

Some New Inequalities of Hermite-Hadamard's Type

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ABSTRACT. In this paper, we establish several new inequalities for some differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Some applications for special means of real numbers are also provided.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [6]):

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

This definition has its origins in Jensen's results from [2] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them. We say that f is concave if $(-f)$ is convex.

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality, (see [1], [3], [4] and [5]) which has generated a wide range of directions for extension and a rich mathematical literature.

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In [4] in order to prove some inequalities related to Hadamard's inequality Kırmacı used the following lemma:

Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, be a differentiable mapping on I° , $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \\ &= (b-a) \left[\int_0^{\frac{1}{2}} t f'(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b) dt \right]. \end{aligned}$$

Also, in [5], Kırmacı and Özdemir obtained the following inequality for differentiable mappings which are connected with Hermite-Hadamard's inequality:

Theorem 1. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$ and $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(3^{1-\frac{1}{q}})}{8} (b-a) (|f'(a)| + |f'(b)|).$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities related to the left side of Hermite-Hadamard inequality. Finally, we gave some applications for special means of real numbers.

2. Main results

We start with the following lemma:

Lemma 2. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} (2.1) \quad & f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{b-a}{2} \int_0^1 \int_0^1 (f'(ta + (1-t)b) - f'(sa + (1-s)b)) (m(s) - m(t)) dt ds. \end{aligned}$$

with

$$m(\cdot) := \begin{cases} t & , t \in [0, \frac{1}{2}] \\ t-1 & , t \in (\frac{1}{2}, 1]. \end{cases}$$

Proof. By definitions of $m(\cdot)$, it follows that

$$\begin{aligned}
& \int_0^1 \int_0^1 (f'(ta + (1-t)b) - f'(sa + (1-s)b)) (m(t) - m(s)) dt ds \\
&= \int_0^1 \left\{ \int_0^1 f'(ta + (1-t)b) (m(t) - m(s)) dt \right. \\
&\quad \left. - \int_0^1 f'(sa + (1-s)b) (m(t) - m(s)) dt \right\} ds \\
&= \int_0^1 \left\{ \int_0^{1/2} f'(ta + (1-t)b) (t - m(s)) dt \right. \\
&\quad \left. + \int_{1/2}^1 f'(ta + (1-t)b) (t - 1 - m(s)) dt \right\} ds \\
&\quad - \int_0^1 \left\{ \int_0^{1/2} f'(sa + (1-s)b) (t - m(s)) dt \right. \\
&\quad \left. + \int_{1/2}^1 f'(sa + (1-s)b) (t - 1 - m(s)) dt \right\} ds \\
&= \int_0^{1/2} \left\{ \int_0^{1/2} f'(ta + (1-t)b) (t - s) dt \right\} ds \\
&\quad + \int_{1/2}^1 \left\{ \int_0^{1/2} f'(ta + (1-t)b) (t - s + 1) dt \right\} ds \\
&\quad + \int_0^{1/2} \left\{ \int_{1/2}^1 f'(ta + (1-t)b) (t - s - 1) dt \right\} ds \\
&\quad + \int_{1/2}^1 \left\{ \int_{1/2}^1 f'(ta + (1-t)b) (t - s) dt \right\} ds \\
&\quad - \int_0^{1/2} \left\{ \int_0^{1/2} f'(sa + (1-s)b) (t - s) dt \right\} ds \\
&\quad - \int_{1/2}^1 \left\{ \int_0^{1/2} f'(sa + (1-s)b) (t - s + 1) dt \right\} ds \\
&\quad - \int_0^{1/2} \left\{ \int_{1/2}^1 f'(sa + (1-s)b) (t - s - 1) dt \right\} ds \\
&\quad - \int_{1/2}^1 \left\{ \int_{1/2}^1 f'(sa + (1-s)b) (t - s) dt \right\} ds \\
&= I_1 + I_2 + I_3 + I_4 - I_5 - I_6 - I_7 - I_8.
\end{aligned}$$

