Ilkhan & Kara, *Cogent Mathematics & Statistics* (2018), 5: 1487500
https://doi.org/10.1080/25742558.2018.1487500

cogent mathematics & statistics

PURE MATHEMATICS | RESEARCH ARTICLE

A new type of statistical Cauchy sequence and its relation to Bourbaki completeness

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Received: 11 May 2018
Accepted: 07 June 2018
First Published: 27 June 2018

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Reviewing editor:
Hari M. Srivastava, University of Victoria, Canada

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Abstract: Bourbaki complete metric spaces are important since they are a class between compact metric spaces and complete metric spaces. The aim of the present paper is to introduce the statistical Bourbaki–Cauchy sequence as a new concept and to give an equivalent condition for a metric space to be Bourbaki complete. Also, Bourbaki complete and Bourbaki-bounded metric spaces are characterized in terms of functions which preserve statistical Bourbaki–Cauchy sequences.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Analysis - Mathematics; Sequences & Series; Functional Analysis; Mathematical Analysis

Keywords: Bourbaki–Cauchy sequences; Bourbaki completeness; Bourbaki boundedness; statistical convergence; asymptotic density

1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will stand for the set of all natural numbers and real numbers, respectively.

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PUBLIC INTEREST STATEMENT

Compact metric spaces and complete metric spaces play an important role in functional analysis. Metric spaces satisfying properties between compactness and completeness have been the subject of research for a number of papers over years. Bourbaki complete metric spaces in which every Bourbaki–Cauchy sequence clusters are complete but not necessarily compact. Bourbaki–Cauchy sequences are defined recently to characterize Bourbaki-bounded metric spaces as Cauchy sequences characterize totally bounded metric spaces. In this paper, some new characterizations of Bourbaki-bounded and Bourbaki complete metric spaces are given.

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Complete metric spaces play an important role in logic, fixed point theory, computer science, quantum mechanics and other branches of science as well as in functional analysis. Also, compactness is central to the theory of metric spaces. It is a well-known fact that a continuous function from a compact metric space to any metric space is uniformly continuous whereas compactness is not necessary. For instance, a continuous function defined on a uniformly discrete metric space with infinitely many points is uniformly continuous, but this discrete space is not compact. Since every compact metric space is complete (whereas the reverse is not the case), metric spaces satisfying properties stronger than completeness but weaker than compactness have been investigated by many mathematicians. The most known of such spaces is the Atsuji space (also called UC space) defined as every real-valued continuous function on it is uniformly continuous. Because of the importance of such space, many authors gave some different characterizations of this space. First, Nagata (Nagata, 1950) studied on Atsuji spaces. Later, in (Atsuji, 1958; Monteiro & Peixoto, 1951), the authors gave many new equivalent conditions for a metric space to be an Atsuji space. In a survey article by Kundu and Jain (Kundu & Jain, 2006), twenty-five equivalent conditions are brought close together. Further, Beer in the papers (Beer, 1985, 1986) investigated Atsuji spaces. Most recently, some new practical and exotic characterizations of these spaces are presented in a different aspect by Aggarwal and Kundu (Aggarwal & Kundu, 2016). For more papers about Atsuji spaces, one can see Aggarwal & Kundu, (2017); Jain & Kundu, (2007).

