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Research Article

Some Parameterized Quantum Simpson's and Quantum Newton's Integral Inequalities via Quantum Differentiable Convex Mappings

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In this work, two generalized quantum integral identities are proved by using some parameters. By utilizing these equalities, we present several parameterized quantum inequalities for convex mappings. These quantum inequalities generalize many of the important inequalities that exist in the literature, such as quantum trapezoid inequalities, quantum Simpson's inequalities, and quantum Newton's inequalities. We also give some new midpoint-type inequalities as special cases. The results in this work naturally generalize the results for the Riemann integral.

1. Introduction

Thomas Simpson has developed crucial methods for the numerical integration and estimation of definite integrals considered as Simpson's rule during 1710–1761. Nevertheless, a similar approximation was used by J. Kepler almost one hundred years earlier, so it is also known as Kepler's rule. Simpson's rule includes the three-point Newton–Cotes quadrature rule, so estimation based on the three-step quadratic kernel is sometimes called Newton-type results.

(1) Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_{\kappa_1}^{\kappa_2} \mathscr{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{6}\left[\mathscr{F}(\kappa_1)+4\mathscr{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathscr{F}(\kappa_2)\right]. \quad (1)$$

(2) Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule):

$$\int_{\kappa_1}^{\kappa_2} \mathscr{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{8}\left[\mathscr{F}(\kappa_1)+3\mathscr{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathscr{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathscr{F}(\kappa_2)\right]. \quad (2)$$

There are a large number of estimations related to these quadrature rules in the literature; one of them is the following estimation known as Simpson's inequality.

Theorem 1. *Suppose that* $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ *is a four-time continuously differentiable mapping on* (κ_1,κ_2), *and let* $\|\mathcal{F}^{(4)}\|_\infty = \sup_{x\in(\kappa_1,\kappa_2)} |\mathcal{F}^{(4)}(x)| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} + 2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\mathrm{d}x\right| \le \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty (\kappa_2-\kappa_1)^4. \quad (3)$$

In recent years, many authors have focused on Simpson's type inequality in various categories of mappings. Specifically, some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map because convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. For example, Dragomir et al. [1] presented the new Simpson's inequalities and their applications in quadrature formulas for numerical integration. In addition, some inequalities of Simpson's type of s-convex functions were determined by Alomari et al. in [2]. Subsequently, Sarikaya et al. noted the variance of Simpson's type inequality based on convexity in [3]. For the further studies of this area, one can consult [4–6].

On the contrary, many well-known integral inequalities have been studied in the setup of q-calculus using the concept of classical convexity [7–28].

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation (see [29]):

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad q \in (0,1). \quad (4)$$

Jackson [30] defined the q-integral of a given function $\mathcal{F}$ from 0 to κ_2 as follows:

$$\int_0^{\kappa_2}\mathcal{F}(x)\mathrm{d}_q x = (1-q)\kappa_2\sum_{n=0}^{\infty} q^n\mathcal{F}(\kappa_2 q^n), \quad \text{where } 0<q<1, \quad (5)$$

provided that the sum converges absolutely. Moreover, he defined the q-integral of a given function over the interval $[\kappa_1,\kappa_2]$ as follows:

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\mathrm{d}_q x = \int_0^{\kappa_2}\mathcal{F}(x)\mathrm{d}_q x - \int_0^{\kappa_1}\mathcal{F}(x)\mathrm{d}_q x. \quad (6)$$

Definition 1 (see [31]). We consider the mapping $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$. Then, the q_{κ_1}-derivative of $\mathcal{F}$ at $x \in [\kappa_1,\kappa_2]$ is defined by the following expression:

$$_{\kappa_1}D_q\mathcal{F}(x) = \frac{\mathcal{F}(x) - \mathcal{F}(qx+(1-q)\kappa_1)}{(1-q)(x-\kappa_1)}, \quad x \ne \kappa_1. \quad (7)$$

If $x=\kappa_1$, we define $_{\kappa_1}D_q\mathcal{F}(\kappa_1) = \lim_{x\longrightarrow\kappa_1} {}_{\kappa_1}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 2 (see [32]). We consider the mapping $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$. Then, the q^{κ_2}-derivative of $\mathcal{F}$ at $x \in [\kappa_1,\kappa_2]$ is defined by

$$^{\kappa_2}D_q\mathcal{F}(x) = \frac{\mathcal{F}(qx+(1-q)\kappa_2) - \mathcal{F}(x)}{(1-q)(\kappa_2-x)}, \quad x \ne \kappa_2. \quad (8)$$

If $x=\kappa_2$, we define $^{\kappa_2}D_q\mathcal{F}(\kappa_2) = \lim_{x\longrightarrow\kappa_2} {}^{\kappa_2}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 3 (see [31]). We consider the mapping $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ is defined by

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)_{\kappa_1}\mathrm{d}_q x = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1) = (\kappa_2-\kappa_1)\int_0^1\mathcal{F}((1-\tau)\kappa_1+\tau\kappa_2)\mathrm{d}_q\tau. \quad (9)$$

In [11, 21], the authors proved quantum Hermite–Hadamard-type inequalities and their estimations by using the notions of the q_{κ_1}-derivative and q_{κ_1}-integral.

