

BALKAN Y.S.¹, AKTAN N.²**ALMOST KENMOTSU f -MANIFOLDS**

In this paper, we consider a generalization almost Kenmotsu f -manifolds. We get basic Riemannian curvature, sectional curvatures and scalar curvature properties such type manifolds. Finally, we give two examples.

Key words and phrases: f -structure, Almost Kenmotsu f -manifolds.

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1 INTRODUCTION

Let M be a real $(2n + s)$ -dimensional smooth manifold. M admits f -structure [8] if there exists a non-null smooth $(1, 1)$ tensor field φ , tangent bundle TM , satisfying $\varphi^3 + \varphi = 0$, $\text{rank } \varphi = 2n$. An f -structure is a generalization of almost complex ($s = 0$) and almost contact ($s = 1$) structure. In the latter case M is orientable [9]. Corresponding to two complementary projection operators P and Q applied to TM , defined by $P = -\varphi^2$ and $Q = \varphi^2 + I$, where I identity operator, there exist two complementary distributions \mathcal{D} and \mathcal{D}^\perp such that $\dim(\mathcal{D}) = 2n$ and $\dim(\mathcal{D}^\perp) = s$. The following relations hold [6]

$$\varphi P = P\varphi = \varphi, \quad \varphi Q = Q\varphi = 0, \quad \varphi^2 P = -P, \quad \varphi^2 Q = 0.$$

Thus, we have an almost complex distribution $(\mathcal{D}, J = \varphi|_{\mathcal{D}}, J^2 = -I)$ and φ acts on \mathcal{D}^\perp as a null operator. It follows that

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp, \quad \mathcal{D} \cap \mathcal{D}^\perp = \{0\}.$$

Assume that \mathcal{D}_p^\perp is spanned by s globally defined orthonormal vector $\{\xi_i\}$ at each point $p \in M$, $(1 \leq i, j \leq s)$, with its dual set $\{\eta^i\}$. Then one obtains

$$\varphi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i.$$

In the above case, M is called a globally framed manifold (or simply an f -manifold) ([1], [5] and [4]) and we denote its frame structure by $M(\varphi, \xi_i)$. From the above conditions one has

$$\varphi \xi_i = 0, \quad \eta^i \circ \varphi = 0, \quad \eta^i(\xi_j) = \delta_i^j.$$

Now we consider Riemannian metric g on M that is compatible with a f -structure such that

$$g(\varphi X, Y) + g(X, \varphi Y) = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^s \eta^i(X) \eta^i(Y), \quad g(X, \xi_i) = \eta^i(X).$$

In the above case, we say that M is a metric f -manifold and its associated structure will be denoted by $M(\varphi, \xi_i, \eta^i, g)$.

A framed structure $M(\varphi, \xi_i)$ is normal [5] if the torsion tensor N_φ of φ is zero i.e., if

$$N_\varphi = N + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0,$$

where N denotes the Nijenhuis tensor field of φ .

Define a 2-form Φ on M by $\Phi(X, Y) = g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f -manifold satisfies the following formula [1]:

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 3d(X, \varphi Y, \varphi Z) - 3d(X, Y, Z) \\ &\quad + g(N(Y, Z), \varphi X) + N_j^2(Y, Z) \eta^j(X) \\ &\quad + 2d\eta^j(\varphi Y, X) \eta^j(Z) - 2d\eta^j(\varphi Z, X) \eta^j(Y), \end{aligned}$$

where the tensor field N_j^2 is defined by

$$N_j^2(X, Y) = (L_{\varphi X} \eta^j)Y - (L_{\varphi Y} \eta^j)X = 2d\eta^j(\varphi X, Y) - 2d\eta^j(\varphi Y, X),$$

for each $j \in (1, \dots, s)$. Following the terminology introduced by Blair [1], we say that a normal metric f -manifold is a K -manifold if its 2-form Φ closed (i.e., $d\Phi = 0$). Since $\eta^1 \wedge \dots \wedge \eta^s \wedge \Phi^n \neq 0$, a K -manifold is orientable. Furthermore, we say that a K -manifold is a C -manifold if each η^i is closed, an S -manifold if $d\eta^1 = d\eta^2 = \dots = d\eta^s = \Phi$.

Note that, if $s = 1$, namely if M is an almost contact metric manifold, the condition $d\Phi = 0$ means that M is quasi-Sasakian. M is said a K -contact manifold if $d\eta = \Phi$ and ξ is Killing.

Falcitelli and Pastore introduced and studied a class of manifolds which is called almost Kenmotsu f -manifold [3]. Such manifolds admit an f -structure with s -dimensional parallelizable kernel. A metric $f.pk$ -manifold of dimension $(2n + s)$, $s \geq 1$, with $f.pk$ -structure $(\varphi, \xi_i, \eta^i, g)$, is said to be a almost Kenmotsu $f.pk$ -manifold if the 1-forms η^i 's are closed and $d\Phi = 2\eta^1 \wedge \Phi$. Several foliations canonically associated with an almost Kenmotsu $f.pk$ -manifold are studied and locally conformal almost Kenmotsu $f.pk$ -manifolds are characterized by Falcitelli and Pastore. Öztürk et al. studied almost α -cosymplectic f -manifolds [6].