For a metric space, features which remain in between compactness and completeness are studied by many authors. One of the nice papers related to this subject is (Beer, 2008) by Beer. As well as being an Atsuji space, being a boundedly compact, a uniformly locally compact, a cofinally complete or a strongly cofinally complete space can be given as examples of such features. Recently, a new pair of these features are presented by Garrido and Meroño (Garrido & Meroño, 2014) such a way that clustering all sequences belonging to a more general class than the class of Cauchy sequences to make this property stronger than completeness. Hence, this property is weaker than compactness since every sequence has a convergent subsequence in a compact metric space. First, they define the concept of a Bourbaki–Cauchy sequence which is more general than a Cauchy sequence. A sequence (Θ_n) in a metric space (X,ρ) is said to be Bourbaki–Cauchy if for every $\varepsilon > 0$ there exist $m, n_0 \in \mathbb{N}$ and $x \in X$ such that $x_n \in \mathcal{B}^m(x,\varepsilon)$ for $n \geq n_0$, where $\mathcal{B}^m(x,\varepsilon)$ consists of points $y \in X$ satisfying $\rho(x,a_1) < \varepsilon, \rho(a_1,a_2) < \varepsilon, \ldots, \rho(a_{m-1},y) < \varepsilon$ for some $a_1, a_2, \ldots, a_{m-1} \in X$. Then, it is obvious that every Cauchy sequence is a Bourbaki–Cauchy sequence (but not reverse). Unlike a complete metric space, Bourbaki completeness is defined as every Bourbaki–Cauchy sequence in X has a convergent subsequence. Since a Bourbaki–Cauchy sequence may have more than one cluster point, the sequence itself cannot be convergent. As an example, the sequence $((-1)^n)$ in $\mathbb{R}$ with the usual metric is a Bourbaki–Cauchy sequence which is not Cauchy and so not convergent but has some convergent subsequences. Second, they define a cofinally Bourbaki–Cauchy sequence and a cofinally Bourbaki complete metric space analogous with Bourbaki complete metric space. The class of Bourbaki–Cauchy sequences and cofinally Bourbaki–Cauchy sequences appeared to characterize a Bourbaki-bounded subset of a metric space in a similar way that a Cauchy sequence characterizes total boundedness of a set. For the first time, Atsuji (Atsuji, 1958) introduced this concept of boundedness under the name of finitely chainable to study metric spaces on which every real-valued uniformly continuous function is bounded. A subset A of a metric space (X,ρ) is said to be Bourbaki bounded if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and finitely many points $x_1, x_2, \ldots, x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} \mathcal{B}^m(x_i,\varepsilon)$. In a metric space, a totally bounded set is Bourbaki bounded and a Bourbaki-bounded set is bounded in the usual sense.

As an extension of usual convergence, the concept of statistical convergence for real-valued sequences was introduced by Fast (Fast, 1951) and Steinhaus (Steinhaus, 1951). However, the idea of statistical convergence (appeared under the name of almost convergence) goes back to Zygmund (Zygmund, 2002) (first edition published in Warsaw 1935). The formal definition is based on the notion of natural density (asymptotic density) of a subset A in $\mathbb{N}$ (Niven, Zuckerman, & Montgomery, 1991). If the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_A(j)$ exists, it is called as the natural density of A and denoted by $\Delta(A)$, where χ_A is the characteristic function of A (that is, $\chi_A(j)=1$ if $j\in A$; else $\chi_A(j)=0$). Along the paper, when $\Delta(A)$ appears, we mean that it is well defined. Also, note that the following statements are true for any subsets A, B in $\mathbb{N}$.

(1) If $\Delta(A)$ exists, then $0\leq\Delta(A)\leq1$ and $\Delta(\mathbb{N}\backslash A)$ also exists with $\Delta(\mathbb{N}\backslash A)=1-\Delta(A)$.

(2) If $\Delta(A)=1$ and $A\subset B$, then $\Delta(B)=1$.

(3) If $\Delta(A)=0$ and $\Delta(B)=0$, then $\Delta(A\cup B)=0$.

(4) If $\Delta(A)=1$ and $\Delta(B)=1$, then $\Delta(A\cap B)=1$.