Thus by integration by parts, we can state:

$$\begin{aligned}
 (2.2) \quad I_1 &= \int_0^{1/2} \left\{ \int_0^{1/2} f'(ta + (1-t)b)(t-s) dt \right\} ds \\
 &= \int_0^{1/2} \left\{ (t-s) \frac{f(ta + (1-t)b)}{a-b} \Big|_0^{1/2} - \frac{1}{(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \right\} ds \\
 &= \int_0^{1/2} \left\{ \left(\frac{1}{2} - s \right) \frac{f(\frac{a+b}{2})}{a-b} + s \frac{f(b)}{a-b} \right\} ds - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \\
 &= \left(\left(\frac{s}{2} - \frac{s^2}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} + \frac{s^2}{2} \frac{f(b)}{a-b} \right) \Big|_0^{1/2} - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \\
 &= \frac{f(\frac{a+b}{2})}{8(a-b)} + \frac{f(b)}{8(a-b)} - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt,
 \end{aligned}$$

$$\begin{aligned}
 (2.3) \quad I_2 &= \int_{1/2}^1 \left\{ \int_0^{1/2} f'(ta + (1-t)b)(t-s+1) dt \right\} ds \\
 &= \int_{1/2}^1 \left\{ (t-s+1) \frac{f(ta + (1-t)b)}{a-b} \Big|_0^{1/2} - \frac{1}{(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \right\} ds \\
 &= \int_{1/2}^1 \left\{ \left(\frac{3}{2} - s \right) \frac{f(\frac{a+b}{2})}{a-b} + (s-1) \frac{f(b)}{a-b} \right\} ds - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \\
 &= \left(\left(\frac{3s}{2} - \frac{s^2}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} + \left(\frac{s^2}{2} - s \right) \frac{f(b)}{a-b} \right) \Big|_{1/2}^1 - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt \\
 &= \frac{3f(\frac{a+b}{2})}{8(a-b)} - \frac{f(b)}{8(a-b)} - \frac{1}{2(a-b)} \int_0^{1/2} f(ta + (1-t)b) dt,
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad I_3 &= \int_0^{1/2} \left\{ \int_{1/2}^1 f'(ta + (1-t)b)(t-s-1) dt \right\} ds \\
 &= \int_0^{1/2} \left\{ (t-s-1) \frac{f(ta + (1-t)b)}{a-b} \Big|_{1/2}^1 - \frac{1}{(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \right\} ds \\
 &= \int_0^{1/2} \left\{ \left(s + \frac{1}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} - s \frac{f(a)}{a-b} \right\} ds - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \\
 &= \left(\left(\frac{s^2}{2} + \frac{s}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} - \frac{s^2}{2} \frac{f(a)}{a-b} \right) \Big|_0^{1/2} - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \\
 &= \frac{3f(\frac{a+b}{2})}{8(a-b)} - \frac{f(a)}{8(a-b)} - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt,
 \end{aligned}$$

$$\begin{aligned}
(2.5) \quad I_4 &= \int_{1/2}^1 \left\{ \int_{1/2}^1 f'(ta + (1-t)b) (t-s) dt \right\} ds \\
&= \int_{1/2}^1 \left\{ (t-s) \frac{f(ta + (1-t)b)}{a-b} \Big|_{1/2}^1 - \frac{1}{(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \right\} ds \\
&= \int_{1/2}^1 \left\{ \left(s - \frac{1}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} + (1-s) \frac{f(a)}{a-b} \right\} ds - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \\
&= \left(\left(\frac{s^2}{2} - \frac{s}{2} \right) \frac{f(\frac{a+b}{2})}{a-b} + \left(s - \frac{s^2}{2} \right) \frac{f(a)}{a-b} \right) \Big|_{1/2}^1 - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt \\
&= \frac{f(\frac{a+b}{2})}{8(a-b)} + \frac{f(a)}{8(a-b)} - \frac{1}{2(a-b)} \int_{1/2}^1 f(ta + (1-t)b) dt,
\end{aligned}$$

