,d)\\
&\quad+\xi\eta(1-\xi)(1-\eta)M_4(\rho,\sigma,d,\varsigma)+\xi(1-\xi)(1-\eta)^2M_5(\rho,\sigma,\varsigma)+\eta(1-\xi)^2(1-\eta)M_6(\sigma,d,\varsigma)\\
&=\xi^2\eta^2F(\rho,d)+\xi^2(1-\eta)^2F(\rho,\varsigma)+(1-\xi)^2\eta^2F(\sigma,d)+(1-\xi)^2(1-\eta)^2F(\sigma,\varsigma)\\
&\quad+\xi^2\eta(1-\eta)P(\rho,\varsigma,d)+\xi\eta(1-\xi)(1-\eta)L(\sigma,\rho,\varsigma,d)+\xi\eta^2(1-\xi)O(\sigma,\rho,d)\\
&\quad+\xi\eta(1-\xi)(1-\eta)N(\sigma,\rho,\varsigma,d)+\xi(1-\xi)(1-\eta)^2M(\sigma,\rho,\varsigma)+\eta(1-\xi)^2(1-\eta)K(\sigma,\varsigma,d).
\end{aligned} \tag{24}$$

Combining (21)–(24), we get

$$\begin{aligned}
&F(\xi\sigma+(1-\xi)\rho,\eta\varsigma+(1-\eta)d)+F(\xi\sigma+(1-\xi)\rho,(1-\eta)\varsigma+\eta d)\\
&\quad+F((1-\xi)\sigma+\xi\rho,\eta\varsigma+(1-\eta)d)+F((1-\xi)\sigma+\xi\rho,(1-\eta)\varsigma+\eta d)\\
&\leq(1-2\xi+2\xi^2)(1-2\eta+2\eta^2)[F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)]\\
&\quad+2\eta(1-\eta)(1-2\xi+2\xi^2)[K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)]\\
&\quad+2\xi(1-\xi)(1-2\eta+2\eta^2)[M(\sigma,\rho,\varsigma)+O(\sigma,\rho,d)]\\
&\quad+4\xi\eta(1-\xi)(1-\eta)[K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)].
\end{aligned} \tag{25}$$

Integrating both sides of (25) over $[0,1]\times[0,1]$ with respect to ξ and η, we obtain

$$\begin{aligned}
\frac{4}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(\varkappa,\gamma)\,d\gamma\,d\varkappa &= \int_0^1\int_0^1 F(\xi\sigma+(1-\xi)\rho,\eta\varsigma+(1-\eta)d)\,d\eta\,d\xi\\
&\quad+\int_0^1\int_0^1 F(\xi\sigma+(1-\xi)\rho,(1-\eta)\varsigma+\eta d)\,d\eta\,d\xi\\
&\quad+\int_0^1\int_0^1 F((1-\xi)\sigma+\xi\rho,\eta\varsigma+(1-\eta)d)\,d\eta\,d\xi\\
&\quad+\int_0^1\int_0^1 F((1-\xi)\sigma+\xi\rho,(1-\eta)\varsigma+\eta d)\,d\eta\,d\xi\\
&\leq[F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)]\\
&\quad\times\int_0^1\int_0^1(1-2\xi+2\xi^2)(1-2\eta+2\eta^2)\,d\eta\,d\xi\\
&\quad+[K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)]\int_0^1\int_0^1 2\eta(1-\eta)(1-2\xi+2\xi^2)\,d\eta\,d\xi
\end{aligned}$$

$$+ [M(\sigma,\rho,\varsigma)+O(\sigma,\rho,d)]\int_0^1\int_0^1 2\xi(1-\xi)(1-2\eta+2\eta^2)\,d\eta\,d\xi$$

$$+ [K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)]\int_0^1\int_0^1 4\xi\eta(1-\xi)(1-\eta)\,d\eta\,d\xi$$

$$= \frac{4}{9}\big[F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)\big]+\frac{2}{9}\big[K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)$$

$$+M(\sigma,\rho,\varsigma)+O(\sigma,\rho,d)\big]+\frac{1}{9}\big[K(\sigma,\varsigma,d)+P(\rho,\varsigma,d)\big].$$

Multiplying the above inequality by $\frac{1}{4}$, the second inequality of (18) is derived. □

Remark 3. *If we consider $h(\varkappa,\gamma)=F(\varkappa,\gamma)$ and $g(\varkappa,\gamma)=1$, then the inequalities (18) reduces to the inequalities*

$$F\left(\frac{\sigma+\rho}{2},\frac{\varsigma+d}{2}\right)\le\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(\varkappa,\gamma)\,d\gamma\,d\varkappa\le\frac{F(\sigma,\varsigma)+F(\sigma,d)+F(\rho,\varsigma)+F(\rho,d)}{4}.$$

This inequality is presented in Theorem 2.

4. Ostrowski-Type Inequalities

In this position, we derive some new Ostrowski-type inequalities for (g,h)-convex functions. To establish the inequalities of the current section, we receive help from a lemma.

Lemma 1 ([22]). *Let $F:\triangle\to\mathbb{R}$ be a twice-partial differentiable function on $\triangle^\circ$. If $\frac{\partial^2}{\partial\xi\,\partial\eta}\in L(\triangle)$, then*

$$\begin{aligned} F(\varkappa,\gamma) &+\tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\\ &=\tfrac{(\varkappa-\sigma)^2(\gamma-\varsigma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\,\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\,d\eta\,d\xi\\ &-\tfrac{(\varkappa-\sigma)^2(d-\gamma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\,\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)d)\,d\eta\,d\xi\\ &-\tfrac{(\rho-\varkappa)^2(\gamma-\varsigma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\,\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)\varsigma)\,d\eta\,d\xi\\ &+\tfrac{(\rho-\varkappa)^2(d-\gamma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\,\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)d)\,d\eta\,d\xi,\end{aligned}\tag{26}$$

for all $(\varkappa,\gamma)\in\triangle$, where A is defined in (7).

Theorem 9. *Under the assumptions of Lemma 1, if $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|$ is co-ordinated (g,h)-convex on $\triangle$, then*

$$\begin{aligned}&\left|F(\varkappa,\gamma)+\tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\right|\le\tfrac{1}{144(\rho-\sigma)(d-\varsigma)}\\ &\times\Bigg\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|\\ &+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\sigma,\gamma,\varsigma)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,\varsigma)+O(\varkappa,\sigma,\varsigma)+P(\sigma,\gamma,\varsigma)\Bigg]\\ &+(\varkappa-\sigma)^2(d-\gamma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|\end{aligned}\tag{27}$$

$$+3K(\varkappa,\gamma,d)+L(\varkappa,\sigma,\gamma,d)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,d)+O(\varkappa,\sigma,d)+P(\sigma,\gamma,d)]$$

$$+(\rho-\varkappa)^2(\gamma-\varsigma)^2\Big[9\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\Big|+3\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\Big|+3\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\Big|+\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\Big|$$
$$+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,\varsigma)+O(\varkappa,\rho,\varsigma)+P(\rho,\gamma,\varsigma)]$$

$$+(\rho-\varkappa)^2(d-\gamma)^2\Big[9\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\Big|+3\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\Big|+3\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\Big|+\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\Big|$$
$$+3K(\varkappa,\gamma,d)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,d)+O(\varkappa,\rho,d)+P(\rho,\gamma,d)]\Big\},$$

where A is defined as in (7), and K, L, M, N, O, P are defined as in Theorem 8.