In this paper, we consider a generalization of almost Kenmotsu f -manifolds. We get some curvature properties.

Throughout this paper we denote by $\bar{\eta} = \eta^1 + \eta^2 + \dots + \eta^s$, $\bar{\xi} = \xi_1 + \xi_2 + \dots + \xi_s$ and $\bar{\delta}_i^j = \delta_i^1 + \delta_i^2 + \dots + \delta_i^s$.

2 ALMOST KENMOTSU f -MANIFOLDS

Almost Kenmotsu f -manifolds firstly defined and study by Aktan et al. as mentioned below [6].

Definition 2.1 ([6]). Let $M(\varphi, \xi_i, \eta^i, g)$ be $(2n + s)$ -dimensional a metric f -manifold. For each $\eta^i, (1 \leq i \leq s)$ 1-forms and each Φ 2-forms, if $d\eta^i = 0$ and $d\Phi = 2\bar{\eta} \wedge \Phi$ satisfy, then M is called almost Kenmotsu f -manifold.

Let M be an almost Kenmotsu f -manifold. Since the distribution \mathcal{D} is integrable, we have $L_{\xi_i} \eta^j = 0, [\xi_i, \xi_j] \in \mathcal{D}$ and $[X, \xi_j] \in \mathcal{D}$ for any $X \in \Gamma(\mathcal{D})$. Then the Levi-Civita connection is given by:

$$2g((\nabla_X \varphi)Y, Z) = 2 \left(\sum_{j=1}^s (g(\varphi X, Y) \xi_j - \eta^j(Y) \varphi X), Z \right) + g(N(Y, Z), \varphi X), \quad (1)$$

for any $X, Y, Z \in \Gamma(TM)$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i} \varphi = 0$ which implies $\nabla_{\xi_i} \xi_j \in \mathcal{D}^\perp$ and then $\nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i$, since $[\xi_i, \xi_j] = 0$.

We put $A_i X = -\nabla_X \xi_i$ and $h_i = \frac{1}{2} (L_{\xi_i} \varphi)$, where L denotes the Lie derivative operator.

Proposition 2.1 ([6]). For any $i \in \{1, \dots, s\}$ the tensor field A_i is a symmetric operator such that

- 1) $A_i(\xi_j) = 0$, for any $j \in \{1, \dots, s\}$,
- 2) $A_i \circ \varphi + \varphi \circ A_i = -2\varphi$,
- 3) $\text{tr}(A_i) = -2n$.

Proof. Equality $d\eta^i = 0$ implies that A_i is symmetric.

1) For any $i, j \in \{1, \dots, s\}$ deriving $g(\xi_i, \xi_j) = \delta_i^j$ with respect to ξ_k , using $\nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i$, we get $2g(\xi_k, A_i(\xi_j)) = 0$. Since $\nabla_{\xi_i} \xi_j \in \mathcal{D}^\perp$, we conclude $A_i(\xi_j) = 0$.

2) For any $Z \in \Gamma(TM)$, we have $\varphi(N(\xi_i, Z)) = (L_{\xi_i} \varphi)Z$ and, on the other hand, since $\nabla_{\xi_i} \varphi = 0$,

$$L_{\xi_i} \varphi = A_i \circ \varphi - \varphi \circ A_i. \quad (2)$$

One can easily obtain from (2)

$$-A_i X = -\varphi^2 X - \varphi h_i X. \quad (3)$$

Applying (1) with $Y = \xi_i$, we have

$$2g(\varphi A_i X, Z) = -2g(\varphi X, Z) - g(\varphi N(\xi_i, Z), X),$$

which implies the desired result.

3) Considering local adapted orthonormal frame $\{X_1, \dots, X_n, \varphi X_1, \dots, \varphi X_n, \xi_1, \dots, \xi_s\}$, by 1) and 2), one has

$$\text{tr} A_i = \sum_{j=1}^n (g(A_i X_j, X_j) + g(A_i \varphi X_j, \varphi X_j)) = -2 \sum_{j=1}^n g(\varphi X_j, \varphi X_j) = -2n.$$

□

Proposition 2.2 ([1]). For any $i \in \{1, \dots, s\}$ the tensor field h_i is a symmetric operator and satisfies

- i) $h_i \xi_j = 0$, for any $j \in \{1, \dots, s\}$,
- ii) $h_i \circ \varphi + \varphi \circ h_i = 0$,
- iii) $\text{tr} h_i = 0$,
- iv) $\text{tr} \varphi h_i = 0$.