$$\begin{aligned}
(2.6) \quad I_5 &= \int_0^{1/2} \left\{ \int_0^{1/2} f'(sa + (1-s)b) (t-s) dt \right\} ds \\
&= \int_0^{1/2} \left\{ \left(\frac{t^2}{2} - st \right) f'(sa + (1-s)b) \Big|_0^{1/2} \right\} ds \\
&= \int_0^{1/2} \left(\frac{1}{8} - \frac{s}{2} \right) f'(sa + (1-s)b) ds \\
&= \left(\frac{1}{8} - \frac{s}{2} \right) \frac{f(sa + (1-s)b)}{a-b} \Big|_0^{1/2} + \frac{1}{2(a-b)} \int_0^{1/2} f(sa + (1-s)b) ds \\
&= -\frac{f(\frac{a+b}{2})}{8(a-b)} - \frac{f(b)}{8(a-b)} + \frac{1}{2(a-b)} \int_0^{1/2} f(sa + (1-s)b) ds,
\end{aligned}$$

$$\begin{aligned}
(2.7) \quad I_6 &= \int_{1/2}^1 \left\{ \int_0^{1/2} f'(sa + (1-s)b) (t-s+1) dt \right\} ds \\
&= \int_{1/2}^1 \left\{ \left(\frac{t^2}{2} - st + t \right) f'(sa + (1-s)b) \Big|_0^{1/2} \right\} ds \\
&= \int_{1/2}^1 \left(\frac{5}{8} - \frac{s}{2} \right) f'(sa + (1-s)b) ds \\
&= \left(\frac{5}{8} - \frac{s}{2} \right) \frac{f(sa + (1-s)b)}{a-b} \Big|_{1/2}^1 + \frac{1}{2(a-b)} \int_{1/2}^1 f(sa + (1-s)b) ds \\
&= -\frac{3f(\frac{a+b}{2})}{8(a-b)} + \frac{f(a)}{8(a-b)} + \frac{1}{2(a-b)} \int_{1/2}^1 f(sa + (1-s)b) ds,
\end{aligned}$$

$$\begin{aligned}
(2.8) \quad I_7 &= \int_0^{1/2} \left\{ \int_{1/2}^1 f'(sa + (1-s)b) (t-s-1) dt \right\} ds \\
&= \int_0^{1/2} \left\{ \left(\frac{t^2}{2} - st - t \right) f'(sa + (1-s)b) \Big|_{1/2}^1 \right\} ds \\
&= \int_0^{1/2} \left(-\frac{1}{8} - \frac{s}{2} \right) f'(sa + (1-s)b) ds \\
&= \left(-\frac{1}{8} - \frac{s}{2} \right) \frac{f(sa + (1-s)b)}{a-b} \Big|_0^{1/2} + \frac{1}{2(a-b)} \int_0^{1/2} f(sa + (1-s)b) ds \\
&= -\frac{3f(\frac{a+b}{2})}{8(a-b)} + \frac{f(b)}{8(a-b)} + \frac{1}{2(a-b)} \int_0^{1/2} f(sa + (1-s)b) ds,
\end{aligned}$$

$$\begin{aligned}
(2.9) \quad I_8 &= \int_{1/2}^1 \left\{ \int_{1/2}^1 f'(sa + (1-s)b) (t-s) dt \right\} ds \\
&= \int_{1/2}^1 \left\{ \left(\frac{t^2}{2} - st \right) f'(sa + (1-s)b) \Big|_{1/2}^1 \right\} ds \\
&= \int_{1/2}^1 \left(\frac{3}{8} - \frac{s}{2} \right) f'(sa + (1-s)b) ds \\
&= \left(\frac{3}{8} - \frac{s}{2} \right) \frac{f(sa + (1-s)b)}{a-b} \Big|_{1/2}^1 + \frac{1}{2(a-b)} \int_{1/2}^1 f(sa + (1-s)b) ds \\
&= -\frac{f(\frac{a+b}{2})}{8(a-b)} - \frac{f(a)}{8(a-b)} + \frac{1}{2(a-b)} \int_{1/2}^1 f(sa + (1-s)b) ds.
\end{aligned}$$