The statistical convergence was generalized to sequences in some other spaces and studied on these spaces. For example, it has been considered in metric spaces (Küçükaslan, Değer, & Dovgoshey, 2014), cone metric spaces (Li, Lin, & Ge, 2015), topological and uniform spaces (Di Maio & Kočinac, 2008) and topological groups (Çakall, 2009). In Schoenberg (1959), Schoenberg gave some basic properties of statistical convergence and also studied the concept as a summability method. Later on it was further investigated and linked with the summability theory by Fridy (Fridy, 1985), Fridy and Orhan (Fridy & Orhan, 1993), Mursaleen and Edely (Mursaleen & Edely, 2004), Acar and Mohiuddine (Acar & Mohiuddine, 2016), M. Aldhaifallah et al (Aldhaifallah, Nisar, Srivastava, & Mursaleen, 2017), Belen and Mohiuddine (Belen & Mohiuddine, 2013), Kirisci and Karaisa (2017), Braha et al (Braha, Srivastava, & Mohiuddine, 2014) and many others. Also, several important applications of statistical convergence is available in different areas of mathematics such as measure theory (Miller, 1995), optimization theory (Pehlivan & Mamedov, 2000), approximation theory (Edely, Mohiuddine, & Noman, 2010, Gadjiev & Orhan, 2002, Kadak, Braha, & Srivastava, 2017, Kadak & Mohiuddine, 2018, Srivastava, Jena, Paikray, & Mishra, 2018), probability theory (Fridy & Khan, 1998), etc. A sequence (Θ_n) in a metric space (X,ρ) statistically converges to a point $x\in X$ if for every $\varepsilon>0$ we have $\Delta(A_\varepsilon)=1$, where $A_\varepsilon=\{n\in\mathbb{N}:\rho(x,\Theta_n)<\varepsilon\}$. A sequence (Θ_n) is a statistical Cauchy sequence in X if for every $\varepsilon>0$ there exists $N=N(\varepsilon)\in\mathbb{N}$ such that $\Delta(A_{N(\varepsilon)})=1$, where $A_{N(\varepsilon)}=\{n\in\mathbb{N}:\rho(\Theta_N,\Theta_n)<\varepsilon\}$. Also, (Θ_n) is said to be statistically bounded in X if there exist $x\in X$ and $M>0$ such that $\Delta(\{n:\mathbb{N}:\rho(\Theta_n,x)\leq M\})=1$.

In this paper, we define the statistical Bourbaki–Cauchy sequence as a new concept in the setting of metric spaces. By their definitions, being a Bourbaki–Cauchy sequence or a statistical Cauchy sequence implies that this sequence is also a statistical Bourbaki–Cauchy sequence. However, a statistical Bourbaki–Cauchy sequence need not be Bourbaki–Cauchy or statistical Cauchy which can be seen in Example 2.2 and Example 2.3. Further, we state a new uniform condition by the aid of a statistical Bourbaki Cauchy sequence and prove that it is equivalent to Bourbaki completeness (see Theorem 2.6). Moreover, we study some new characterizations of Bourbaki completeness and Bourbaki boundedness of a metric space by using functions which preserve statistical Bourbaki Cauchy sequences.

2. Statistical Bourbaki–Cauchy sequence and some results related to this concept

We start this section with the definition of a statistical Bourbaki–Cauchy sequence by using the concept of natural density of a set in $\mathbb{N}$. Later on, we examine the relations between this new sequence with some other sequences defined earlier in the literature.

Definition 2.1. *A sequence (Θ_n) in a metric space (X,ρ) is said to be statistical Bourbaki–Cauchy if for every $\varepsilon>0$ there exist $m\in\mathbb{N}$ and $x\in X$ such that $\Delta\{n\in\mathbb{N}:\Theta_n\in\mathcal{B}^m(x,\varepsilon)\}=1$.*

Equivalently, this definition can be given as for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $x \in X$ such that $\Delta\{n \in \mathbb{N} : \Theta_n \notin \mathcal{B}^m(x, \varepsilon)\} = 0$ owing to the fact that $\Delta(\mathbb{N}\backslash A) = 1 - \Delta(A)$ for all subset A of $\mathbb{N}$.

From the definitions, it is clear that every Bourbaki–Cauchy sequence in a metric space is also statistical Bourbaki–Cauchy. But the reverse implication is not true as the following example shows. One of the most interesting difference between these two type of Cauchy sequences is that statistical Bourbaki–Cauchy sequences are not generally bounded whereas Bourbaki–Cauchy sequences are bounded in the sense of the metric. On the other hand, if (Θ_n) is a statistical Bourbaki–Cauchy sequence in a metric space X, then given any $\varepsilon > 0$, we have

$$\Delta(\{n \in \mathbb{N} : \Theta_n \in \mathcal{B}(x, m\varepsilon)\}) \geq \Delta(\{n \in \mathbb{N} : \Theta_n \in \mathcal{B}^m(x, \varepsilon)\}) = 1$$

for some $x \in X$ and $m \in \mathbb{N}$ which shows that (Θ_n) is statistically bounded.