Proposition 2.3. ∇_φ satisfies the following relation [6]:

$$(\nabla_{X\varphi})Y + (\nabla_{\varphi X}\varphi)\varphi Y = \sum_{i=1}^s \left[-\left(\eta^i(Y)\varphi X + 2g(X, \varphi Y)\xi_i \right) - \eta^i(Y)h_i X \right]. \quad (4)$$

Proof. By direct computations, we get

$$\varphi N(X, Y) + N(\varphi X, Y) = 2 \sum_{i=1}^s \eta^i(X)h_i Y,$$

and

$$\eta^i(N(\varphi X, Y)) = 0.$$

From (1) and the equations above, the proof is completed. \square

3 ALMOST KENMOTSU f -MANIFOLDS WITH ξ BELONGING TO THE (κ, μ, ν) -NULLITY DISTRIBUTION

Definition 3.1. Let M be an almost Kenmotsu f -manifold, κ, μ and ν are real constants. We say that M verifies the (κ, μ, ν) -nullity condition if and only if for each $i \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$\begin{aligned} R(X, Y)\xi_i &= \kappa \left(\bar{\eta}(X)\varphi^2 Y - \bar{\eta}(Y)\varphi^2 X \right) + \mu \left(\bar{\eta}(Y)h_i X - \bar{\eta}(X)h_i Y \right) \\ &\quad + \nu \left(\bar{\eta}(Y)\varphi h_i X - \bar{\eta}(X)\varphi h_i Y \right). \end{aligned} \quad (5)$$

Lemma 3.1. Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then we have:

- (i) $h_i \circ h_j = h_j \circ h_i$ for each $i, j \in \{1, 2, \dots, s\}$,
- (ii) $\kappa \leq -1$,
- (iii) if $\kappa < -1$ then, for each $i \in \{1, 2, \dots, s\}$, h_i has eigenvalues $0, \pm\sqrt{-(\kappa+1)}$.

Proof. From (5) it follows that for each $X \in \Gamma(TM)$, $i, j \in \{1, 2, \dots, s\}$

$$R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2\kappa\varphi^2 X.$$

Using

$$R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2 \left[-\varphi^2 X + (h_i \circ h_j) X \right]$$

we obtain

$$(h_i \circ h_j) X = (\kappa + 1)\varphi^2 X = (h_j \circ h_i) X \quad (6)$$

and then (i) is verified. Next from (6) we get

$$h_i^2 X = (\kappa + 1)\varphi^2 X, \quad (7)$$

$$h_i^2 X = -(\kappa + 1)X, \quad X \in \Gamma(\mathcal{D}). \quad (8)$$

Then, using Proposition 2 and (8) we obtain that the eigenvalues of h_i^2 are 0 and $-(\kappa + 1)$. Moreover h_i is symmetric: $\|h_i X\|^2 = -(\kappa + 1)\|X\|^2$. Hence $\kappa \leq -1$. Finally let t be a real eigenvalue of h_i and X be an eigenvector corresponding to t . Then $t^2\|X\|^2 = -(\kappa + 1)\|X\|^2$ and $t = \pm\sqrt{-(\kappa + 1)}$. Taking Proposition 2 into account we get (iii). \square

Proposition 3.1. *Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then*

$$h_1 = \dots = h_s. \quad (9)$$

Proof. If $\kappa = -1$ then from (7) and the symmetry of each h_i we have $h_1 = \dots = h_s = 0$. Let now $\kappa < -1$. We fix $x \in M$ and $i \in \{1, 2, \dots, s\}$. Since h_i is symmetric then we have $\mathcal{D}_x = (\mathcal{D}_+)_x \oplus (\mathcal{D}_-)_x$, where $(\mathcal{D}_+)_x$ is the eigenspace of h_i corresponding to the eigenvalue $\lambda = \sqrt{-(\kappa+1)}$ and $(\mathcal{D}_-)_x$ is the eigenspace of h_i corresponding to the eigenvalue $-\lambda$. If $X \in \mathcal{D}_x$ then we can write $X = X_+ + X_-$, where $X_+ \in (\mathcal{D}_+)_x$, $X_- \in (\mathcal{D}_-)_x$ so that $h_i X = \lambda(X_+ + X_-)$. We fix $j \in \{1, 2, \dots, s\}$, $j \neq i$. Then from (6) we get $h_j X = h_j(X_+ + X_-) = h_j(\frac{1}{\lambda}h_i X_+ - \frac{1}{\lambda}h_i X_-) = \frac{1}{\lambda}(h_j \circ h_i)(X_+ + X_-) = \lambda(X_+ + X_-) = h_i X$. Taking Proposition 2 into account we obtain (9). \square

Remark 3.1. *Throughout all this paper whenever (5) holds we put $h := h_1 = \dots = h_s$. Then (5) becomes*

$$\begin{aligned} R(X, Y) \xi_i &= \kappa \left(\bar{\eta}(X) \varphi^2 Y - \bar{\eta}(Y) \varphi^2 X \right) + \mu \left(\bar{\eta}(Y) hX - \bar{\eta}(X) hY \right) \\ &\quad + \nu \left(\bar{\eta}(Y) \varphi hX - \bar{\eta}(X) \varphi hY \right). \end{aligned} \quad (10)$$

Furthermore, using (10), the symmetry properties of the curvature tensor and the symmetry of φ^2 and h , we get

$$\begin{aligned} R(\xi_i, X)Y &= \kappa \left(\bar{\eta}(Y) \varphi^2 X - g(X, \varphi^2 Y) \bar{\xi} \right) + \mu \left(g(X, hY) \bar{\xi} - \bar{\eta}(Y) hX \right) \\ &\quad + \nu \left(g(\varphi hX, Y) \bar{\xi} - \bar{\eta}(Y) \varphi hX \right) \end{aligned} \quad (11)$$