Proof. The conclusion of Lemma 1 yields

$$\begin{aligned}
&\Big|F(\varkappa,\gamma)+\tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\Big|\\
&\le \tfrac{(\varkappa-\sigma)^2(\gamma-\varsigma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi\\
&\quad+\tfrac{(\varkappa-\sigma)^2(d-\gamma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi\\
&\quad+\tfrac{(\rho-\varkappa)^2(\gamma-\varsigma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi\\
&\quad+\tfrac{(\rho-\varkappa)^2(d-\gamma)^2}{(\rho-\sigma)(d-\varsigma)}\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi.
\end{aligned}\tag{28}$$

By the co-ordinated (g,h)-convexity of $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|$, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi\\
&\le\int_0^1\int_0^1\xi\eta\Big[\xi^2\eta^2\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\Big|+\xi^2(1-\eta)^2\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\Big|\\
&\quad+(1-\xi)^2\eta^2\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\Big|+(1-\xi)^2(1-\eta)^2\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\Big|\\
&\quad+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,\varsigma)+\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,\sigma,\gamma,\varsigma)\\
&\quad+\xi\eta^2(1-\xi)M_3(\varkappa,\sigma,\gamma)+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,\sigma,\gamma,\varsigma)\\
&\quad+\xi(1-\xi)(1-\eta)^2M_5(\varkappa,\sigma,\varsigma)+\eta(1-\xi)^2(1-\eta)M_6(\sigma,\gamma,\varsigma)\Big]\,d\eta\,d\xi\\
&=\tfrac{1}{16}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\Big|+\tfrac{1}{48}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\Big|+\tfrac{1}{48}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\Big|+\tfrac{1}{144}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\Big|\\
&\quad+\tfrac{1}{48}K(\varkappa,\gamma,\varsigma)+\tfrac{1}{144}L(\varkappa,\sigma,\gamma,\varsigma)+\tfrac{1}{48}M(\varkappa,\sigma,\gamma)\\
&\quad+\tfrac{1}{144}N(\varkappa,\sigma,\gamma,\varsigma)+\tfrac{1}{144}O(\varkappa,\sigma,\varsigma)+\tfrac{1}{144}P(\sigma,\gamma,\varsigma).
\end{aligned}\tag{29}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1\xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi\\
&\le\tfrac{1}{16}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\Big|+\tfrac{1}{48}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\Big|+\tfrac{1}{48}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\Big|+\tfrac{1}{144}\Big|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\Big|
\end{aligned}\tag{30}$$

$$+\tfrac{1}{48}K(\varkappa,\gamma,d)+\tfrac{1}{144}L(\varkappa,\sigma,\gamma,d)+\tfrac{1}{48}M(\varkappa,\sigma,\gamma)$$
$$+\tfrac{1}{144}N(\varkappa,\sigma,\gamma,d)+\tfrac{1}{144}O(\varkappa,\sigma,d)+\tfrac{1}{144}P(\sigma,\gamma,d),$$

$$\begin{aligned}\int_0^1\int_0^1 \quad & \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi \\ & \le \tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|+\tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right| \\ & +\tfrac{1}{48}K(\varkappa,\gamma,\varsigma)+\tfrac{1}{144}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{48}M(\varkappa,\rho,\gamma) \\ & +\tfrac{1}{144}N(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{144}O(\varkappa,\rho,\varsigma)+\tfrac{1}{144}P(\rho,\gamma,\varsigma),\end{aligned} \tag{31}$$

and

$$\begin{aligned}\int_0^1\int_0^1 \quad & \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi \\ & \le \tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|+\tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right| \\ & +\tfrac{1}{48}K(\varkappa,\gamma,d)+\tfrac{1}{144}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{48}M(\varkappa,\rho,\gamma) \\ & +\tfrac{1}{144}N(\varkappa,\rho,\gamma,d)+\tfrac{1}{144}O(\varkappa,\rho,d)+\tfrac{1}{144}P(\rho,\gamma,d).\end{aligned} \tag{32}$$

Substituting the inequalities (29)–(32) in the inequality (28), we obtain

$$\begin{aligned}&\left|F(\varkappa,\gamma)+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\right|\le\frac{1}{144(\rho-\sigma)(d-\varsigma)} \\ &\times\left\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\left[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|\right.\right. \\ &+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\sigma,\gamma,\varsigma)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,\varsigma)+O(\varkappa,\sigma,\varsigma)+P(\sigma,\gamma,\varsigma)] \\ &+(\varkappa-\sigma)^2(d-\gamma)^2\left[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|\right. \\ &+3K(\varkappa,\gamma,d)+L(\varkappa,\sigma,\gamma,d)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,d)+O(\varkappa,\sigma,d)+P(\sigma,\gamma,d)] \\ &+(\rho-\varkappa)^2(\gamma-\varsigma)^2\left[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|\right. \\ &+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,\varsigma)+O(\varkappa,\rho,\varsigma)+P(\rho,\gamma,\varsigma)] \\ &+(\rho-\varkappa)^2(d-\gamma)^2\left[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|\right. \\ &\left.+3K(\varkappa,\gamma,d)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,d)+O(\varkappa,\rho,d)+P(\rho,\gamma,d)]\right\}.\end{aligned}$$

The proof is completed. □

Remark 4. *Set* $h(\varkappa,\gamma)=\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|$, $g(\varkappa,\gamma)=1$ *and* $\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|\le M$ *for all* $(\varkappa,\gamma)\in\triangle$. *In that case, the inequality (28) reduces to the inequality (6).*

Theorem 10. *Under the assumptions of Lemma 1, if* $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|^q$ *is co-ordinated* (g,h)*-convex on* $\triangle$ *for* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p,q>1$*, then*

$$
\begin{aligned}
&\left|F(\varkappa,\gamma)+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\right|\le\frac{1}{(\rho-\sigma)(d-\varsigma)(p+1)^{2/p}}\left(\frac{1}{36}\right)^{1/q}\\
&\times\Bigg\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q\\
&+2K(\varkappa,\gamma,\varsigma)+L(\varkappa,\sigma,\gamma,\varsigma)+2M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,\varsigma)+2O(\varkappa,\sigma,\varsigma)+2P(\sigma,\gamma,\varsigma)\Bigg]^{1/q}\\
&+(\varkappa-\sigma)^2(d-\gamma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q\\
&+2K(\varkappa,\gamma,d)+L(\varkappa,\sigma,\gamma,d)+2M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,d)+2O(\varkappa,\sigma,d)+2P(\sigma,\gamma,d)\Bigg]^{1/q}\\
&+(\rho-\varkappa)^2(\gamma-\varsigma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q\\
&+2K(\varkappa,\gamma,\varsigma)+L(\varkappa,\rho,\gamma,\varsigma)+2M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,\varsigma)+2O(\varkappa,\rho,\varsigma)+2P(\rho,\gamma,\varsigma)\Bigg]^{1/q}\\
&+(\rho-\varkappa)^2(d-\gamma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q\\
&+2K(\varkappa,\gamma,d)+L(\varkappa,\rho,\gamma,\varsigma)+2M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,d)+2O(\varkappa,\rho,d)+2P(\rho,\gamma,d)\Bigg]^{1/q}\Bigg\},
\end{aligned}
\tag{33}
$$

where A is defined as in (7) and K, L, M, N, O, P are defined as in Theorem 8.