Example 2.2. *Consider $\mathbb{R}$ with the usual metric. The sequence (Θ_n) defined in the following way*

$$\Theta_n = \begin{cases} n & , \quad n \text{ is a prime number}, \\ 1 & , \quad \text{otherwise} \end{cases}$$

is in fact statistical Cauchy and so statistical Bourbaki–Cauchy due to the fact that the natural density of the set of all prime numbers equals to zero (see Kováč, 2005*). However, it is not a Bourbaki–Cauchy sequence since it is not bounded with respect to the usual metric.*

Obviously, statistical convergence of a sequence in a metric space implies that the sequence is a statistical Bourbaki–Cauchy sequence. But there are statistical Bourbaki–Cauchy sequences in some metric spaces which are not statistically convergent such as given in the following example.

Example 2.3. *The sequence $((-1)^n)$ in $\mathbb{R}$ with the usual metric is a Bourbaki–Cauchy sequence and therefore it is a statistical Bourbaki–Cauchy sequence. But it is not statistical Cauchy. Although it has statistically convergent subsequences, the sequence $((-1)^n)$ itself is not statistically convergent.*

These last two examples show that if a statistical Bourbaki–Cauchy sequence has a statistically convergent subsequence, then the sequence itself does not have to be statistically convergent likewise a Bourbaki–Cauchy sequence. Also, it can be seen that there is no relation between statistically Cauchy and Bourbaki–Cauchy sequences. Consequently, we have the following diagram where the reverse implications do not hold.

$$\begin{array}{lll} \text{Cauchy} & \Rightarrow & \text{statistical Cauchy} \\ \Downarrow & & \Downarrow \\ \text{Bourbaki} - \text{Cauchy} & \Rightarrow & \text{statisticalBourbaki} - \text{Cauchy} \end{array}$$

In the following theorem, some relations between Bourbaki–Cauchy and statistical Bourbaki–Cauchy sequences are obtained.

Theorem 2.4. *For a sequence (Θ_n) in a metric space (X, ρ), the following statements are equivalent.*

(1) *(Θ_n) is a statistical Bourbaki–Cauchy sequence in X.*

(2). *There exists a Bourbaki–Cauchy subsequence (Θ_{n_j}) of (Θ_n) such that $\Delta(\{n_j \in \mathbb{N} : j \in \mathbb{N}\}) = 1$.*

(3). *There exists a statistical Bourbaki–Cauchy subsequence (Θ_{n_j}) of (Θ_n) such that $\Delta(\{n_j \in \mathbb{N} : j \in \mathbb{N}\}) = 1$.*

Proof. $(1) \Rightarrow (2)$ Let (Θ_n) be a statistical Bourbaki–Cauchy sequence in X. Then, there exist $m_1 \in \mathbb{N}$ and $i_1 \in \mathbb{N}$ such that $\Delta(A_1) = 1$, where $A_1 = \{k \in \mathbb{N} : \Theta_k \in \mathcal{B}^{m_1}(\Theta_{i_1}, \frac{1}{2})\}$. Similarly, there exist $m_2 \in$