Remark 3.2. *Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition, with $\kappa \neq -1$. We denote by \mathcal{D}_+ and \mathcal{D}_- the n -dimensional distributions of the eigenspaces of $\lambda = \sqrt{-(\kappa+1)}$ and $-\lambda$, respectively. We have that \mathcal{D}_+ and \mathcal{D}_- are mutually orthogonal. Moreover, since φ anticommutes with h , we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, \mathcal{D}_+ is a Legendrian distribution and \mathcal{D}_- is the conjugate Legendrian distribution of \mathcal{D}_+ .*

Proposition 3.2. *Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then M is a Kenmotsu f -manifold if and only if $\kappa = -1$.*

Proof. We observed in the proof of Proposition 3.1 that if $\kappa = -1$ then $h = 0$. It follows that (10) reduces to $R(X, Y) \xi_i = \bar{\eta}(Y) \varphi^2 X - \bar{\eta}(X) \varphi^2 Y$. From ([2], Proposition 3.4 and Theorem 4.3) we get the claim. \square

4 PROPERTIES OF THE CURVATURE

Let $M(\varphi, \xi_i, \eta^i, g)$ be a $(2n + s)$ -dimensional almost Kenmotsu f -manifold. We consider the $(1, 1)$ -tensor fields defined by

$$l_{ij}(\cdot) = R_{\xi_i} \xi_j$$

for each $i, j \in \{1, \dots, s\}$ and put $l_i = l_{ii}$.

Lemma 4.1. For each $i, j, k \in \{1, \dots, s\}$ the following identities hold:

$$\varphi \circ l_{ji} \circ \varphi - l_{ji} = 2 \left[h_i \circ h_j - \varphi^2 \right], \quad (12)$$

$$\eta_k \circ l_{ji} = 0, \quad (13)$$

$$l_{ji}(\xi_k) = 0, \quad (14)$$

$$\nabla_{\xi_j} h_i = -\varphi \circ l_{ji} - \varphi - (h_j + h_i) - \varphi \circ h_i \circ h_j, \quad (15)$$

$$\nabla_{\xi_i} h_i = -\varphi \circ l_{ji} - \varphi - 2h_i - \varphi h_i^2. \quad (16)$$

Proof. Identity (12) is a rewriting of [7, (3.4)]. Formulas (13) and (14) are an immediate consequence of (12). Next from (3) and $\eta_l \circ (\nabla_{\xi_i} h_k) = 0$ we get

$$l_{ij} = \left(\varphi \left(\nabla_{\xi_j} h_i \right) + \varphi^2 + \varphi \circ h_i + \varphi \circ h_j - h_j \circ h_i \right).$$

Applying φ both sides we get

$$\left(\nabla_{\xi_j} h_i \right) = \left(-\varphi \circ l_{ij} - \varphi - h_i - h_j - \varphi \circ h_j \circ h_i \right),$$

from which it follows (15). Finally, identity (16) is (15) when $i = j$. \square

Remark 4.1. Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then for each $i, j \in \{1, \dots, s\}$ we have

$$l_{ji} = -\kappa \varphi^2 + \mu h + \nu \varphi h. \quad (17)$$

It follows that all the l_{ji} 's coincide. We put $l = l_{ji}$.

Lemma 4.2. Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then for each $i \in \{1, \dots, s\}$, the following identities hold:

$$\nabla_{\xi_i} h = -\mu \varphi h + \nu h - 2h, \quad (18)$$

$$l\varphi - \varphi l = 2\mu h\varphi + 2\nu h, \quad (19)$$

$$l\varphi + \varphi l = 2\kappa\varphi, \quad (20)$$

$$Q\xi_i = 2n\kappa\bar{\xi}, \quad (21)$$

Proof. From (16), using (17), we obtain (18). Identities (19) and (20) follow directly from (17) using $h \circ \varphi = -\varphi \circ h$. For the proof of (21) we fix $x \in M$ and $\{E_1, \dots, E_{2n+s}\}$ a local φ -basis around x with $E_{2n+1} = \xi_1, \dots, E_{2n+s} = \xi_s$. Then using (11) and $\text{trace}(h) = 0$ we get

$$Q\xi_i = \sum_{j=1}^{2n} R_{\xi_i E_j} E_j = \sum_{j=1}^{2n} \kappa g(\varphi^2 E_j, E_j) \bar{\xi} = \kappa \sum_{j=1}^{2n} \delta_{jj} \bar{\xi}. \quad \square$$