Adding (2.2)-(2.9) and rewritting, we easily deduce:

$$\begin{aligned}
&\int_0^1 \int_0^1 (f'(ta + (1-t)b) - f'(sa + (1-s)b)) (m(t) - m(s)) dt ds \\
&= I_1 + I_2 + I_3 + I_4 - I_5 - I_6 - I_7 - I_8 \\
&= 2 \frac{f(\frac{a+b}{2})}{a-b} - \frac{2}{(a-b)} \int_0^1 f(ta + (1-t)b) dt.
\end{aligned}$$

Using the change of the variable $x = ta + (1-t)b$ for $t \in [0, 1]$, and multiplying the both sides by $(a-b)/2$, we obtain (2.1), which completes the proof. \square

Theorem 2. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^2$ is convex on $[a, b]$, then the following inequality holds:

$$(2.10) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{\sqrt{6}} \left(\frac{|f'(a)|^2 + |f'(b)|^2}{2} \right)^{\frac{1}{2}}.$$

Proof. From Lemma 2, using the Cauchy-Schwartz for double integrals, we get

$$\begin{aligned}
 (2.11) \quad & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 &= \frac{b-a}{2} \left| \int_0^1 \int_0^1 (f'(ta+(1-t)b) - f'(sa+(1-s)b))(m(s) - m(t)) dt ds \right| \\
 &\leq \frac{b-a}{2} \left[\int_0^1 \int_0^1 |f'(ta+(1-t)b) - f'(sa+(1-s)b)| |m(t) - m(s)| dt ds \right] \\
 &\leq \frac{b-a}{2} \left[\int_0^1 \int_0^1 |f'(ta+(1-t)b)| |m(t) - m(s)| dt ds \right. \\
 &\quad \left. + \int_0^1 \int_0^1 |f'(sa+(1-s)b)| |m(t) - m(s)| dt ds \right] \\
 &= (b-a) \int_0^1 \int_0^1 |f'(ta+(1-t)b)| |m(t) - m(s)| dt ds \\
 &\leq (b-a) \left[\left(\int_0^1 \int_0^1 (m(t) - m(s))^2 dt ds \right)^{\frac{1}{2}} \left(\int_0^1 \int_0^1 |f'(ta+(1-t)b)|^2 dt ds \right)^{\frac{1}{2}} \right]
 \end{aligned}$$

By definitions of $m(t)$ and $m(s)$ and by simple computation, we get

$$\begin{aligned}
 (2.12) \quad & \int_0^1 \int_0^1 (m(t) - m(s))^2 dt ds \\
 &= \int_0^1 \left\{ \int_0^{1/2} (t - m(s))^2 dt + \int_{1/2}^1 (t - 1 - m(s))^2 dt \right\} ds \\
 &= \int_0^{1/2} \left\{ \int_0^{1/2} (t - s)^2 dt \right\} ds + \int_{1/2}^1 \left\{ \int_0^{1/2} (t - s + 1)^2 dt \right\} ds \\
 &\quad + \int_0^{1/2} \left\{ \int_{1/2}^1 (t - s - 1)^2 dt \right\} ds + \int_{1/2}^1 \left\{ \int_{1/2}^1 (t - s)^2 dt \right\} ds \\
 &= \int_0^{1/2} \left\{ \frac{(t-s)^3}{3} \Big|_0^{1/2} \right\} ds + \int_{1/2}^1 \left\{ \frac{(t-s+1)^3}{3} \Big|_0^{1/2} \right\} ds \\
 &\quad + \int_0^{1/2} \left\{ \frac{(t-s-1)^3}{3} \Big|_{1/2}^1 \right\} ds + \int_{1/2}^1 \left\{ \frac{(t-s)^3}{3} \Big|_{1/2}^1 \right\} ds \\
 &= \int_0^{1/2} \left\{ \frac{(1-2s)^3}{24} + \frac{s^3}{3} \right\} ds + \int_{1/2}^1 \left\{ \frac{(3-2s)^3}{24} + \frac{(s-1)^3}{3} \right\} ds \\
 &\quad + \int_0^{1/2} \left\{ \frac{(2s+1)^3}{24} - \frac{s^3}{3} \right\} ds + \int_{1/2}^1 \left\{ \frac{(2s-1)^3}{3} + \frac{(1-s)^3}{3} \right\} ds = \frac{1}{6}
 \end{aligned}$$