Proof. By the Hölder inequality for double integrals and co-ordinated (g,h)-convexity of $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|^q$, with the help of Lemma 1, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi\\
&\le\left(\int_0^1\int_0^1\xi^p\eta^p\,d\eta\,d\xi\right)^{1/p}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right|^q d\eta\,d\xi\right)^{1/q}\\
&\le\frac{1}{(p+1)^{2/p}}\left(\int_0^1\int_0^1\xi^2\eta^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\xi^2(1-\eta)^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q\right.\\
&\quad+(1-\xi)^2\eta^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+(1-\xi)^2(1-\eta)^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q\\
&+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,\varsigma)+\xi\eta(1-\xi)(1-\eta)M_2(\varkappa,\sigma,\gamma,\varsigma)+\xi\eta^2(1-\xi)M_3(\varkappa,\sigma,\gamma)\\
&\left.+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,\sigma,\gamma,\varsigma)+\xi(1-\xi)(1-\eta)^2M_5(\varkappa,\sigma,\varsigma)+\eta(1-\xi)^2(1-\eta)M_6(\sigma,\gamma,\varsigma)\,d\eta\,d\xi\right)^{1/q}\\
&=\frac{1}{(p+1)^{2/p}}\left(\frac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\frac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+\frac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+\frac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q\right.\\
&+\frac{1}{18}K(\varkappa,\gamma,\varsigma)+\frac{1}{36}L(\varkappa,\sigma,\gamma,\varsigma)+\frac{1}{18}M(\varkappa,\sigma,\gamma)\\
&\left.+\frac{1}{36}N(\varkappa,\sigma,\gamma,\varsigma)+\frac{1}{18}O(\varkappa,\sigma,\varsigma)+\frac{1}{18}P(\sigma,\gamma,\varsigma)\right)^{1/q}.
\end{aligned}
\tag{34}
$$

Similarly, we obtain

$$\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi$$
$$\le \tfrac{1}{(p+1)^{2/p}}\Big(\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q$$
$$+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q+\tfrac{1}{18}K(\varkappa,\gamma,d)+\tfrac{1}{36}L(\varkappa,\sigma,\gamma,d)+\tfrac{1}{18}M(\varkappa,\sigma,\gamma)$$
$$+\tfrac{1}{36}N(\varkappa,\sigma,\gamma,d)+\tfrac{1}{18}O(\varkappa,\sigma,d)+\tfrac{1}{18}P(\sigma,\gamma,d)\Big)^{1/q}, \tag{35}$$

$$\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi$$
$$\le \tfrac{1}{(p+1)^{2/p}}\Big(\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q$$
$$+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q+\tfrac{1}{18}K(\varkappa,\gamma,\varsigma)+\tfrac{1}{36}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{18}M(\varkappa,\rho,\gamma)$$
$$+\tfrac{1}{36}N(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{18}O(\varkappa,\rho,\varsigma)+\tfrac{1}{18}P(\rho,\gamma,\varsigma)\Big)^{1/q}, \tag{36}$$

and

$$\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi$$
$$\le \tfrac{1}{(p+1)^{2/p}}\Big(\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q$$
$$+\tfrac{1}{9}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q+\tfrac{1}{18}K(\varkappa,\gamma,d)+\tfrac{1}{36}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{18}M(\varkappa,\rho,\gamma)$$
$$+\tfrac{1}{36}N(\varkappa,\rho,\gamma,d)+\tfrac{1}{18}O(\varkappa,\rho,d)+\tfrac{1}{18}P(\rho,\gamma,d)\Big)^{1/q}. \tag{37}$$

If we substitute the inequalities (35)–(37) in the inequality (28), we obtain

$$\left|F(\varkappa,\gamma)+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\right|\le\frac{1}{(\rho-\sigma)(d-\varsigma)(p+1)^{2/p}}\left(\frac{1}{36}\right)^{1/q}$$
$$\times\Bigg\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q$$
$$+2K(\varkappa,\gamma,\varsigma)+L(\varkappa,\sigma,\gamma,\varsigma)+2M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,\varsigma)+2O(\varkappa,\sigma,\varsigma)+2P(\sigma,\gamma,\varsigma)\Bigg]^{1/q}$$
$$+(\varkappa-\sigma)^2(d-\gamma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q$$
$$+2K(\varkappa,\gamma,d)+L(\varkappa,\sigma,\gamma,d)+2M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,d)+2O(\varkappa,\sigma,d)+2P(\sigma,\gamma,d)\Bigg]^{1/q}$$
$$+(\rho-\varkappa)^2(\gamma-\varsigma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q$$
$$+2K(\varkappa,\gamma,\varsigma)+L(\varkappa,\rho,\gamma,\varsigma)+2M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,\varsigma)+2O(\varkappa,\rho,\varsigma)+2P(\rho,\gamma,\varsigma)\Bigg]^{1/q}$$
$$+(\rho-\varkappa)^2(d-\gamma)^2\Bigg[4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+4\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q$$
$$+2K(\varkappa,\gamma,d)+L(\varkappa,\rho,\gamma,\varsigma)+2M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,d)+2O(\varkappa,\rho,d)+2P(\rho,\gamma,d)\Bigg]^{1/q}\Bigg\}.$$

The proof is ended. □

Remark 5. *Set* $h(\varkappa,\gamma) = \left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q$, $g(\varkappa,\gamma) = 1$ *and* $\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right| \leq M$ *for all* $(\varkappa,\gamma) \in \triangle$. *Then, the inequality (33) reduces to the inequality (8).*