Lemma 4.3. Let $(M, \varphi, \xi_i, \eta_j, g)$ be a $(2n+s)$ -dimensional almost Kenmotsu f -manifold. Then the curvature tensor satisfies the identities

$$\begin{aligned} g(R_{\xi_i X} Y, Z) &= \sum_{j=1}^s \eta_j(Z) g(\varphi^2 Y, X) - \sum_{j=1}^s \eta_j(Y) g(\varphi^2 Z, X) \\ &+ \sum_{j=1}^s \eta_j(Z) g(\varphi h_j Y, X) - \sum_{j=1}^s \eta_j(Y) g(\varphi h_j Z, X) + g((\nabla_Z \varphi h_i) Y - (\nabla_Y \varphi h_i) Z, X) \end{aligned} \quad (22)$$

and

$$\begin{aligned} &g(R_{\xi_i X} Y, Z) - g(R_{\xi_i X} \varphi Y, \varphi Z) + g(R_{\xi_i \varphi X} Y, \varphi Z) + g(R_{\xi_i \varphi X} \varphi Y, Z) \\ &= 2g((\nabla_{h_i X} \varphi) Y, Z) + 2\bar{\eta}(Z) g(h_i X - \varphi X, \varphi Y) - 2\bar{\eta}(Y) g(h_i X - \varphi X, \varphi Z) \end{aligned} \quad (23)$$

for each $i = 1, \dots, s$ and $X, Y, Z \in \Gamma(TM)$.

Proof. Using the Riemannian curvature tensor and (8), we obtain (22).

We introduce the operators A and B_i , $i \in \{1, \dots, s\}$ defined by

$$A(X, Y, Z) := 2\bar{\eta}(Y) g(\varphi X, \varphi Z) - 2\bar{\eta}(Z) g(\varphi X, \varphi Y)$$

and

$$B_i(X, Y, Z) := -g(\varphi X, (\nabla_Y(\varphi \circ h_i))(\varphi Z)) - g(\varphi X, (\nabla_Y(\varphi \circ h_i))Z) \\ - g(X, (\nabla_Y(\varphi \circ h_i))Z) + g(X, (\nabla_{\varphi Y}(\varphi \circ h_i))(\varphi Z))$$

for each $X, Y, Z \in \Gamma(TM)$. By a direct computation and using (22) we get that the left hand side of (23) equals $A(X, Y, Z) + B_i(X, Y, Z) - B_i(X, Z, Y)$. Since

$$\eta_j((\nabla_{\varphi Y} h_i)Z) = \eta_j(\nabla_{\varphi Y}(h_i Z)),$$

we can write

$$B_i(X, Y, Z) = -g(X, (\nabla_Y(\varphi \circ h_i)Z)) + g(X, (\varphi \circ h_i)(\nabla_Y Z)) \\ + g(X, (\nabla_{\varphi Y}(\varphi \circ h_i \circ \varphi)Z)) + g(X, (\varphi \circ h_i)(\nabla_{\varphi Y} \varphi Z)) \\ - g(\varphi X, (\nabla_Y(\varphi \circ h_i \circ \varphi)Z)) + g(\varphi X, (\varphi \circ h_i)(\nabla_Y(\varphi Z))) \\ - g(\varphi X, (\nabla_{\varphi Y}(\varphi \circ h_i)Z)) + g(\varphi X, (\varphi \circ h_i)(\nabla_{\varphi Y}(h_i Z))) \\ = -g(X, (\nabla_Y \varphi)(h_i Z)) + g(X, h_i((\nabla_Y \varphi)Z)) \\ + g(X, (h_i \circ \varphi)((\nabla_{\varphi Y} \varphi)Z)) + g(X, \varphi((\nabla_{\varphi Y} \varphi)(h_i Z))) \\ + \sum_{j=1}^s \eta_j((\nabla_{\varphi Y} h_i)Z) \eta_j(X). \quad (24)$$

Moreover, from (3), (4) and Proposition 1 it follows that

$$(\varphi \circ (\nabla_{\varphi X} \varphi))Y = (\nabla_{\varphi X} \varphi^2)Y - (\nabla_{\varphi X} \varphi)(\varphi Y) \\ = \sum_{j=1}^s ((\nabla_{\varphi X} \eta_j)Y \xi_j) + \sum_{j=1}^s (\eta_j(Y) \nabla_{\varphi X} \xi_j) \\ - (\nabla_{\varphi X} \varphi)(\varphi Y) = \sum_{j=1}^s ((\nabla_{\varphi X})(g(\xi_j, Y)) \xi_j \\ - g(\nabla_{\varphi X} Y, \xi_j) \xi_j) + \sum_{j=1}^s \eta_j(Y) (\varphi X - h_j X) \\ + \sum_{j=1}^s \eta_j(Y) h_j X + \bar{\eta}(Y) \varphi X + 2g(X, \varphi Y) \bar{\xi} + (\nabla_X \varphi)Y.$$

Hence

$$(\varphi \circ (\nabla_{\varphi X} \varphi))Y = \sum_{j=1}^s g(X, \varphi Y) \xi_j - \sum_{j=1}^s g(Y, h_j X) \xi_j \\ + 2 \sum_{j=1}^s \eta_j(Y) \varphi X + (\nabla_X \varphi)Y.$$

Furthermore, from (4), for each $j \in \{1, \dots, s\}$ we have

$$\begin{aligned} \eta_i ((\nabla_{\varphi Y} h_j) Z) &= \eta_i (\nabla_{\varphi Y} (h_j Z)) = (\nabla_{\varphi Y} \eta_i) (h_j Z) \\ &= -g(h_j Z, \nabla_{\varphi Y} \xi_i) = g(h_j Z, h_i Y - \varphi Y). \end{aligned} \quad (25)$$