On the contrary, in [32], Bermudo et al. gave the following definition and obtained the related Hermite–Hadamard-type inequalities:

Definition 4 (see [32]). We consider the mapping $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$. Then, the q^{κ_2}-definite integral on $[\kappa_1,\kappa_2]$ is defined by

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)^{\kappa_2}\mathrm{d}_q x = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2) = (\kappa_2-\kappa_1)\int_0^1\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau. \quad (10)$$

Theorem 2 (see [32]). *Let* $\mathcal{F}\colon [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ *be a convex function on* $[\kappa_1,\kappa_2]$ *and* $0<q<1$. *Then,* q^{κ_2}*-Hermite–Hadamard inequalities are given as follows:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}d_qx\leq\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}. \tag{11}$$

In [33], Budak proved the left and right bounds of inequality (11).

The present paper aims to generalize the results proved in [13, 33, 34]. The key benefit of our paper is that it includes multiple inequalities at the same time, such as Simpson's inequalities, Newton's inequalities, midpoint inequalities, and trapezoidal inequities.

3. Crucial Identities

We deal with the three identities which are necessary to obtain our main results in this section.

Let us start with the following useful lemma.

Lemma 1. *If $\mathcal{F}\colon[\kappa_1,\kappa_2]\subset\mathbb{R}\longrightarrow\mathbb{R}$ is a q^{κ_2}-differentiable function on (κ_1,κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$, then we have the following identity:*

$$\begin{aligned}&\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&=(\kappa_2-\kappa_1)\left[\int_0^{1/2}(q\tau-\beta)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau+\int_{1/2}^{1}(q\tau-\alpha)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau\right.,\end{aligned} \tag{12}$$

where $q\in(0,1)$.

Proof. By Definition 2, we see that

$${}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)=\frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}. \tag{13}$$

After applying the fundamental properties of quantum integrals, we deduce that

$$\begin{aligned}&\int_0^{1/2}(q\tau-\beta)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau+\int_{1/2}^{1}(q\tau-\alpha)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau\\&=\int_0^{1/2}(\alpha-\beta)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau+\int_0^{1}(q\tau-\alpha)\,{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau\\&=(\alpha-\beta)\int_0^{1/2}\frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}\mathrm{d}_q\tau\\&\quad+q\int_0^{1}\frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)}\mathrm{d}_q\tau\\&\quad-\alpha\int_0^{1}\frac{\mathcal{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}\mathrm{d}_q\tau.\end{aligned} \tag{14}$$

From Definition 4, we conclude that

$$\begin{aligned}
&\int_0^{1/2} \frac{\mathscr{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}\mathrm{d}_q\tau \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}\mathscr{F}\left(\frac{q^{n+1}}{2}\kappa_1+\left(1-\frac{q^{n+1}}{2}\right)\kappa_2\right)-\sum_{n=0}^{\infty}\mathscr{F}\left(\frac{q^n}{2}\kappa_1+\left(1-\frac{q^n}{2}\right)\kappa_2\right)\right] \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\mathscr{F}(\kappa_2)-\mathscr{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right],
\end{aligned} \tag{15}$$

$$\int_0^{1} \frac{\mathscr{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}\mathrm{d}_q\tau = \frac{1}{\kappa_2-\kappa_1}\left[\mathscr{F}(\kappa_2)-\mathscr{F}(\kappa_1)\right], \tag{16}$$

and

$$\begin{aligned}
&\int_0^{1} \frac{\mathscr{F}(q\tau\kappa_1+(1-q\tau)\kappa_2)-\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)}{(1-q)(\kappa_2-\kappa_1)\tau}\mathrm{d}_q\tau \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathscr{F}\left(q^{n+1}\kappa_1+\left(1-q^{n+1}\right)\kappa_2\right)-\sum_{n=0}^{\infty}q^n\mathscr{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\sum_{n=1}^{\infty}q^n\mathscr{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)-\sum_{n=0}^{\infty}q^n\mathscr{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\sum_{n=0}^{\infty}q^n\mathscr{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)-\frac{1}{q}\mathscr{F}(\kappa_1)-\sum_{n=0}^{\infty}q^n\mathscr{F}\left(q^n\kappa_1+(1-q^n)\kappa_2\right)\right] \\
&= \frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathscr{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\frac{1}{q}\mathscr{F}(\kappa_1)\right].
\end{aligned} \tag{17}$$

We can obtain the required identity (12) by putting the computed integrals (15)–(17) in (14). □

Remark 1. If we assume $\beta = (1/6)$ and $\alpha = (5/6)$ in Lemma 1, then we obtain Lemma 2 of [13].

Remark 2. In Lemma 1, by taking the limit $q \longrightarrow 1^-$, we have Lemma 2.1 of [34] for $m = 1$.

Remark 3. In Lemma 1, if we choose $\beta = \alpha = (q/[2]_q)$, then we obtain the following identity:

$$\frac{\mathscr{F}(\kappa_1)+q\mathscr{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathscr{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx = \frac{q(\kappa_2-\kappa_1)}{[2]_q}\int_0^1\left(1-[2]_q\tau\right){}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau, \tag{18}$$

which is proved by Budak in Lemma 1 of [33].