$\mathbb{N}$ and $i_2 \in \mathbb{N}$ such that $\Delta(B_1) = 1$, where $B_1 = \{k \in \mathbb{N} : \Theta_k \in \mathcal{B}^{m_2}(\Theta_{i_2}, \frac{1}{2^2})\}$. Put $A_2 = A_1 \cap B_1$. Then, we have $\Delta(A_2) = 1$, $A_2 \subset A_1$ and $\Theta_{k_2} \in \mathcal{B}^{2m_2}(\Theta_{k_1}, \frac{1}{2^2})$ for all $k_1, k_2 \in A_2$. By continuing this process, we obtain a decreasing sequence $A_1 \supset A_2 \supset \ldots \supset A_j \supset \ldots$ of subsets of $\mathbb{N}$ with $\Delta(A_j) = 1$ and $\Theta_{k_2} \in \mathcal{B}^{2m_j}(\Theta_{k_1}, \frac{1}{2^j})$ for all $k_1, k_2 \in A_j$. Let $n_1 \in A_1$ and choose $n_2 \in A_2$ with $n_2 > n_1$ such that $\frac{1}{n}\sum_{k=1}^{n} \chi_{A_2}(k) > 1 - \frac{1}{2}$ for all $n \geq n_2$. In this manner, we construct an increasing sequence (n_j) in $\mathbb{N}$ such that $\frac{1}{n}\sum_{k=1}^{n} \chi_{A_j}(k) > 1 - \frac{1}{j}$ for all $n \geq n_j$, where $n_j \in A_j$ for each $j \in \mathbb{N}$. Set $A = \{k : 1 \leq k \leq n_1\} \cup \left[\bigcup_{j \in \mathbb{N}} \{k : n_j < k \leq n_{j+1}\} \cap A_j\right]$. For any $j \in \mathbb{N}$ and $n_j < n \leq n_{j+1}$, we have $\frac{1}{n}\sum_{k=1}^{n} \chi_A(k) \geq \frac{1}{n}\sum_{k=1}^{n} \chi_{A_j}(k) > 1 - \frac{1}{j}$ which implies that $\Delta(A) = 1$. Now, given any $\varepsilon > 0$, we can find a natural number $j_0 \in \mathbb{N}$ satisfying $\frac{1}{2^{j_0}} < \varepsilon$. Choose fixed $k \in A$ and an arbitrary $l \in A$ with $l > k > n_{j_0}$. Then, there exist $r, s \in \mathbb{N}$ with $s \geq r \geq j_0$ such that $k \in A_r$, $n_r < k \leq n_{r+1}$ and $l \in A_s$, $n_s < l \leq n_{s+1}$. Hence, we have $l, k \in A_r$ and so $x_l \in \mathcal{B}^{2m_r}(\Theta_k, \frac{1}{2^r}) \subset \mathcal{B}^{2m_r}(\Theta_k, \varepsilon)$ which means that $(\Theta_l)_{l \in A}$ is the desired Bourbaki–Cauchy subsequence.

$(2) \Rightarrow (3)$ The implication follows from the fact that a Bourbaki–Cauchy sequence is a statistical Bourbaki–Cauchy sequence.

$(3) \Rightarrow (1)$ Let (Θ_{n_j}) be a statistical Bourbaki–Cauchy subsequence of (Θ_n), where $\Delta(\{n_j \in \mathbb{N} : j \in \mathbb{N}\}) = 1$. Then, given any $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $x \in X$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \chi_A(j) \geq \lim_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} \chi_{\widetilde{A}}(n_j) = 1,$$

where $A = \{j \in \mathbb{N} : \Theta_j \in \mathcal{B}^m(x, \varepsilon)\}$ and $\widetilde{A} = \{n_j \in \mathbb{N} : \Theta_{n_j} \in \mathcal{B}^m(x, \varepsilon)\}$, respectively. We conclude that $\Delta(A) = 1$ which proves that the sequence (Θ_n) is statistical Bourbaki–Cauchy in X. □

As a consequence of this theorem, we have the following result.

Corollary 2.5. *Every statistical Bourbaki–Cauchy sequence has a Bourbaki–Cauchy subsequence in a metric space.*

The next result states a condition which is equivalent to Bourbaki completeness.

Theorem 2.6. *A metric space is Bourbaki complete if and only if every statistical Bourbaki–Cauchy sequence has a statistical convergent subsequence.*

Proof. Suppose that a metric space X is Bourbaki complete. Let (Θ_n) be a statistical Bourbaki–Cauchy sequence in X. By the previous theorem, it has a Bourbaki–Cauchy subsequence. Then, Bourbaki completeness of X implies that it has a usual convergent and so statistical convergent subsequence.