and since $|f'|^2$ is convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(ta + (1-t)b)|^2 \leq t|f'(a)|^2 + (1-t)|f'(b)|^2,$$

hence

$$(2.13) \quad \left(\int_0^1 \int_0^1 |f'(ta + (1-t)b)|^2 dt ds \right)^{\frac{1}{2}} \leq \left(\int_0^1 \int_0^1 (t|f'(a)|^2 + (1-t)|f'(b)|^2) dt ds \right)^{\frac{1}{2}} \\ = \left(\frac{|f'(a)|^2 + |f'(b)|^2}{2} \right)^{\frac{1}{2}}.$$

Therefore, using (2.12) and (2.13) in (2.11), we obtain (2.10).

Theorem 3. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds:

$$(2.14) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2 and Hölder's integral inequality, we observe that

$$(2.15) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ = \frac{b-a}{2} \left| \int_0^1 \int_0^1 (f'(ta + (1-t)b) - f'(sa + (1-s)b)) (m(s) - m(t)) dt ds \right| \\ \leq \frac{b-a}{2} \left[\int_0^1 \int_0^1 |f'(ta + (1-t)b) - f'(sa + (1-s)b)| |m(t) - m(s)| dt ds \right] \\ \leq \frac{b-a}{2} \left[\int_0^1 \int_0^1 |f'(ta + (1-t)b)| |m(t) - m(s)| dt ds \right. \\ \left. + \int_0^1 \int_0^1 |f'(sa + (1-s)b)| |m(t) - m(s)| dt ds \right] \\ \leq (b-a) \int_0^1 \int_0^1 |f'(ta + (1-t)b)| |m(t) - m(s)| dt ds \\ \leq (b-a) \left[\left(\int_0^1 \int_0^1 |m(t) - m(s)|^p dt ds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 |f'(ta + (1-t)b)|^q dt ds \right)^{\frac{1}{q}} \right].$$

By definitions of $m(t)$ and $m(s)$, we get

$$\begin{aligned} & \int_0^1 \int_0^1 |m(t) - m(s)|^p dt ds \\ &= \int_0^1 \left\{ \int_0^{1/2} |t - m(s)|^p dt + \int_{1/2}^1 |t - 1 - m(s)|^p dt \right\} ds \\ &= \int_0^{1/2} \int_0^{1/2} |t - s|^p dt ds + \int_{1/2}^1 \int_0^{1/2} |t - s + 1|^p dt ds \\ &\quad + \int_0^{1/2} \int_{1/2}^1 |t - s - 1|^p dt ds + \int_{1/2}^1 \int_{1/2}^1 |t - s|^p dt ds \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Thus, by simple computation we obtain

$$\begin{aligned} (2.16) \quad J_1 &= \int_0^{1/2} \int_0^{1/2} |t - s|^p dt ds = \int_0^{1/2} \left\{ \int_0^s (s - t)^p dt + \int_s^{1/2} (t - s)^p dt \right\} ds \\ &= \frac{1}{p+1} \int_0^{1/2} \left\{ s^{p+1} + \left(\frac{1}{2} - s\right)^{p+1} \right\} ds = \frac{2}{2^{p+1} (p+1) (p+2)}, \end{aligned}$$