Corollary 1. *In Lemma 1, if we choose $\beta = 0$ and $\alpha = 1$, then we obtain the following new identity:*

$$\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathscr{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\mathscr{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) = (\kappa_2-\kappa_1)\left[\int_0^{1/2}q\tau\,{}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)\mathrm{d}_q\tau+\int_{1/2}^{1}(q\tau-1)\,{}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1+(1-\tau)\kappa_2)d_q\tau\right]. \tag{19}$$

Lemma 2. *If $\mathscr{F}: [\kappa_1, \kappa_2] \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a q^{κ_2}-differentiable function on (κ_1, κ_2) such that ${}^{\kappa_2}D_q\mathscr{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$, then we have the following identity:*

$$\frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathscr{F}(x)\, {}^{\kappa_2}\mathrm{d}_q x - \left[\beta\mathscr{F}(\kappa_2) + (\alpha - \beta)\mathscr{F}\left(\frac{\kappa_1 + 2\kappa_2}{3}\right) + (\gamma - \alpha)\mathscr{F}\left(\frac{2\kappa_1 + \kappa_2}{3}\right) + (1 - \gamma)\mathscr{F}(\kappa_1)\right]$$
$$= (\kappa_2 - \kappa_1)\left[\int_0^{1/3} (q\tau - \beta)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau + \int_{1/3}^{2/3} (q\tau - \alpha)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau \right. \tag{20}$$
$$\left. + \int_{\frac{2}{3}}^{1} (q\tau - \gamma)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)d_q\tau\right],$$

where $q \in (0, 1)$.

Proof. After applying the fundamental properties of quantum integrals, we deduce that

$$\begin{aligned}
&\int_0^{1/3} (q\tau - \beta)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau + \int_{1/3}^{2/3} (q\tau - \alpha)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau \\
&\quad + \int_{2/3}^{1} (q\tau - \gamma)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau \\
&\quad = \int_0^{1/2} (\alpha - \beta)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau + \int_0^{2/3} (\gamma - \alpha)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau \\
&\quad + \int_0^{1} (q\tau - \gamma)\, {}^{\kappa_2}D_q\mathscr{F}(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}_q\tau.
\end{aligned} \tag{21}$$

If the same steps in the proof of Lemma 1 are applied for the rest of this proof, we can obtain the desired identity (20). □

Remark 4. By assuming $\beta = 1/8$, $\alpha = 1/2$, and $\gamma = 7/8$ in Lemma 2, we obtain Lemma 3 of [13].

Remark 5. If we take $\beta = \alpha = \gamma = q/[2]_q$ in Lemma 2, then we recapture identity (18).

Corollary 2. *If we take the limit $q \longrightarrow 1^-$ in Lemma 2, then we obtain the following new identity:*

$$\frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathscr{F}(x)\mathrm{d}x - \left[\beta\mathscr{F}(\kappa_2) + (\alpha - \beta)\mathscr{F}\left(\frac{\kappa_1 + 2\kappa_2}{3}\right) + (\gamma - \alpha)\mathscr{F}\left(\frac{2\kappa_1 + \kappa_2}{3}\right) + (1 - \gamma)\mathscr{F}(\kappa_1)\right]$$
$$= (\kappa_2 - \kappa_1)\left[\int_0^{1/3} (\tau - \beta)\mathscr{F}'(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}\tau + \int_{1/3}^{2/32} (\tau - \alpha)\mathscr{F}'(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}\tau + \int_{2/3}^{1} (\tau - \gamma)\mathscr{F}'(\tau\kappa_1 + (1 - \tau)\kappa_2)\mathrm{d}\tau\right]. \tag{22}$$

Now, we calculate the integrals in the following lemma which will be used in our next section.

Lemma 3. *The subsequent quantum integrals are true:*

$$\Xi_{11} = \int_0^{1/2} |q\tau - \beta| \mathrm{d}_q\tau = \begin{cases} \dfrac{8\beta^2 + q}{4[2]_q} - \dfrac{\beta}{2}, & q > 2\beta, \\ \dfrac{\beta}{2} - \dfrac{q}{4[2]_q}, & q \leq 2\beta, \end{cases} \tag{23}$$

$$\Xi_{12} = \int_{1/2}^{1} |q\tau - \alpha| \mathrm{d}_q\tau = \begin{cases} \dfrac{\alpha}{2} - \dfrac{3q}{4[2]_q}, & q < \alpha, \\ \dfrac{8\alpha^2 + 5q}{4[2]_q} - \dfrac{3\alpha}{2}, & \alpha \leq q \leq 2\alpha, \\ \dfrac{3q}{4[2]_q} - \dfrac{\alpha}{2}, & q > 2\alpha, \end{cases} \tag{24}$$

$$\Xi_{13} = \int_0^{1/3} |q\tau - \beta| d_q\tau = \begin{cases} \dfrac{2\beta^2}{[2]_q} + \dfrac{q}{9[2]_q} - \dfrac{\beta}{3}, & q > 3\beta, \\ \dfrac{\beta}{3} - \dfrac{q}{9[2]_q}, & q \leq 3\beta, \end{cases} \tag{25}$$

$$\Xi_{14} = \int_{\frac{1}{3}}^{2/3} |q\tau - \alpha| \mathrm{d}_q\tau = \begin{cases} \dfrac{\alpha}{3} - \dfrac{q}{3[2]_q}, & q < \dfrac{3\alpha}{2}, \\ \dfrac{18\alpha^2 + 5q}{9[2]_q} - \alpha, & \dfrac{3\alpha}{2} \leq q \leq 3\alpha, \\ \dfrac{q}{3[2]_q} - \dfrac{\alpha}{3}, & q > 3\alpha, \end{cases} \tag{26}$$