Then, using (24) and (25) we get

$$\begin{aligned} B_i(X, Y, Z) &= -g(X, (\nabla_Y \varphi)(h_i Z)) + g(X, h_i((\nabla_Y \varphi) Z)) + 2\bar{\eta}(Z) g(h_i X, \varphi Y) \\ &\quad + g(h_i X, (\nabla_Y \varphi) Z) + \bar{\eta}(X) g(Y, \varphi h_i Z) - \sum_{j=1}^s \eta_j(X) g(h_i Z, h_j Y) \\ &\quad + \sum_{j=1}^s \eta_j(X) g(h_i Z, h_j Y) + g(X, (\nabla_Y \varphi)(h_i Z)) - \bar{\eta}(X) g(\varphi Y, h_i Z) \\ &= 2(g(h_i X, (\nabla_Y \varphi) Z) + \bar{\eta}(Z) g(h_i X, \varphi Y) - \bar{\eta}(X) g(\varphi Y, h_i Z)). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &A(X, Y, Z) + B_i(X, Y, Z) - B_i(X, Z, Y) \\ &= 2(\nabla_Y \Phi)(h_i X, Z) - 2(\nabla_Z \Phi)(h_i X, Y) \\ &\quad + 2\bar{\eta}(Z) g(h_i X - \varphi X, \varphi Y) - 2\bar{\eta}(Y) g(h_i X - \varphi X, \varphi Z) \end{aligned}$$

and hence (23) follows. \square

Remark 4.2. Let M be an almost Kenmotsu f -manifold. Then from (23) using $(\nabla_{h_i X} \Phi)(Y, Z) = -g((\nabla_{h_i X} \varphi) Y, Z)$, for each $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned} (\nabla_{h_i X} \varphi) Y &= \frac{1}{2} (\varphi R_{\xi_i \varphi X} Y - R_{\xi_i \varphi X} \varphi Y - \varphi R_{\xi_i X} \varphi Y - R_{\xi_i X} Y) \\ &\quad + g(h_i X - \varphi X, \varphi Y) \bar{\xi} + \bar{\eta}(Y) (\varphi h_i X - \varphi^2 X). \end{aligned} \quad (26)$$

Lemma 4.4. Let M be an almost Kenmotsu f -manifold verifying (κ, μ, ν) -nullity condition. Then the following identities hold:

$$(\nabla_X \varphi) Y = g(\varphi X + hX, Y) \bar{\xi} - \eta(Y) (\varphi X + hX), \quad (27)$$

$$\begin{aligned} (\nabla_X h) Y - (\nabla_Y h) X &= (\kappa + 1) (\eta(Y) \varphi X - \eta(X) \varphi Y + 2g(\varphi X, Y) \bar{\xi}) \\ &\quad + \mu (\eta(Y) \varphi hX - \eta(X) \varphi hY) + (1 - \nu) (\eta(Y) hX - \eta(X) hY). \end{aligned} \quad (28)$$

Proof. From (26) we obtain

$$(\nabla_{hX} \varphi) Y = -(\kappa + 1) g(X, Y) \bar{\xi} + (\kappa + 1) \bar{\eta}(Y) X + \bar{\eta}(Y) \varphi hX + g(hX, \varphi Y) \bar{\xi}.$$

Here we replace X with hX and by a direct computation, taking (3), (7) into account, we get (27). From (27), since h and φ^2 are self-adjoint, we have

$$(\nabla_X (\varphi \circ h)) Y - (\nabla_Y (\varphi \circ h)) X = \varphi ((\nabla_X h) Y - (\nabla_Y h) X).$$

It follows that for each $Z \in \Gamma(TM)$

$$\begin{aligned} g(R_{XY} \xi_i, Z) &= \bar{\eta}(Y) g(\varphi^2 X + \varphi hX, Z) - \bar{\eta}(X) g(\varphi^2 Y + \varphi hY, Z) \\ &\quad + g(\varphi ((\nabla_Y h) X - (\nabla_X h) Y), Z), \end{aligned} \quad (29)$$

where we use 5 of [6] and (27). From (29) and the symmetry of h and φ^2 it follows that

$$\varphi((\nabla_Y h)X - (\nabla_X h)Y) = R_{XY}\xi_i - \bar{\eta}(Y)(\varphi^2 X + \varphi hX) + \bar{\eta}(X)(\varphi^2 Y + \varphi hY).$$

Then, applying φ to both the sides of the last identity, using (10) and

$$\eta_l((\nabla_Y h)X - (\nabla_X h)Y) = -2(\kappa + 1)g(\varphi X, Y), \quad l \in \{1, \dots, s\},$$

we get (28). \square

Theorem 1. Let $Z=(M, \varphi, \xi_i, \eta_j, g)$ be a $(2n + s)$ -dimensional almost Kenmotsu f -manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ be an almost f -structure on M obtained by a \mathcal{D} -homothetic transformation of constant α . If Z verifies the (κ, μ, ν) -nullity condition for certain real constants (κ, μ, ν) then $(M, \tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ verifies the $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$ -nullity condition, where

$$\tilde{\kappa} = \frac{\kappa}{\alpha}, \quad \tilde{\mu} = \frac{\mu}{\alpha}, \quad \tilde{\nu} = \frac{\nu}{\alpha}.$$