Theorem 11. *Under the assumptions of Lemma 1, if* $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|^q$ *is co-ordinated* (g,h)*-convex on* $\triangle$ *for* $q \geq 1$, *then*

$$\begin{aligned}
&\left|F(\varkappa,\gamma) + \tfrac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du - A\right| \leq \tfrac{1}{4(\rho-\sigma)(d-\varsigma)}\left(\tfrac{1}{36}\right)^{1/q} \\
&\times\Bigg\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q + \left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q \\
&+3K(\varkappa,\gamma,\varsigma) + L(\varkappa,\sigma,\gamma,\varsigma) + 3M(\varkappa,\sigma,\gamma) + N(\varkappa,\sigma,\gamma,\varsigma) + O(\varkappa,\sigma,\varsigma) + P(\sigma,\gamma,\varsigma)\Bigg]^{1/q} \\
&+(\varkappa-\sigma)^2(d-\gamma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q + \left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q \\
&+3K(\varkappa,\gamma,d) + L(\varkappa,\sigma,\gamma,d) + 3M(\varkappa,\sigma,\gamma) + N(\varkappa,\sigma,\gamma,d) + O(\varkappa,\sigma,d) + P(\sigma,\gamma,d)\Bigg]^{1/q} \\
&+(\rho-\varkappa)^2(\gamma-\varsigma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q + \left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q \\
&+3K(\varkappa,\gamma,\varsigma) + L(\varkappa,\rho,\gamma,\varsigma) + 3M(\varkappa,\rho,\gamma) + N(\varkappa,\rho,\gamma,\varsigma) + O(\varkappa,\rho,\varsigma) + P(\rho,\gamma,\varsigma)\Bigg]^{1/q} \\
&+(\rho-\varkappa)^2(d-\gamma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q + 3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q + \left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q \\
&+3K(\varkappa,\gamma,d) + L(\varkappa,\rho,\gamma,\varsigma) + 3M(\varkappa,\rho,\gamma) + N(\varkappa,\rho,\gamma,d) + O(\varkappa,\rho,d) + P(\rho,\gamma,d)\Bigg]^{1/q}\Bigg\},
\end{aligned} \tag{38}$$

where A *is defined as in* (7) *and* K, L, M, N, O, P *are defined as in Theorem 8.*

Proof. By vitue of the power mean inequality in relation to double integrals and by the co-ordinated (g,h)-convexity of $\left|\frac{\partial^2 F}{\partial\xi\,\partial\eta}\right|^q$, with the help of Lemma 1, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right| d\eta\,d\xi \\
&\leq \left(\int_0^1\int_0^1 \xi\eta\,d\eta\,d\xi\right)^{1-1/q}\left(\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)\varsigma)\right|^q d\eta\,d\xi\right)^{1/q} \\
&\leq \left(\tfrac{1}{4}\right)^{1-1/q}\Bigg(\int_0^1\int_0^1 \xi\eta\Bigg[\xi^2\eta^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + \xi^2(1-\eta)^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q \\
&\quad+(1-\xi)^2\eta^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q + (1-\xi)^2(1-\eta)^2\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q \\
&\quad+\xi^2\eta(1-\eta)M_1(\varkappa,\gamma,\varsigma) + \xi\eta(1-\xi)(1-\eta)M_2(\varkappa,\sigma,\gamma,\varsigma) + \xi\eta^2(1-\xi)M_3(\varkappa,\sigma,\gamma) \\
&\quad+\xi\eta(1-\xi)(1-\eta)M_4(\varkappa,\sigma,\gamma,\varsigma) + \xi(1-\xi)(1-\eta)^2M_5(\varkappa,\sigma,\varsigma) + \eta(1-\xi)^2(1-\eta)M_6(\sigma,\gamma,\varsigma)\Bigg]\,d\eta\,d\xi\Bigg)^{1/q} \\
&= \left(\tfrac{1}{4}\right)^{1-1/q}\Bigg(\tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q + \tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q + \tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q + \tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q \\
&\quad+\tfrac{1}{48}K(\varkappa,\gamma,\varsigma) + \tfrac{1}{144}L(\varkappa,\sigma,\gamma,\varsigma) + \tfrac{1}{48}M(\varkappa,\sigma,\gamma) \\
&\quad+\tfrac{1}{144}N(\varkappa,\sigma,\gamma,\varsigma) + \tfrac{1}{144}O(\varkappa,\sigma,\varsigma) + \tfrac{1}{144}P(\sigma,\gamma,\varsigma)\Bigg)^{1/q}.
\end{aligned} \tag{39}$$

In a similar way, one can obtain

$$\begin{aligned}
&\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\sigma,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi\\
&\leq\left(\tfrac{1}{4}\right)^{1-1/q}\Bigg(\tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+\tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q\\
&\quad+\tfrac{1}{48}K(\varkappa,\gamma,d)+\tfrac{1}{144}L(\varkappa,\sigma,\gamma,d)+\tfrac{1}{48}M(\varkappa,\sigma,\gamma)\\
&\quad+\tfrac{1}{144}N(\varkappa,\sigma,\gamma,d)+\tfrac{1}{144}O(\varkappa,\sigma,d)+\tfrac{1}{144}P(\sigma,\gamma,d)\Bigg)^{1/q},
\end{aligned}\tag{40}$$

$$\begin{aligned}
&\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)\varsigma)\right|d\eta\,d\xi\\
&\leq\left(\tfrac{1}{4}\right)^{1-1/q}\Bigg(\tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+\tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q\\
&\quad+\tfrac{1}{48}K(\varkappa,\gamma,\varsigma)+\tfrac{1}{144}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{48}M(\varkappa,\rho,\gamma)\\
&\quad+\tfrac{1}{144}N(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{144}O(\varkappa,\rho,\varsigma)+\tfrac{1}{144}P(\rho,\gamma,\varsigma)\Bigg)^{1/q},
\end{aligned}\tag{41}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 \xi\eta\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\xi\varkappa+(1-\xi)\rho,\eta\gamma+(1-\eta)d)\right|d\eta\,d\xi\\
&\leq\left(\tfrac{1}{4}\right)^{1-1/q}\Bigg(\tfrac{1}{16}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+\tfrac{1}{48}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+\tfrac{1}{144}\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q\\
&\quad+\tfrac{1}{48}K(\varkappa,\gamma,d)+\tfrac{1}{144}L(\varkappa,\rho,\gamma,\varsigma)+\tfrac{1}{48}M(\varkappa,\rho,\gamma)\\
&\quad+\tfrac{1}{144}N(\varkappa,\rho,\gamma,d)+\tfrac{1}{144}O(\varkappa,\rho,d)+\tfrac{1}{144}P(\rho,\gamma,d)\Bigg)^{1/q}.
\end{aligned}\tag{42}$$