$$\Xi_{15} = \int_{2/3}^{1} |q\tau - \gamma| \mathrm{d}_q\tau = \begin{cases} \dfrac{\gamma}{3} - \dfrac{5q}{9[2]_q}, & q < \gamma, \\ \dfrac{18\gamma^2 + 13q}{9[2]_q} - \dfrac{5\gamma}{3}, & \gamma \leq q \leq \dfrac{3\gamma}{2}, \\ \dfrac{5q}{9[2]_q} - \dfrac{\gamma}{3}, & q > \dfrac{3\gamma}{2}, \end{cases} \tag{27}$$

$$\Xi_1 = \int_0^{1/2} \tau|q\tau - \beta| \mathrm{d}_q\tau = \begin{cases} \dfrac{2\beta^3}{[2]_q[3]_q} + \dfrac{q}{8[3]_q} - \dfrac{\beta}{4[2]_q}, & q > 2\beta, \\ \dfrac{\beta}{4[2]_q} - \dfrac{q}{8[3]_q}, & q \leq 2\beta, \end{cases} \tag{28}$$

$$\begin{aligned} \Xi_2 &= \int_0^{1/2} (1-\tau)|q\tau - \beta| \mathrm{d}_q\tau \\ &= \Xi_{11} - \Xi_1 \\ &= \begin{cases} \dfrac{8\beta^2 + \beta + q}{4[2]_q} - \dfrac{\beta}{2} - \dfrac{q}{8[3]_q} - \dfrac{2\beta^3}{[2]_q[3]_q}, & q > 2\beta, \\ \dfrac{\beta}{2} - \dfrac{\beta + q}{4[2]_q} + \dfrac{q}{8[3]_q}, & q \leq 2\beta, \end{cases} \end{aligned} \tag{29}$$

$$\Xi_3 = \int_{1/2}^{1} \tau|q\tau - \alpha| \mathrm{d}_q\tau = \begin{cases} \dfrac{3\alpha}{4[2]_q} - \dfrac{7q}{8[3]_q}, & q < \alpha, \\ \dfrac{2\alpha^3}{[2]_q[3]_q} - \dfrac{5\alpha}{4[2]_q} + \dfrac{9q}{8[3]_q}, & \alpha \leq q \leq 2\alpha, \\ \dfrac{7q}{8[3]_q} - \dfrac{3\alpha}{4[2]_q}, & q > 2\alpha, \end{cases} \tag{30}$$

$$\begin{aligned} \Xi_4 &= \int_{1/2}^{1} (1-\tau)|q\tau - \alpha| d_q\tau \\ &= \Xi_{12} - \Xi_3 \\ &= \begin{cases} \dfrac{\alpha}{2} - \dfrac{3(\alpha + q)}{4[2]_q} + \dfrac{7q}{8[3]_q}, & q < \alpha, \\ \dfrac{8\alpha^2 + 5q - 5\alpha}{4[2]_q} - \dfrac{3\alpha}{2} - \dfrac{9q}{8[3]_q} - \dfrac{2\alpha^3}{[2]_q[3]_q}, & \alpha \leq q \leq 2\alpha, \\ \dfrac{3(\alpha + q)}{4[2]_q} - \dfrac{\alpha}{2} - \dfrac{7q}{8[3]_q}, & q > 2\alpha, \end{cases} \end{aligned} \tag{31}$$

$$\Xi_5 = \int_0^{1/3} \tau|q\tau - \beta| \mathrm{d}_q\tau = \begin{cases} \dfrac{2\beta^3}{[2]_q[3]_q} + \dfrac{q}{27[3]_q} - \dfrac{\beta}{9[2]_q}, & q > 3\beta, \\ \dfrac{\beta}{9[2]_q} - \dfrac{q}{27[3]_q}, & q \leq 3\beta, \end{cases} \tag{32}$$

$$\begin{aligned}\Xi_6 &= \int_0^{1/3}(1-\tau)|q\tau-\beta|\mathrm{d}_q\tau \\ &= \Xi_{13}-\Xi_5 \\ &= \begin{cases} \dfrac{18\beta^2+\beta+q}{9[2]_q}-\dfrac{\beta}{3}-\dfrac{q}{27[3]_q}-\dfrac{2\beta^3}{[2]_q[3]_q}, & q>2\beta, \\ \dfrac{\beta}{3}-\dfrac{\beta+q}{9[2]_q}+\dfrac{q}{27[3]_q}, & q\le 2\beta, \end{cases}\end{aligned} \tag{33}$$

$$\begin{aligned}\Xi_7 &= \int_{1/3}^{2/3}\tau|q\tau-\alpha|\mathrm{d}_q\tau \\ &= \begin{cases} \dfrac{\alpha}{3[2]_q}-\dfrac{7q}{27[3]_q}, & q<\dfrac{3\alpha}{2}, \\ \dfrac{2\alpha^3}{[2]_q[3]_q}-\dfrac{5\alpha}{9[2]_q}+\dfrac{q}{3[3]_q}, & \dfrac{3\alpha}{2}\le q\le 3\alpha, \\ \dfrac{7q}{27[3]_q}-\dfrac{\alpha}{3[2]_q}, & q>3\alpha, \end{cases}\end{aligned} \tag{34}$$