$$\begin{aligned} (2.17) \quad J_2 &= \int_{1/2}^1 \int_0^{1/2} |t - s + 1|^p dt ds = \int_{1/2}^1 \int_0^{1/2} (t - s + 1)^p dt ds, \quad (\text{for } t - s + 1 \geq 0) \\ &= \frac{1}{p+1} \int_{1/2}^1 \left\{ \left(\frac{3}{2} - s\right)^{p+1} - (1 - s)^{p+1} \right\} ds \\ &= \frac{1}{(p+1) (p+2)} - \frac{1}{2^{p+1} (p+1) (p+2)}, \end{aligned}$$

$$\begin{aligned} (2.18) \quad J_3 &= \int_0^{1/2} \int_{1/2}^1 |t - s - 1|^p dt ds = \int_0^{1/2} \int_{1/2}^1 (-t + s + 1)^p dt ds, \quad (\text{for } t - s - 1 \leq 0) \\ &= \frac{1}{p+1} \int_0^{1/2} \left\{ -s^{p+1} + \left(s + \frac{1}{2}\right)^{p+1} \right\} ds \\ &= \frac{1}{(p+1) (p+2)} - \frac{1}{2^{p+1} (p+1) (p+2)}, \end{aligned}$$

$$\begin{aligned} (2.19) \quad J_4 &= \int_{1/2}^1 \int_{1/2}^1 |t - s|^p dt ds = \int_{1/2}^1 \left\{ \int_{1/2}^s (s - t)^p dt + \int_s^1 (t - s)^p dt \right\} ds \\ &= \frac{1}{p+1} \int_{1/2}^1 \left\{ \left(s - \frac{1}{2}\right)^{p+1} + (1 - s)^{p+1} \right\} ds \\ &= \frac{1}{2^{p+1} (p+1) (p+2)}. \end{aligned}$$

Adding (2.16)-(2.19), we have

$$(2.20) \quad \left(\int_0^1 \int_0^1 |m(t) - m(s)|^p dt ds \right)^{\frac{1}{p}} = \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}}.$$

Since $|f'|^q$ is convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(ta + (1-t)b)|^q \leq t|f'(a)|^q + (1-t)|f'(b)|^q,$$

hence

$$(2.21) \quad \left(\int_0^1 \int_0^1 |f'(ta + (1-t)b)|^q dt ds \right)^{\frac{1}{q}} \leq \left(\int_0^1 \int_0^1 (t|f'(a)|^q + (1-t)|f'(b)|^q) dt ds \right)^{\frac{1}{q}} \\ = \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Therefore, using (2.20) and (2.21) in (2.15), we obtain (2.14). \square

3. Applications to some special means

We now consider the applications of our Theorems to the following special means:

(a) The arithmetic mean: $A = A(a, b) := \frac{a+b}{2}$, $a, b \geq 0$,

(b) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0,$$

(c) The Identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0,$$

(d) The p -logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; a, b > 0.$$

The following proposition holds:

Proposition 1. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, we have

$$|A^n(a, b) - L_n^n(a, b)| \leq |n| \frac{(b-a)}{\sqrt{6}} A\left(a^{2(n-1)}, b^{2(n-1)}\right).$$

Proof. The proof is immediate from Theorem 2 applied for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $|n| \geq 1$. \square

Proposition 2. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$. Then, for all $q > 1$, we have

$$|A(a^n, b^n) - L_n^n(a, b)| \leq |n| (b-a) \left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} \left[A\left(|a|^{q(n-1)}, |b|^{q(n-1)}\right)\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 3 applied for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $|n| \geq 1$. \square

Proposition 3. Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q > 1$, we have

$$\ln \left[\frac{I(a, b)}{A(a, b)} \right] \leq \frac{(b-a)}{ab} \left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} [A(|b|^q, |a|^q)]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 3 applied to $f : (0, \infty) \rightarrow (-\infty, 0)$, $f(x) = -\ln(x)$ and the details are omitted. \square

Proposition 4. Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, the following inequality holds:

$$|A^{-1}(a, b) - L^{-1}(a, b)| \leq \frac{(b-a)}{(ab)^2} \left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}} \left[A\left(|a|^{2q}, |b|^{2q}\right)\right]^{\frac{1}{q}}.$$

Proof. The proof is obvious from Theorem 3 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. \square

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