Proof. From (18) and (9) it follows that $\tilde{h}_1 = \dots = \tilde{h}_s$. Then, using (27), by a direct calculation we get the claim. \square

Lemma 4.5. Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition. Then

$$X, Y \in \Gamma(\mathcal{D}_+) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+), \quad (30)$$

$$X, Y \in \Gamma(\mathcal{D}_-) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_-), \quad (31)$$

$$X \in \Gamma(\mathcal{D}_+), Y \in \Gamma(\mathcal{D}_-) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_- \oplus \ker(\varphi)), \quad (32)$$

$$X \in \Gamma(\mathcal{D}_-), Y \in \Gamma(\mathcal{D}_+) \Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+ \oplus \ker(\varphi)). \quad (33)$$

Proof. From (28) we get $g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = 0$, for each $X, Y, Z \in \Gamma(\mathcal{D}_+)$. On the other hand, since h is symmetric, from Remark 2 we have

$$g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = -2\lambda g(\nabla_X(\varphi Z), Y).$$

Then

$$g(\varphi Z, \nabla_X Y) = -g(\nabla_X(\varphi Z), Y),$$

that is $\nabla_X Y$ is normal to \mathcal{D}_- . Moreover from (3) and Remark 2 it follows that, for each $i \in \{1, \dots, s\}$, $g(\nabla_X Y, \xi_i) = -g(Y, \nabla_X \xi_i) = 0$. Then we have (30). The proof of (31) is analogous. If $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$ then from (30) and Remark 2 we get that, for each $Z \in \Gamma(\mathcal{D}_+)$ $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = 0$ and then we have (32). Analogously we prove (33). \square

Remark 4.3. It follows from (30) and (31) that \mathcal{D}_\pm define two orthogonal totally geodesic Legendrian foliations F_\pm on M .

Lemma 4.6. Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition. Then for each $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned} (\nabla_X h)Y &= (\kappa + 1)g(\varphi X, Y)\bar{\xi} - g(hX, Y)\bar{\xi} - \eta(Y)h(X + h\varphi X) \\ &\quad - \mu\eta(X)\varphi hY + (\nu - 2)\eta(X)hY. \end{aligned} \quad (34)$$

Proof. Let be $X, Y \in \Gamma(\mathcal{D})$. From Proposition 2. *i*) we get $g(h_i Y, \xi_j) = 0$. Taking the derivative of this equality of the direction X we obtain

$$(\nabla_X h) Y = -g\left(Y, h_i X + h_i^2 \varphi X\right) \xi_j.$$

Then, we write any vector field X on M as $X = X_+ + \eta_i(X) \xi_j$, X_+ denoting positive component of X in \mathcal{D} , and, using (18) and (8), we have

$$\begin{aligned} (\nabla_X h) Y &= (\nabla_{X_+} h) Y_+ + \bar{\eta}(Y) (\nabla_{X_+} h) \bar{\xi} + \bar{\eta}(X) (-\mu \varphi h + \nu h - 2h) Y \\ &\quad - g\left(Y, h X + h^2 \varphi X\right) \bar{\xi} - \bar{\eta}(Y) (h X + h^2 \varphi X) + \bar{\eta}(X) (-\mu \varphi h Y + \nu h Y - 2h Y). \end{aligned}$$

□

Remark 4.4. Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition. Then using (27), (34) and (8) we get, for all $X, Y \in \Gamma(TM)$

$$\begin{aligned} (\nabla_X \varphi h) Y &= (\kappa + 1) g\left(\varphi^2 X, Y\right) \bar{\xi} + g(\varphi X, h Y) \bar{\xi} - \bar{\eta}(Y) \varphi h X \\ &\quad + (\kappa + 1) \bar{\eta}(Y) \varphi^2 X + \mu \bar{\eta}(X) h Y + (\nu - 2) \eta(X) \varphi h Y. \end{aligned} \quad (35)$$

Lemma 4.7. Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition. Then for each $X, Y, Z \in \Gamma(\mathcal{D})$ we have

$$\begin{aligned} R_{XY} h Z - h R_{XY} Z &= s[\kappa\{g(Z, \varphi Y) X - g(Z, \varphi Y) \varphi h X - g(Z, \varphi X) Y + g(Z, \varphi X) \varphi h Y \\ &\quad + g(Z, X) \varphi Y - g(Z, \varphi h X) \varphi Y - g(Z, Y) \varphi X + g(Z, \varphi h Y) \varphi X\} \\ &\quad + g(Z, \varphi Y) X - g(Z, \varphi Y) \varphi h X - g(Z, \varphi X) Y + g(Z, \varphi X) \varphi h Y \\ &\quad - g(Z, h Y) X + g(Z, h Y) \varphi h X + g(Z, h X) Y - g(Z, h X) \varphi h Y \\ &\quad - g(Z, X) h Y + g(Z, X) \varphi Y + g(Z, \varphi h X) h Y - g(Z, \varphi h X) \varphi Y \\ &\quad + g(Z, Y) h X - g(Z, Y) \varphi X - g(Z, \varphi h Y) h X + g(Z, \varphi h Y) \varphi X]. \end{aligned} \quad (36)$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by a direct computation we get