By substituting the inequalities (39)–(42) in the inequality (28), we obtain

$$\begin{aligned}
&\left|F(\varkappa,\gamma)+\frac{1}{(\rho-\sigma)(d-\varsigma)}\int_\sigma^\rho\int_\varsigma^d F(u,v)\,dv\,du-A\right|\leq\frac{1}{4(\rho-\sigma)(d-\varsigma)}\left(\frac{1}{36}\right)^{1/q}\\
&\times\Bigg\{(\varkappa-\sigma)^2(\gamma-\varsigma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\varsigma)\right|^q\\
&+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\sigma,\gamma,\varsigma)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,\varsigma)+O(\varkappa,\sigma,\varsigma)+P(\sigma,\gamma,\varsigma)\Bigg]^{1/q}\\
&+(\varkappa-\sigma)^2(d-\gamma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,\gamma)\right|^q+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\sigma,d)\right|^q\\
&+3K(\varkappa,\gamma,d)+L(\varkappa,\sigma,\gamma,d)+3M(\varkappa,\sigma,\gamma)+N(\varkappa,\sigma,\gamma,d)+O(\varkappa,\sigma,d)+P(\sigma,\gamma,d)\Bigg]^{1/q}\\
&+(\rho-\varkappa)^2(\gamma-\varsigma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\varsigma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\varsigma)\right|^q\\
&+3K(\varkappa,\gamma,\varsigma)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,\varsigma)+O(\varkappa,\rho,\varsigma)+P(\rho,\gamma,\varsigma)\Bigg]^{1/q}\\
&+(\rho-\varkappa)^2(d-\gamma)^2\Bigg[9\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,d)\right|^q+3\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,\gamma)\right|^q+\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\rho,d)\right|^q
\end{aligned}$$

$$+\ 3K(\varkappa,\gamma,d)+L(\varkappa,\rho,\gamma,\varsigma)+3M(\varkappa,\rho,\gamma)+N(\varkappa,\rho,\gamma,d)+O(\varkappa,\rho,d)+P(\rho,\gamma,d)]^{1/q}\Big\}.$$

The proof is completed. □

Remark 6. *Set* $h(\varkappa,\gamma)=\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|^q$, $g(\varkappa,\gamma)=1$ *and* $\left|\frac{\partial^2}{\partial\xi\,\partial\eta}F(\varkappa,\gamma)\right|\leq M$ *for all* $(\varkappa,\gamma)\in\triangle$. *In that case, the inequality (38) reduces to (9).*

5. Examples

In the section of examples, we give some examples to demonstrate and confirm the consistency of our main findings.

Example 2. *Let* $g,h,F:[0,1]\times[0,1]\to\mathbb{R}$ *be given by* $g(\varkappa,\gamma)=\varkappa\gamma$, $h(\varkappa,\gamma)=\mathrm{e}^{-\varkappa-\gamma}$ *and* $F(\varkappa,\gamma)=\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}$. *By Example 1,* F *is co-ordinated* (g,h)*-convex. Applying Theorem 8, the first inequality of (18) is*

$$\begin{aligned}0.038126\ldots&=\frac{1}{\mathrm{e}}-2\left(\frac{-2}{\mathrm{e}}+1\right)\left(\frac{1}{\mathrm{e}}\right)-\left(\frac{1}{\mathrm{e}^2}\right)\\&=4F\left(\frac{1}{2},\frac{1}{2}\right)-\int_0^1\int_0^1\mathrm{e}^{-\varkappa-\gamma}[\varkappa(1-\gamma)+(1-\varkappa)\gamma+(1-\varkappa)(1-\gamma)]\,d\gamma\,d\varkappa\\&\leq\int_0^1\int_0^1\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}\,d\gamma\,d\varkappa=\frac{4}{\mathrm{e}^2}-\frac{4}{\mathrm{e}}+1=0.069823\ldots.\end{aligned}$$

Additionally, the second inequality of (18) is

$$\begin{aligned}0.069823\ldots&=\int_0^1\int_0^1\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}\,d\gamma\,d\varkappa\leq\frac{1}{9}\Big[F(0,0)+F(0,1)+F(1,0)+F(1,1)\\&+\frac{K(0,0,1)+P(1,0,1)+M(0,1,0)+O(0,1,1)}{2}+\frac{L(0,1,0,1)+N(0,1,0,1)}{4}\Big]\\&=\frac{1}{9}\left[\frac{1}{\mathrm{e}^2}+\frac{1}{\mathrm{e}}+\frac{1}{4}\right]=0.083690\ldots.\end{aligned}$$

It is clear that

$$0.038126\ldots\leq0.069823\ldots\leq0.083690\ldots.$$

This confirms the correctness of the result established in Theorem 8.

Example 3. *Let* $g,h,F:[0,1]\times[0,1]\to\mathbb{R}$ *be formulated by* $g(\varkappa,\gamma)=(1-\varkappa)(1-\gamma)$, $h(\varkappa,\gamma)=\mathrm{e}^{-\varkappa-\gamma}$ *and* $F(\varkappa,\gamma)=\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}$ *(see Figure 1). Then,* F *is partially differentiable on* $(0,1)\times(0,1)$*, and its partial derivative is integrable and co-ordinated* (g,h)*-convex on* $[0,1]\times[0,1]$*. By utilizing Theorem 9, we have*

$$\begin{aligned}&\left|\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}+\left(\frac{4}{\mathrm{e}^2}-\frac{4}{\mathrm{e}}+1\right)+\left(\varkappa\,\mathrm{e}^{-\varkappa}\right)\left(\frac{2}{\mathrm{e}}-1\right)+\left(\gamma\,\mathrm{e}^{-\gamma}\right)\left(\frac{2}{\mathrm{e}}-1\right)\right|\\&=\left|\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}+\int_0^1\int_0^1uv\,\mathrm{e}^{-u-v}\,dv\,du-\int_0^1u\gamma\,\mathrm{e}^{-u-\gamma}\,du-\int_0^1\varkappa v\,\mathrm{e}^{-\varkappa-v}\,dv\right|\\&\leq\frac{1}{144}\Big\{\varkappa^2\gamma^2[9(1-\varkappa)(1-\gamma)\,\mathrm{e}^{-\varkappa-\gamma}+3(1-\varkappa)\,\mathrm{e}^{-\varkappa}+3(1-\gamma)\,\mathrm{e}^{-\gamma}+1\\&\quad+3(1-\varkappa)(1-\gamma)\,\mathrm{e}^{-\varkappa}+3(1-\varkappa)\,\mathrm{e}^{-\varkappa-\gamma}+(1-\gamma)\,\mathrm{e}^{-\varkappa}+(1-\varkappa)\,\mathrm{e}^{-\gamma}\\&\quad+3(1-\gamma)\,\mathrm{e}^{-\varkappa-\gamma}+3(1-\varkappa)(1-\gamma)\,\mathrm{e}^{-\gamma}+\mathrm{e}^{-\varkappa-\gamma}+(1-\varkappa)(1-\gamma)\end{aligned}$$