$$\begin{aligned}\Xi_8 &= \int_{1/3}^{2/3}(1-\tau)|q\tau-\alpha|\mathrm{d}_q\tau \\ &= \Xi_{14}-\Xi_7 \\ &= \begin{cases} \dfrac{\alpha}{3}-\dfrac{q+\alpha}{3[2]_q}+\dfrac{7q}{27[3]_q}, & q<\dfrac{3\alpha}{2}, \\ \dfrac{18\alpha^2+5q+5\alpha}{9[2]_q}-\alpha-\dfrac{q}{3[3]_q}-\dfrac{2\alpha^3}{[2]_q[3]_q}, & \dfrac{3\alpha}{2}\le q\le 3\alpha, \\ \dfrac{q+\alpha}{3[2]_q}-\dfrac{\alpha}{3}-\dfrac{7q}{27[3]_q}, & q>3\alpha, \end{cases}\end{aligned} \tag{35}$$

$$\begin{aligned}\Xi_9 &= \int_{2/3}^{1}\tau|q\tau-\gamma|\mathrm{d}_q\tau \\ &= \begin{cases} \dfrac{5\gamma}{9[2]_q}-\dfrac{19q}{27[3]_q}, & q<\gamma, \\ \dfrac{2\gamma^3}{[2]_q[3]_q}-\dfrac{13\gamma}{9[2]_q}+\dfrac{35q}{27[3]_q}, & \gamma\le q\le\dfrac{3\gamma}{2}, \\ \dfrac{19q}{27[3]_q}-\dfrac{5\gamma}{9[2]_q}, & q>\dfrac{3\gamma}{2}, \end{cases}\end{aligned} \tag{36}$$

$$\begin{aligned}\Xi_{10} &= \int_{2/3}^{1}(1-\tau)|q\tau-\gamma|\,\mathrm{d}_q\tau\\ &= \Xi_{15}-\Xi_9\\ &= \begin{cases}\dfrac{\gamma}{3}-\dfrac{5(q+\gamma)}{9[2]_q}+\dfrac{19q}{27[3]_q}, & q<\gamma,\\[2ex] \dfrac{18\gamma^2+13q+13\gamma}{9[2]_q}-\dfrac{5\gamma}{3}-\dfrac{35q}{27[3]_q}-\dfrac{2\gamma^3}{[2]_q[3]_q}, & \gamma\le q\le\dfrac{3\gamma}{2},\\[2ex] \dfrac{5(q+\gamma)}{9[2]_q}-\dfrac{\gamma}{3}-\dfrac{19q}{27[3]_q}, & q>\dfrac{3\gamma}{2}.\end{cases}\end{aligned} \tag{37}$$

Proof. Case I: let $q>2\beta$.

By the definition of the q-integral, we have

$$\begin{aligned}\Xi_1 &= \int_0^{1/2}\tau|q\tau-\beta|\,\mathrm{d}_q\tau\\ &= \int_0^{\beta/q}\tau(\beta-q\tau)\,\mathrm{d}_q\tau+\int_{\beta/q}^{1/2}\tau(q\tau-\beta)\,\mathrm{d}_q\tau\\ &= 2\int_0^{\beta/q}\tau(\beta-q\tau)\,\mathrm{d}_q\tau+\int_0^{1/2}\tau(q\tau-\beta)\,\mathrm{d}_q\tau\\ &= \frac{2\beta^3}{[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{\beta}{4[2]_q}.\end{aligned} \tag{38}$$

Case II: let $q\le 2\beta$.

From the definition of the q-integral, we get

$$\begin{aligned}\Xi_1 &= \int_0^{1/2}\tau|q\tau-\beta|\,\mathrm{d}_q\tau\\ &= \int_0^{1/2}\tau(\beta-q\tau)\,\mathrm{d}_q\tau\\ &= \frac{\beta}{4[2]_q}-\frac{q}{8[3]_q}.\end{aligned} \tag{39}$$

This gives the proof of equality (28). In a similar way, we can prove the others. □

4. Simpson's Type Inequalities for Quantum Integrals

An extension of quantum Simpson's inequalities for quantum differentiable convex functions using the quantum integrals is given in this section.

Theorem 3. *We assume that the given conditions of Lemma 1 hold. If the mapping $|{}^{\kappa_2}D_q\mathscr{F}|$ is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathscr{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\left[\beta\mathscr{F}(\kappa_2)+(1-\alpha)\mathscr{F}(\kappa_1)+(\alpha-\beta)\mathscr{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\ &\quad\le(\kappa_2-\kappa_1)\left[(\Xi_1+\Xi_3)\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_1)\right|+(\Xi_2+\Xi_4)\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_2)\right|\right],\end{aligned} \tag{40}$$

where Ξ_1–Ξ_4 are given in (28)–(30), respectively.

Proof. By Lemma 1 and the convexity of $|{}^{\kappa_2}D_q\mathscr{F}|$, we conclude that

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad\le(\kappa_2-\kappa_1)\left[\int_0^{1/2}|q\tau-\beta||{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)|\mathrm{d}_q\tau+\int_{1/2}^{1}|q\tau-\alpha||{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)|\mathrm{d}_q\tau\right] \\
&\quad\le(\kappa_2-\kappa_1)\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|\left\{\int_0^{1/2}\tau|q\tau-\beta|\mathrm{d}_q\tau+\int_{1/2}^{1}\tau|q\tau-\alpha|\mathrm{d}_q\tau\right\}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right. \\
&\quad\quad\left.\cdot\left\{\int_0^{1/2}(1-\tau)|q\tau-\beta|\mathrm{d}_q\tau+\int_{1/2}^{1}(1-\tau)|q\tau-\alpha|\mathrm{d}_q\tau\right\}\right] \\
&\quad=(\kappa_2-\kappa_1)\left[(\Xi_1+\Xi_3)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+(\Xi_2+\Xi_4)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right],
\end{aligned}\tag{41}$$

which is the desired conclusion. □

Remark 6. By taking the limit $q\longrightarrow 1^-$ in Theorem 3, we have Theorem 2.1 of [34] for $s=m=1$.