For the converse, take a Bourbaki–Cauchy sequence (Θ_n) in X. Since it is also statistical Bourbaki–Cauchy, by hypothesis there exists a statistical convergent subsequence of (Θ_n). Since every statistical convergent sequence has a convergent subsequence (see Lemma 1.1 in Šalát, 1980), it follows that X is a Bourbaki complete metric space. □

In a recent paper (Kundu, Aggarwal, & Hazra, 2017), Kundu et al. studied three new characterizations of Bourbaki-bounded metric spaces. For this purpose, they used various types of functions defined in (Aggarwal & Kundu, 2017). One of them is a Bourbaki–Cauchy regular function required for our main results. A function $f : (X, \rho) \to (X', \rho')$ is said to be Bourbaki–Cauchy regular if $(f(\Theta_n))$ is a Bourbaki–Cauchy sequence in X' whenever (Θ_n) is a Bourbaki–Cauchy sequence in X. Also, we need to recall some more definitions. In a metric space, (X,ρ), for $\varepsilon > 0$, the ordered set $\{x_0, x_1, \ldots, x_m\}$ in X is called an ε-chain of length m from x_0 to x_m if $\rho(x_{i-1}, x_i) < \varepsilon$ holds for $i = 1, 2, \ldots, m$. It is said that X is ε-chainable if each two points of X can be joined by an ε-chain, and X is chainable if X is ε-chainable for every $\varepsilon > 0$.

In the next theorem, some new characterizations of Bourbaki completeness are given by using functions which preserve statistical Bourbaki–Cauchy sequences and can be named as a statistical Bourbaki–Cauchy regular function. Further, in the last theorem, Bourbaki boundedness of a metric space is characterized in terms of these functions. Before characterizing Bourbaki completeness, we examine the relation between statistical Bourbaki–Cauchy regular and Bourbaki–Cauchy regular functions.

Lemma 2.7. *Each Bourbaki–Cauchy regular function is statistical Bourbaki–Cauchy regular.*

Proof. Let $f : (X,\rho) \to (X',\rho')$ be a Bourbaki–Cauchy regular function and (Θ_n) be a statistical Bourbaki–Cauchy sequence in X. Then, by Theorem 2.4, it has a Bourbaki–Cauchy subsequence (Θ_{n_j}) such that $\Delta(\{n_j \in \mathbb{N} : j \in \mathbb{N}\}) = 1$. Hence, our assumption implies that the sequence $(f(\Theta_{n_j}))$ is also Bourbaki–Cauchy. It follows again from Theorem 2.4 that the sequence $(f(\Theta_n))$ is statistical Bourbaki–Cauchy. Thus, we conclude that the function f is statistical Bourbaki–Cauchy regular.

Hereby, we give one of our main results.

Theorem 2.8. *The following statements are equivalent for a metric space* (X,ρ).

(1) (X,ρ) *is Bourbaki complete.*

(2) *Every continuous function from* (X,ρ) *into a chainable metric space* (X',ρ') *is Bourbaki–Cauchy regular.*

(3). *Every continuous function from* (X,ρ) *into a chainable metric space* (X',ρ') *is statistical Bourbaki–Cauchy regular.*

(4) *Every continuous function from* (X,ρ) *into* $\mathbb{R}$ *is statistical Bourbaki–Cauchy regular.*

Proof. $(1) \Rightarrow (2)$ It is proved in [3, Theorem 2.4].

$(2) \Rightarrow (3)$ The proof comes from the fact that every Bourbaki–Cauchy regular function is statistical Bourbaki–Cauchy regular which is proved in Lemma 2.7.

$(3) \Rightarrow (4)$ It is clear since $\mathbb{R}$ is chainable with respect to the usual metric.