$$+ e^{-\varkappa} + (1-\varkappa) + e^{-\gamma} + (1-\gamma)] + \varkappa^2(1-\gamma)^2\big[9(1-\varkappa)(1-\gamma)\,e^{-\varkappa-\gamma} + 3(1-\gamma)\,e^{-\gamma}$$
$$+ 3(1-\varkappa)(1-\gamma)\,e^{-\varkappa-1} + (1-\gamma)\,e^{-\varkappa-1} + 3(1-\gamma)\,e^{-\varkappa-\gamma} + 3(1-\varkappa)(1-\gamma)\,e^{-\gamma}$$
$$+(1-\varkappa)(1-\gamma)\,e^{-1} + (1-\gamma)\,e^{-1}\Big] + (1-\varkappa)^2\gamma^2\big[9(1-\varkappa)(1-\gamma)\,e^{-\varkappa-\gamma} + 3(1-\varkappa)\,e^{-\varkappa}$$
$$+ 3(1-\varkappa)(1-\gamma)\,e^{-\varkappa} + 3(1-\varkappa)\,e^{-\varkappa-\gamma} + (1-\varkappa)\,e^{-\gamma-1} + 3(1-\varkappa)(1-\gamma)\,e^{-\gamma-1}$$
$$+ (1-\varkappa)(1-\gamma)\,e^{-\gamma-1} + (1-\varkappa)(1-\gamma)\,e^{-1} + (1-\varkappa)\,e^{-1}\Big]$$
$$+ (1-\varkappa)^2(1-\gamma)^2\Big[9(1-\varkappa)(1-\gamma)\,e^{-\varkappa-\gamma} + 3(1-\varkappa)(1-\gamma)\,e^{-\varkappa-1} + (1-\gamma)\,e^{-\varkappa-1}$$
$$+3(1-\varkappa)(1-\gamma)\,e^{-\gamma-1}\Big]\Big\}.$$

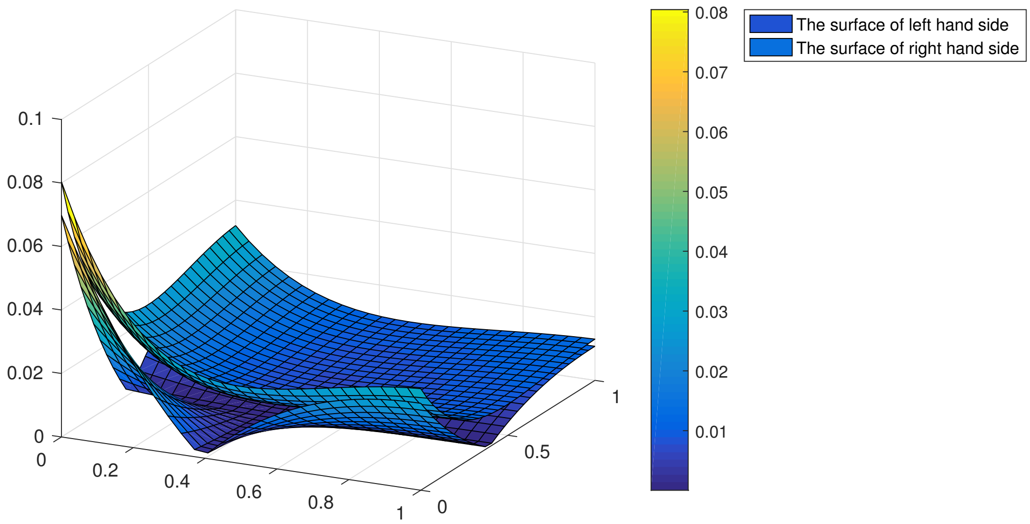

Figure 1. The image description for Theorem 9.

Example 4. *Let $g, h, \digamma : [0,1]\times[0,1] \to \mathbb{R}$ be defined by $g(\varkappa,\gamma) = (1-\varkappa)^2(1-\gamma)^2$, $h(\varkappa,\gamma) = e^{-2\varkappa-2\gamma}$ and $\digamma(\varkappa,\gamma) = \varkappa\gamma\,e^{-\varkappa-\gamma}$ (see Figure 2). Then, $\digamma$ is partially differentiable on $(0,1)\times(0,1)$, its partial derivative is integrable and $\left|\frac{\partial^2\digamma}{\partial\xi\,\partial\eta}\right|^2$ is co-ordinated (g,h)-convex on $[0,1]^2$. Via Theorem 10 with $p=q=2$, we have*

$$\left|\varkappa\gamma\,e^{-\varkappa-\gamma} + \left(\frac{4}{e^2} - \frac{4}{e} + 1\right) + (\varkappa\,e^{-\varkappa})\left(\frac{2}{e} - 1\right) + (\gamma\,e^{-\gamma})\left(\frac{2}{e} - 1\right)\right|$$
$$= \left|\varkappa\gamma\,e^{-\varkappa-\gamma} + \int_0^1\int_0^1 uv\,e^{-u-v}\,dv\,du - \int_0^1 u\gamma\,e^{-u-\gamma}\,du - \int_0^1 \varkappa v\,e^{-\varkappa-v}\,dv\right|$$
$$\le \frac{1}{18}\Big\{\varkappa^2\gamma^2\Big[4(1-\varkappa)^2(1-\gamma)^2\,e^{-2\varkappa-2\gamma} + 4(1-\varkappa)^2\,e^{-2\varkappa} + 4(1-\gamma)^2\,e^{-2\gamma} + 4$$
$$+ 2(1-\varkappa)^2(1-\gamma)^2\,e^{-2\varkappa} + 2(1-\varkappa)^2\,e^{-2\varkappa-2\gamma} + (1-\gamma)^2\,e^{-2\varkappa} + (1-\varkappa)^2\,e^{-2\gamma}$$
$$+ 2(1-\gamma)^2\,e^{-2\varkappa-2\gamma} + 2(1-\varkappa)^2(1-\gamma)^2\,e^{-2\gamma} + e^{-2\varkappa-2\gamma} + (1-\varkappa)^2(1-\gamma)^2$$
$$+ 2e^{-2\varkappa} + 2(1-\varkappa)^2 + 2e^{-2\gamma} + 2(1-\gamma)^2\Big]^{1/2} + \varkappa^2(1-\gamma)^2\Big[4(1-\varkappa)^2(1-\gamma)^2\,e^{-2\varkappa-2\gamma} + 4(1-\gamma)^2\,e^{-2\gamma}$$
$$+ 2(1-\varkappa)^2(1-\gamma)^2\,e^{-2\varkappa-2} + (1-\gamma)^2\,e^{-2\varkappa-2} + 2(1-\gamma)^2\,e^{-2\varkappa-2\gamma} + (1-\varkappa)^2(1-\gamma)^2\,e^{-2\gamma}$$

$$+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2}+2(1-\gamma)^2\,\mathrm{e}^{-2}\Big]^{1/2}+(1-\varkappa)^2\gamma^2\Big[4(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+4(1-\varkappa)^2\,\mathrm{e}^{-2\varkappa}$$
$$+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa}+2(1-\varkappa)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+(1-\varkappa)^2\,\mathrm{e}^{-2\gamma-2}+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\gamma-2}$$
$$+(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\gamma-2}+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2}+2(1-\varkappa)^2\,\mathrm{e}^{-2}\Big]^{1/2}$$
$$+(1-\varkappa)^2(1-\gamma)^2\Big[4(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2}+(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2}$$
$$+2(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\gamma-2}\Big]^{1/2}\Big\}.$$