Remark 7. If we assume $\beta=\alpha=q/[2]_q$ in Theorem 3, then we obtain the following trapezoidal type inequality:

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x\right| \\
&\quad\le(\kappa_2-\kappa_1)\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|\frac{q^2([3]_q+3q)}{[3]_q[2]_q^4}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\frac{q^2(1+3q^2+2q^3)}{[3]_q[2]_q^4}\right],
\end{aligned}\tag{42}$$

which is established by Budak in Theorem 3 of [33].

Corollary 3. *In Theorem 3, if we choose $\beta=0$ and $\alpha=1$, then we obtain the following midpoint-type inequality:*

$$\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x\right|\le(\kappa_2-\kappa_1)\left[\frac{3}{4[2]_q[3]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+\frac{2q^2+2q-1}{4[2]_q[3]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right].\tag{43}$$

Remark 8. In Theorem 3, if we assume $\beta=1/6$ and $\alpha=5/6$, then Theorem 3 reduces to Theorem 4 of [13].

Theorem 4. *We assume that the given conditions of Lemma 1 hold. If the mapping $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, $p_1\ge 1$, is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\quad\le(\kappa_2-\kappa_1)\left[\Xi_{11}^{1-(1/p_1)}\left(\Xi_1\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_2\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{(1/p_1)}+\Xi_{12}^{1-(1/p_1)}\left(\Xi_3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_4\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{(1/p_1)}\right],
\end{aligned}\tag{44}$$

where Ξ_{11}, Ξ_{12}, and Ξ_1–Ξ_4 are given in (23), (24), and (28)–(30), respectively.

Proof. From Lemma 1 and the power mean inequality, we obtain

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{1/2}|q\tau-\beta|d_q\tau\right)^{1-(1/p_1)}\left(\int_0^{1/2}|q\tau-\beta|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{(1/p_1)}\right. \tag{45}$$
$$\left.+\left(\int_{1/2}^{1}|q\tau-\alpha|d_q\tau\right)^{1-(1/p_1)}\left(\int_{1/2}^{1}|q\tau-\alpha|\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}d_q\tau\right)^{(1/p_1)}\right].$$

By applying the convexity of $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, we obtain

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{1/2}|q\tau-\beta|\mathrm{d}_q\tau\right)^{1-(1/p_1)}\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{1/2}\tau|q\tau-\beta|\mathrm{d}_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{1/2}(1-\tau)|q\tau-\beta|\mathrm{d}_q\tau\right)^{(1/p_1)}\right.$$
$$\left.+\left(\int_{1/2}^{1}|q\tau-\alpha|\mathrm{d}_q\tau\right)^{1-(1/p_1)}\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{1/2}^{1}\tau|q\tau-\alpha|\mathrm{d}_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{1/2}^{1}(1-\tau)|q\tau-\alpha|\mathrm{d}_q\tau\right)^{(1/p_1)}\right]$$
$$=(\kappa_2-\kappa_1)\left[\Xi_{11}^{1-(1/p_1)}\left(\Xi_1\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_2\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{(1/p_1)}+\Xi_{12}^{1-(1/p_1)}\left(\Xi_3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_4\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{(1/p_1)}\right], \tag{46}$$

and the proof is completed. □

Remark 9. If we take the limit $q\longrightarrow 1^-$ in Theorem 4, then we have Theorem 2.3 of [34] for $s=m=1$.

Remark 10. If we assume $\beta=\alpha=q/[2]_q$ in Theorem 4, then we obtain the following trapezoidal type inequality:

$$\left|\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx\right|$$
$$\leq\frac{q(\kappa_2-\kappa_1)}{[2]_q}\left(\frac{q(2+q+q^3)}{[2]_q^3}\right)^{1-(1/p_1)} \tag{47}$$
$$\times\left[\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q([3]_q+3q)}{[3]_q[2]_q^3}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q(1+3q^2+2q^3)}{[3]_q[2]_q^3}\right],$$

which is established by Budak in Theorem 4 of [33].

Remark 11. If we assume $\beta=1/6$ and $\alpha=5/6$ in Theorem 4, then Theorem 4 reduces to Theorem 6 of [13].

Corollary 4. *In Theorem 4, if we choose $\beta = 0$ and $\alpha = 1$, then we obtain the following midpoint-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}d_qx-\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\
&\leq(\kappa_2-\kappa_1)\left(\frac{q}{4[2]_q}\right)^{1-(1/p_1)}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q}{8[3]_q}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q\left([3]_q+q^2\right)}{8[2]_q[3]_q}\right)^{(1/p_1)} \\
&\quad+\left(\frac{2-q}{4[2]_q}\right)^{1-(1/p_1)}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\left(\frac{1}{2}-\frac{3q}{4[2]_q}-\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}\right)\right)^{(1/p_1)}.
\end{aligned} \tag{48}$$

Theorem 5. *Assume that the given conditions of Lemma 1 hold. If the mapping $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, $p_1>1$, is convex on $[\kappa_1,\kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq(\kappa_2-\kappa_1)\left[\Xi_{16}^{1/r_1}\left(\frac{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{(1/p_1)}+\Xi_{17}^{1/r_1}\left(\frac{3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{(1/p_1)}\right],
\end{aligned} \tag{49}$$

where $p_1^{-1}+r_1^{-1}=1$ and

$$\begin{aligned}
\Xi_{16}&=\int_0^{1/2}|q\tau-\beta|^{r_1}\mathrm{d}_q\tau, \\
\Xi_{17}&=\int_{1/2}^{1}|q\tau-\alpha|^{r_1}\mathrm{d}_q\tau.
\end{aligned} \tag{50}$$