$(4) \Rightarrow (1)$ Let (Θ_n) be a Bourbaki–Cauchy sequence in X. We can say that (Θ_n) has a Bourbaki–Cauchy subsequence whose terms are distinct; otherwise, there is nothing to prove. Now, suppose that a Bourbaki–Cauchy sequence (Θ_n) with distinct terms has no convergent subsequence. It follows that $Y = \{\Theta_n : n \in \mathbb{N}\}$ is a closed subset of X. Also, the subspace topology on the set Y is discrete topology since it consists of only isolated points. Define a real-valued function g on Y with $g(\Theta_n) = n$ for all $n \in \mathbb{N}$. Then g is a continuous function since every function defined on a discrete topological space is continuous. Accordingly, Tietze extension theorem implies that there is a continuous function $f : (X,\rho) \to \mathbb{R}$ with $f(\Theta_n) = g(\Theta_n)$ for all $n \in \mathbb{N}$. But this function cannot be statistical Bourbaki–Cauchy regular since the sequence $(f(\Theta_n)) = (n)$ is not a statistical Bourbaki–Cauchy sequence in $\mathbb{R}$ whereas (Θ_n) is statistical Bourbaki–Cauchy in X. Therefore, every Bourbaki–

Cauchy sequence in X must have a convergent subsequence which means that X is Bourbaki complete. □

Theorem 2.9. *The following statements are equivalent for a metric space* (X,ρ).

(1) *Every sequence in* X *has a statistical Bourbaki–Cauchy subsequence.*

(2) *If* $f:(X,\rho)\to(X',\rho')$ *is a statistical Bourbaki–Cauchy regular function, where* (X',ρ') *is any metric space, then* f *is bounded.*

(3). *If* $f:(X,\rho)\to(\tilde{X},\tilde{\rho})$ *is a statistical Bourbaki–Cauchy regular function, where* $(\tilde{X},\tilde{\rho})$ *is an unbounded chainable metric space, then* f *is bounded.*

(4) (X,ρ) *is Bourbaki bounded.*

Proof. $(1)\Rightarrow(2)$ Suppose that $f:(X,\rho)\to(X',\rho')$ is a statistical Bourbaki–Cauchy regular function but not a bounded function. Then for all $n\in\mathbb{N}$, we can construct a sequence (Θ_n) in X satisfying $\rho'(f(\Theta_{n+1}),f(\Theta_i))>n$ $(i=1,\dots,n)$ since the set $f(X)$ is not bounded. By hypothesis, the sequence (Θ_n) has a statistical Bourbaki–Cauchy subsequence, say (Θ_{n_k}). However, $(f(\Theta_{n_k}))$ is not a statistical Bourbaki–Cauchy sequence. Indeed, given any $m\in\mathbb{N}$ and $x'\in X'$, the set $\{k\in\mathbb{N}:f(\Theta_{n_k})\in\mathcal{B}^m(x',1)\}$ is finite. Otherwise, for a fixed $p\in A$, the inclusion

$$\mathcal{B}^m(x',1)\subset\mathcal{B}^{2m}(f(\Theta_{n_p}),1)\subset\mathcal{B}(f(\Theta_{n_p}),2m)$$

implies that $\rho'(f(\Theta_{n_k}),f(\Theta_{n_p}))<2m$ for infinitely many $k\in\mathbb{N}$. Hence, $(f(\Theta_{n_k}))$ is not a statistical Bourbaki–Cauchy sequence which contradicts the fact that f is statistical Bourbaki–Cauchy regular. Thus, f must be a bounded function.

$(2)\Rightarrow(3)$ This is obvious.

$(3)\Rightarrow(4)$ Suppose that (X,ρ) is not Bourbaki bounded. Then, there exists an $\varepsilon_0>0$ such that for all $m\in\mathbb{N}$, X cannot be covered by a union of finitely many sets $\mathcal{B}^m(x,\varepsilon_0)$ $(x\in X)$. Fix $\Theta_0\in X$. Then, we can choose $\Theta_1\in X$ such that $\Theta_1\notin\mathcal{B}^1(\Theta_0,\varepsilon_0)$. In the same manner, we can choose $\Theta_2\in X$ such that $\Theta_2\notin\mathcal{B}^2(\Theta_0,\varepsilon_0)\cup\mathcal{B}^2(\Theta_1,\varepsilon_0)$. By continuing this process, we obtain a sequence (Θ_j) in X such that $\Theta_j\notin\mathcal{B}^j(\Theta_i,\varepsilon_0)$ for every $j\in\mathbb{N}$ and $i=0,\dots,j-1$. Let $\tilde{x}_0\in\tilde{X}$. Since $(\tilde{X},\tilde{\rho})$ is an unbounded metric space, there is a point $\tilde{x}_n\in\tilde{X}$ such that $\tilde{\rho}(\tilde{x}_n,\tilde{x}_0)>n$ for all $n\in\mathbb{N}$. By virtue of this fact, we define an unbounded function $f:(X,\rho)\to(\tilde{X},\tilde{\rho})$ as:

$$f(x)=\begin{cases}\tilde{x}_j & , \quad \text{if } x=\Theta_j \text{for some } j\in\mathbb{N},\\ \tilde{x}_0 & , \quad \text{else}.\end{cases}$$

However, this function is statistical Bourbaki–Cauchy regular. To observe this, take a statistical Bourbaki–Cauchy sequence (Φ_n) in X. Then for this $\varepsilon_0>0$, there exit a natural number $m_0\in\mathbb{N}$ and a point $x_0\in X$ such that $\Delta(\{n\in\mathbb{N}:\Phi_n\in\mathcal{B}^{m_0}(x_0,\varepsilon_0)\})=1$; that is $\mathcal{B}^{m_0}(x_0,\varepsilon_0)$ contains infinitely many terms of the sequence (Φ_n). On the other hand, for only finitely many $j\in\mathbb{N}$, $\Theta_j\in\{\Phi_n:n\in A\}$, where $A=\{n\in\mathbb{N}:\Phi_n\in\mathcal{B}^{m_0}(x_0,\varepsilon_0)\}$. Otherwise, since the inclusion

$$\mathcal{B}^{m_0}(x_0,\varepsilon_0)\subset\mathcal{B}^{2m_0}(\Theta_{j_0},\varepsilon_0)$$

holds for infinitely many $j_0\in\mathbb{N}$, we contradict with the construction of the sequence (Θ_j). Hence, $\{f(\Phi_n):n\in A\}$ is a finite subset of $\tilde{X}$. It follows that given any $\varepsilon>0$, $f(\Phi_n)\in\mathcal{B}^M(\tilde{x}_0,\varepsilon)$, where $M=\max\{m_n:n\in A\}$ and m_n is the length of the ε-chain from $\tilde{x}_0$ to $f(\Phi_n)$ for every $n\in A$. Thus, we conclude that the subsequence $(f(\Phi_n))_{n\in A}$ is Bourbaki–Cauchy with $\Delta(A)=1$ which means the sequence itself $(f(\Phi_n))$ is a statistical Bourbaki–Cauchy sequence in $\tilde{X}$. Consequently, we obtain an unbounded statistical Bourbaki–Cauchy regular function from X into unbounded chainable metric space $\tilde{X}$ opposite to hypothesis and so X is Bourbaki bounded.

$(4) \Rightarrow (1)$ It is proved in [18, Theorem 4] that if X is Bourbaki bounded, then every sequence in X has a Bourbaki–Cauchy subsequence and so it has a statistical Bourbaki–Cauchy subsequence.

3. Conclusion

Compact metric spaces and complete metric spaces with their basic properties are well known by all mathematicians and metric spaces satisfying properties between compactness and completeness have been the subject of research for many papers over years. One such well-known metric space is Atsuji or UC space on which every real-valued continuous function is uniformly continuous. Also, a Bourbaki complete metric space can be given as an example of such an intermediate property defined and studied in the recent time. It has been proved that every UC metric space is Bourbaki complete. In this present paper, we state a new condition equivalent to Bourbaki completeness by defining a new class of sequences named as a statistical Bourbaki–Cauchy sequence. Hence, we conclude that every sequence in any UC space has a statistical Bourbaki–Cauchy subsequence. Further, since compactness has been characterized by Bourbaki boundedness and Bourbaki completeness, we can say that a metric space X is compact if and only if X is Bourbaki bounded and every sequence in X has a statistical Bourbaki–Cauchy subsequence.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: A new type of statistical Cauchy sequence and its relation to Bourbaki completeness, Merve Ilkhan & Emrah Evren Kara, *Cogent Mathematics & Statistics* (2018), 5: 1487500.

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