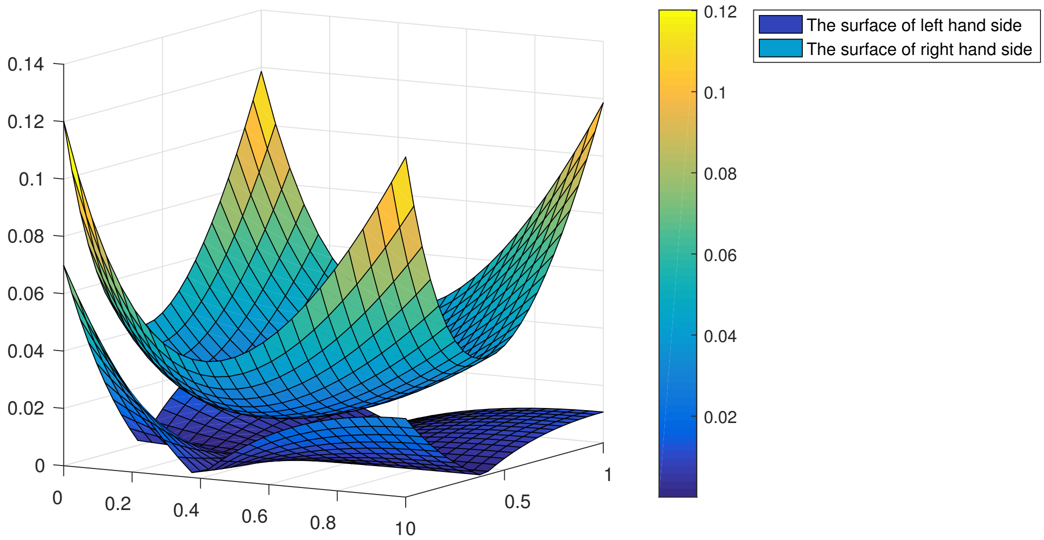

Figure 2. The image description for Theorem 10.

Example 5. *Consider the functions* $g, h, \digamma : [0,1]^2 \to \mathbb{R}$ *formulated as* $g(\varkappa,\gamma) = (1-\varkappa)(1-\gamma)$, $h(\varkappa,\gamma) = \mathrm{e}^{-\varkappa-\gamma}$ *and* $\digamma(\varkappa,\gamma) = \varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}$ *(see Figure 3). Then,* $\digamma$ *is partially differentiable on* $(0,1)\times(0,1)$*, its partial derivative is integrable and* $\left|\frac{\partial^2\digamma}{\partial\xi\,\partial\eta}\right|^2$ *is co-ordinated* (g,h)*-convex on* $[0,1]^2$*. The utilization of Theorem 11 with* $q=2$ *gives*

$$\left|\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}+\left(\frac{4}{\mathrm{e}^2}-\frac{4}{\mathrm{e}}+1\right)+\left(\varkappa\,\mathrm{e}^{-\varkappa}\right)\left(\frac{2}{\mathrm{e}}-1\right)+\left(\gamma\,\mathrm{e}^{-\gamma}\right)\left(\frac{2}{\mathrm{e}}-1\right)\right|$$
$$=\left|\varkappa\gamma\,\mathrm{e}^{-\varkappa-\gamma}+\int_0^1\!\!\int_0^1 uv\,\mathrm{e}^{-u-v}\,dv\,du-\int_0^1 u\gamma\,\mathrm{e}^{-u-\gamma}\,du-\int_0^1 \varkappa v\,\mathrm{e}^{-\varkappa-v}\,dv\right|$$
$$\le\frac{1}{24}\Big\{\varkappa^2\gamma^2\Big[9(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+3(1-\varkappa)^2\,\mathrm{e}^{-2\varkappa}+3(1-\gamma)^2\,\mathrm{e}^{-2\gamma}+1$$
$$+3(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa}+3(1-\varkappa)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+(1-\gamma)^2\,\mathrm{e}^{-2\varkappa}+(1-\varkappa)^2\,\mathrm{e}^{-2\gamma}$$
$$+3(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+3(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\gamma}+\mathrm{e}^{-2\varkappa-2\gamma}+(1-\varkappa)^2(1-\gamma)^2$$
$$+\mathrm{e}^{-2\varkappa}+(1-\varkappa)^2+\mathrm{e}^{-2\gamma}+(1-\gamma)^2\Big]^{1/2}+\varkappa^2(1-\gamma)^2\Big[9(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+3(1-\gamma)^2\,\mathrm{e}^{-2\gamma}$$
$$+3(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2}+(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2}+3(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+3(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\gamma}$$
$$+(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2}+(1-\gamma)^2\,\mathrm{e}^{-2}\Big]^{1/2}+(1-\varkappa)^2\gamma^2\Big[9(1-\varkappa)^2(1-\gamma)^2\,\mathrm{e}^{-2\varkappa-2\gamma}+3(1-\varkappa)^2\,\mathrm{e}^{-2\varkappa}$$

$$+ 3(1-\varkappa)^2(1-\gamma)^2 \, e^{-2\varkappa} + 3(1-\varkappa)^2 \, e^{-2\varkappa-2\gamma} + (1-\varkappa)^2 \, e^{-2\gamma-2} + 3(1-\varkappa)^2(1-\gamma)^2 \, e^{-2\gamma-2}$$

$$+ (1-\varkappa)^2(1-\gamma)^2 \, e^{-2\gamma-2} + (1-\varkappa)^2(1-\gamma)^2 \, e^{-2} + (1-\varkappa)^2 \, e^{-2}\Big]^{1/2}$$

$$+ (1-\varkappa)^2(1-\gamma)^2\Big[9(1-\varkappa)^2(1-\gamma)^2 \, e^{-2\varkappa-2\gamma} + 3(1-\varkappa)^2(1-\gamma)^2 \, e^{-2\varkappa-2} + (1-\gamma)^2 \, e^{-2\varkappa-2}$$

$$+3(1-\varkappa)^2(1-\gamma)^2 \, e^{-2\gamma-2}\Big]^{1/2}\Big\}.$$

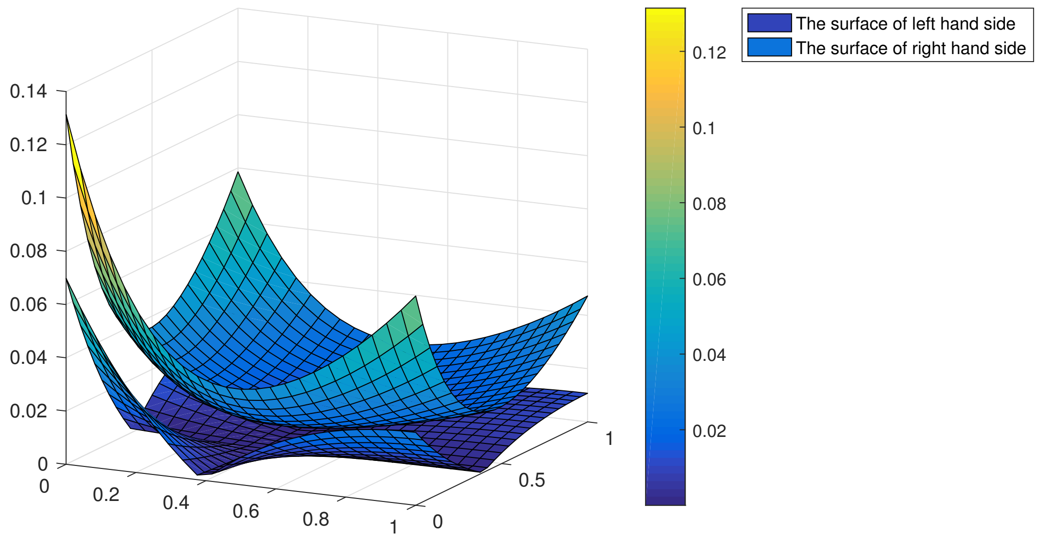

Figure 3. The image description for Theorem 11.