Proof. From Lemma 1 and the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}d_qx-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
&\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{1/2}|q\tau-\beta|^{r_1}d_q\tau\right)^{1/r_1}\left(\int_0^{1/2}\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}\mathrm{d}_q\tau\right)^{(1/p_1)}+\left(\int_{1/2}^{1}|q\tau-\alpha|^{r_1}\mathrm{d}_q\tau\right)^{1/r_1}\right. \\
&\quad\left.\left(\int_{1/2}^{1}\left|{}^{\kappa_2}D_q\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\right|^{p_1}\mathrm{d}_q\tau\right)^{(1/p_1)}\right].
\end{aligned} \tag{51}$$

By using the convexity of $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x-\left[\beta\mathcal{F}(\kappa_2)+(1-\alpha)\mathcal{F}(\kappa_1)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{1/2}|q\tau-\beta|^{r_1}\mathrm{d}_q\tau\right)^{1/r_1}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{1/2}\tau\mathrm{d}_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{1/2}(1-\tau)\mathrm{d}_q\tau\right)^{1/p_1}\right.\\
&\left.+\left(\int_{1/2}^{1}|q\tau-\alpha|^{r_1}d_q\tau\right)^{1/r_1}\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{1/2}^{1}\tau\mathrm{d}_q\tau+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{1/2}^{1}(1-\tau)\mathrm{d}_q\tau\right)^{1/p_1}\right]\\
&=(\kappa_2-\kappa_1)\left[\Xi_{16}^{1/r_1}\left(\frac{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{1/p_1}\right.\\
&\left.+\Xi_{17}^{1/r_1}\left(\frac{3\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}\right)^{1/p_1}\right],
\end{aligned}\tag{52}$$

and the proof is completed. □

Remark 12. If we take the limit $q \longrightarrow 1^-$ in Theorem 5, then Theorem 5 becomes Theorem 2.2 of [34] for $s = m = 1$.

Remark 13. If we assume $\beta = 1/6$ and $\alpha = 5/6$ in Theorem 5, then Theorem 5 becomes Theorem 5 of [13].

5. Newton's Type Inequalities for Quantum Integrals

In this section, a new extension of quantum Newton's inequalities for quantum differentiable convex functions is given.

Theorem 6. *We assume that the given conditions of Lemma 2 hold. If the mapping $|{}^{\kappa_2}D_q\mathcal{F}|$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|\\
&\leq(\kappa_2-\kappa_1)\left[(\Xi_5+\Xi_7+\Xi_9)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+(\Xi_6+\Xi_8+\Xi_{10})\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right],
\end{aligned}\tag{53}$$

where Ξ_5–Ξ_{10} are given in (31)–(36), respectively.

Proof. If we consider Lemma 2 and apply the same method that is used in the proof of Theorem 3, then we can obtain the desired inequality (53). □

Corollary 5. *If we take the limit $q \longrightarrow 1^-$ in Theorem 6, then we obtain the following Newton's type inequality:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|\\
&\leq(\kappa_2-\kappa_1)\left[(\Xi_5^*+\Xi_7^*+\Xi_9^*)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|+(\Xi_6^*+\Xi_8^*+\Xi_{10}^*)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\right],
\end{aligned}\tag{54}$$

where

$$\begin{aligned}
\Xi_5^* &= \int_0^{1/3} \tau|\tau-\beta|d\tau \\
&= \frac{\beta^3}{3}+\frac{1}{81}-\frac{\beta}{18}, \\
\Xi_6^* &= \int_0^{1/3} (1-\tau)|\tau-\beta|\mathrm{d}\tau \\
&= \frac{18\beta^2+\beta+1}{18}-\frac{28}{81}-\frac{\beta^3}{3}, \\
\Xi_7^* &= \int_{1/3}^{2/3} \tau|q\tau-\alpha|d\tau \\
&= \frac{\alpha^3}{3}-\frac{5\alpha}{18}+\frac{1}{9} \\
\Xi_8^* &= \int_{1/3}^{2/3} (1-\tau)|\tau-\alpha|\mathrm{d}\tau \\
&= \frac{18\alpha^2+5+5\alpha}{18}-\alpha-\frac{1}{9}-\frac{\alpha^3}{3}, \\
\Xi_9^* &= \int_{2/3}^{1} \tau|\tau-\gamma|\mathrm{d}\tau \\
&= \frac{\gamma^3}{3}-\frac{13\gamma}{18}+\frac{35}{81}, \\
\Xi_{10}^* &= \int_{2/3}^{1} (1-\tau)|\tau-\gamma|d\tau \\
&= \frac{18\gamma^2+13+13\gamma}{18}-\frac{5\gamma}{3}-\frac{35}{81}-\frac{\gamma^3}{3}.
\end{aligned} \tag{55}$$

Remark 14. If we take $\beta = 1/8$, $\alpha = 1/2$, and $\gamma = 7/8$ in Theorem 6, then Theorem 6 reduces to Theorem 7 of [13].

Remark 15. If we assume $\beta = \alpha = \gamma = q/[2]_q$ in Theorem 6, then we recapture inequality (42).