6. Conclusions

We gave a definition of newly defined co-ordinated (g, h)-convexity, which is a generalization of co-ordinated convexity. We also proved some of its significant properties. We established some new Hermite–Hadamard- and Ostrowski-type inequalities in relation to such co-ordinated (g, h)-convex functions. We established that the inequalities derived in this paper generalize the results given in earlier works. Lastly, we gave some examples to demonstrate and show the correctness of the main results. In the next works, one can extend similar theorems to other types of generalized convexity.

Author Contributions: Conceptualization, M.A.A., F.W., H.B.; formal analysis, M.A.A., F.W., H.B., S.E. and S.R.; funding acquisition, S.R.; methodology, M.A.A., F.W., H.B., S.E. and S.R.; software, M.A.A., F.W., H.B. and S.E. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated nor analyzed during the current study.

Acknowledgments: S.E. and S.R. would like to thank Azarbaijan Shahid Madani University. Additionally, the authors would like to thank dear reviewers for their constructive comments to improve the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ye, H.; Gao, J.; Ding, Y. A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1075–1081. [CrossRef]
2. Rezapour, S.; Ntouyas, S.K.; Amara, A.; Etemad, S.; Tariboon, J. Some existence and dependence criteria of solutions to a fractional integro-differential boundary value problem via the generalized Gronwall inequality. *Mathematics* **2021**, *9*, 1165. [CrossRef]
3. Rahman, G.; Ullah, Z.; Khan, A.; Set, E.; Nisar, K.S. Certain Chebyshev-type inequalities involving fractional conformable integral operators. *Mathematics* **2019**, *7*, 364. [CrossRef]
4. Alzabut, J.; Abdeljawad, T. A generalized discrete fractional Gronwall inequality and its application on the uniqueness of solutions for nonlinear delay fractional difference system. *Appl. Anal. Discr. Math.* **2018**, *12*, 36–48. [CrossRef]
5. Mohammadi, H.; Baleanu, D.; Etemad, S.; Rezapour, S. Criteria for existence of solutions for a Liouville–Caputo boundary value problem via generalized Gronwall's inequality. *J. Inequal. Appl.* **2021**, *2021*, 36. [CrossRef]
6. Wang, Y.; Liang, S.; Xia, C. A Lyapunov-type inequality for a fractional differential equation under Sturm-Liouville boundary conditions. *Math. Inequal. Appl.* **2017**, *20*, 139–148. [CrossRef]
7. Matar, M.M.; Abu Jarad, M.; Ahmad, M.; Zada, A.; Etemad, S.; Rezapour, S. On the existence and stability of two positive solutions of a hybrid differential system of arbitrary fractional order via Avery–Anderson–Henderson criterion on cones. *Adv. Differ. Equ.* **2021**, *2021*, 423. [CrossRef]
8. Mohammad, H.; Kumar, S.; Rezapour, S.; Etemad, S. A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. *Chaos Solitons Fractals* **2021**, *144*, 110668. [CrossRef]
9. Etemad, S.; Avcı, İ.; Kumar, P.; Baleanu, D.; Rezapour, S. Some novel mathematical analysis on the fractal–fractional model of the AH1N1/09 virus and its generalized Caputo-type version. *Chaos Solitons Fractals* **2022**, *162*, 112511. [CrossRef]
10. Najafi, H.; Etemad, S.; Patanarapeelert, N.; Asamoah, J.K.K.; Rezapour, S.; Sitthiwirattham, T. A study on dynamics of $CD4^{+}$ T-cells under the effect of HIV-1 infection based on a mathematical fractal-fractional model via the Adams-Bashforth scheme and Newton polynomials. *Mathematics* **2022**, *10*, 1366. [CrossRef]
11. Baleanu, D.; Etemad, S.; Rezapour, S. A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. *Bound. Value Probl.* **2020**, *2020*, 64. [CrossRef]
12. Hermite, C. Sur deux limites d'une integrale de finie. *Mathesis* **1883**, *3*, 82.
13. Hadamard, J. Etude sur les fonctions entiees et en particulier d'une fonction consideree par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
14. Ostrowski, A. Uber die absolutabweichung einer differentiebaren funcktion von iherem integralmittelwert. *Comment. Math. Helv.* **1938**, *10*, 226–227. [CrossRef]
15. Ozdemir, M.E.; Avci, M.; Set, E. On some inequalities of Hermite-Hadamard type via m-convexity. *Math. Lett.* **2010**, *23*, 1065–1070. [CrossRef]
16. Ozdemir, M.E.; Avci, M.; Kavurmaci, H. Hermite-Hadamard-type inequalities via (σ, m)-convexity. *Comput. Math. Appl.* **2011**, *61*, 2614–2620. [CrossRef]
17. Lü, Z. On sharp inequalities of Simpson type and Ostrowski type in two independent variables. *Comput. Math. Appl.* **2008**, *56*, 2043–2047. [CrossRef]
18. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
19. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are η-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
20. İşcan, İ.; Wu, S. Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals. *Appl. Math. Comput.* **2014**, *238*, 237–244. [CrossRef]
21. Dragomir, S.S. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwan. J. Math.* **2001**, *5*, 775–788. [CrossRef]
22. Latif, M.A.; Hussain, S.; Dragomir, S.S. New Ostrowski type inequalities for co-ordinated convex functions. *TJMM* **2012**, *4*, 125–136.
23. Samet, B. A convexity concept with respect to a pair of functions. *Numer. Funct. Anal. Optim.* **2022**, *43*, 522–540. [CrossRef]
24. Ali, M.A.; Soontharanon, J.; Budak, H.; Sitthiwirattham, T.; Fečkon, M. Fractional Hermite-Hadamard inequality and error estimates for Simpson's formula through convexity with respect to pair of functions. *Miskolc Math. Notes* 2022, *in press*.
25. Xie, J.; Ali, M.A.; Budak, H.; Fečkon, M.; Sitthiwirattham, T. Fractional Hermite-Hadamard inequality, Simpson's and Ostrowski's type inequalities for convex functions with respect to a pair of functions. *Rocky Mt. J. Math.* 2022, *in press*.