Theorem 7. *We assume that the given conditions of Lemma 2 hold. If the mapping $|{}^{\kappa_2}D_q\mathscr{F}|^{p_1}$, $p_1 \geq 1$, is convex on $[\kappa_1, \kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathscr{F}(x)\,{}^{\kappa_2}\mathrm{d}_q x-\left[\beta\mathscr{F}(\kappa_2)+(\alpha-\beta)\mathscr{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathscr{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathscr{F}(\kappa_1)\right]\right| \\
&\quad\leq(\kappa_2-\kappa_1)\left[\Xi_{13}^{1-(1/p_1)}\left(\Xi_5\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_1)\right|^{p_1}+\Xi_6\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}+\Xi_{14}^{1-(1/p_1)}\left(\left(\Xi_7\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_1)\right|^{p_1}+\Xi_8\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}\right)\right. \\
&\qquad\left.+\Xi_{15}^{1-(1/p_1)}\left(\Xi_9\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_1)\right|^{p_1}+\Xi_{10}\left|{}^{\kappa_2}D_q\mathscr{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}\right],
\end{aligned} \tag{56}$$

where Ξ_5–Ξ_{10} and Ξ_{13}–Ξ_{15} are given in (31)–(36) and (25)–(27), respectively.

Proof. If we apply the steps used in the proof of Theorem 4 and taking into account Lemma 2, we can obtain the required inequality (56). □

Corollary 6. *If we take the limit $q \longrightarrow 1^-$ in Theorem 7, then we obtain the following Newton's type inequality:*

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\mathrm{d}x-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|$$
$$\leq(\kappa_2-\kappa_1)\left[\Pi_{11}^{1-(1/p_1)}\left(\Xi_5^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_6^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}+\Pi_{12}^{1-(1/p_1)}\left(\left(\Xi_7^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_8^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}\right)\right.$$
$$\left.+\Pi_{13}^{1-(1/p_1)}\left(\Xi_9^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\Xi_{10}^*\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{1/p_1}\right],\tag{57}$$

where Ξ_5^*–Ξ_{10}^* are defined in Corollary 5 and

$$\begin{aligned}\Pi_{11}&=\int_0^{1/3}|\tau-\beta|\mathrm{d}\tau\\&=\beta^2+\frac{1}{9[2]_q}-\frac{\beta}{3},\\\Pi_{12}&=\int_{1/3}^{2/3}|\tau-\alpha|\mathrm{d}\tau\\&=\frac{18\alpha^2+5}{18}-\alpha,\\\Pi_{13}&=\int_{2/3}^{1}|\tau-\gamma|\mathrm{d}\tau\\&=\frac{18\gamma^2+13}{18}-\frac{5\gamma}{3}.\end{aligned}\tag{58}$$

Remark 16. If we take $\beta=1/8$, $\alpha=1/2$, and $\gamma=7/8$ in Theorem 7, then Theorem 7 reduces to Theorem 9 of [13].

Remark 17. If we assume $\beta=\alpha=\gamma=q/[2]_q$ in Theorem 7, then we recapture inequality (47).

Theorem 8. *We assume that the given conditions of Lemma 2 hold. If the mapping $|{}^{\kappa_2}D_q\mathcal{F}|^{p_1}$, $p_1>1$, is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}^{\kappa_2}\mathrm{d}_qx\leq-\left[\beta\mathcal{F}(\kappa_2)+(\alpha-\beta)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(\gamma-\alpha)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(1-\gamma)\mathcal{F}(\kappa_1)\right]\right|$$
$$(\kappa_2-\kappa_1)\left[\Xi_{18}^{1/r_1}\left(\frac{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}+\frac{(3q+2)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q}\right)^{1/p_1}+\Xi_{19}^{1/r_1}\left(\frac{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{3[2]_q}+\frac{q\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{3[2]_q}\right)^{1/p_1}\right.\tag{59}$$
$$\left.+\Xi_{20}^{1/r_1}\left(\frac{5\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}+\frac{(3q-2)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q}\right)^{1/p_1}\right],$$

where $p_1^{-1}+r_1^{-1}=1$ and

$$\Xi_{18} = \int_0^{1/3} |q\tau - \beta|^{r_1} \mathrm{d}_q\tau,$$
$$\Xi_{19} = \int_{1/3}^{2/3} |q\tau - \alpha|^{r_1} \mathrm{d}_q\tau, \quad (60)$$
$$\Xi_{20} = \int_{2/3}^{1} |q\tau - \gamma|^{r_1} \mathrm{d}_q\tau.$$

Proof. If we apply the steps used in the proof of Theorem 5 and taking into account Lemma 2, we can obtain the required inequality (59). □

Remark 18. If we take $\beta = 1/8$, $\alpha = 1/2$, and $\gamma = 7/8$ in Theorem 8, then Theorem 8 becomes Theorem 8 of [13].

6. Conclusions

In this work, using quantum integrals, we developed a new extension of quantum trapezoid, quantum Simpson's, and quantum Newton's type estimations for quantum differentiable convex functions. It is also shown that the results presented here are generalizations of the findings presented in [13, 33, 34]. In the special cases of newly developed findings, we also obtained several new Simpson's type, Newton's type, midpoint-type, and trapezoidal type inequalities. It is an interesting and innovative problem that future researchers may investigate in order to achieve similar inequalities for convex and coordinated convex functions via different quantum integrals.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper and read and approved the final manuscript.

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