$$\begin{aligned} (\nabla_X \nabla_Y h) Z &= (\kappa + 1) [g(\nabla_X Z, \varphi Y) \bar{\xi} + g(Z, (\nabla_X \varphi) Y) \bar{\xi} + g(Z, \varphi(\nabla_X Y)) \bar{\xi} \\ &\quad + g(Z, \varphi Y) (-\varphi^2 X - \varphi h X)] - g(\nabla_X Z, h Y) \bar{\xi} - g(Z, (\nabla_X h) Y) \bar{\xi} \\ &\quad - g(Z, h(\nabla_X Y)) \bar{\xi} + g(Z, h Y) (\varphi^2 X + \varphi h X) - g(\nabla_X Z, \bar{\xi}) (h Y + h^2 \varphi Y) \\ &\quad - g(Z, \nabla_X \bar{\xi}) (h Y + h^2 \varphi Y) - \bar{\eta}(Z) (\nabla_X h) Y - \bar{\eta}(Z) h(\nabla_X Y) \\ &\quad (\kappa + 1) [\bar{\eta}(Z) (\nabla_X \varphi) Y + \bar{\eta}(Z) \varphi(\nabla_X Y)] - \mu [g(\nabla_X Y, \bar{\xi}) \varphi h Z \\ &\quad - g(Y, \nabla_X \bar{\xi}) \varphi h Z - \bar{\eta}(Y) (\nabla_X \varphi h) Z - \bar{\eta}(Y) \varphi h(\nabla_X Z)] \\ &\quad + (\nu - 2) [g(\nabla_X Y, \bar{\xi}) h Z + g(Y, \nabla_X \bar{\xi}) h Z + \bar{\eta}(Y) (\nabla_X h) Z + \bar{\eta}(Y) h(\nabla_X Z)], \end{aligned}$$

where we (34), (8) and the antisymmetry of $\nabla_X \varphi$. Hence, using the Ricci identity

$$R_{XY} h Z - h R_{XY} Z = (\nabla_X \nabla_Y h) Z - (\nabla_Y \nabla_X h) Z - (\nabla_{[X, Y]} h) Z,$$

(34), the symmetry of $\nabla_X (h \circ \varphi)$ and (3), we obtain

$$\begin{aligned}
R_{XY}hZ - hR_{XY}Z &= (\kappa + 1) [g(Z, (\nabla_X \varphi)Y - (\nabla_Y \varphi)X) \bar{\xi} - g(Z, \varphi Y) (\varphi^2 X + \varphi hX) \\
&+ g(Z, \varphi X) (\varphi^2 Y + \varphi hY)] - g(Z, (\nabla_X h)Y - (\nabla_Y h)X) \bar{\xi} + g(Z, hY) (\varphi^2 X + \varphi hX) \\
&- g(Z, hX) (\varphi^2 Y + \varphi hY) - g(Z, \nabla_X \bar{\xi}) (hY + h^2 \varphi Y) + g(Z, \nabla_Y \bar{\xi}) (hX + h^2 \varphi X) \quad (37) \\
&- \bar{\eta}(Z) ((\nabla_X h)Y - (\nabla_Y h)X) + (\kappa + 1) \bar{\eta}(Z) ((\nabla_X \varphi)Y - (\nabla_Y \varphi)X) \\
&+ \mu [(g(X, \nabla_Y \bar{\xi}) - g(Y, \nabla_X \bar{\xi})) \varphi hZ - \bar{\eta}(Y) (\nabla_X \varphi h)Z + \bar{\eta}(X) (\nabla_Y \varphi h)Z] \\
&+ (\nu - 2) [(g(Y, \nabla_X \bar{\xi}) - g(X, \nabla_Y \bar{\xi})) hZ + \bar{\eta}(Y) (\nabla_X h)Z - \bar{\eta}(X) (\nabla_Y h)Z].
\end{aligned}$$

If we take $X, Y, Z \in \Gamma(\mathcal{D})$ then from (37), using identities (35), (27) and (8), we get (36). \square

Lemma 4.8. *Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition. Then for each $X, Y, Z \in \Gamma(TM)$ we have*

$$\begin{aligned}
R_{XY}\varphi Z - \varphi R_{XY}Z &= [\kappa (\bar{\eta}(Y) g(\varphi X, Z) - \bar{\eta}(X) g(\varphi Y, Z)) \\
&+ \mu (\bar{\eta}(Y) g(\varphi hX, Z) - \bar{\eta}(X) g(\varphi hY, Z)) - \nu (\bar{\eta}(Y) g(hX, Z) - \bar{\eta}(X) g(hY, Z))] \bar{\xi} \\
&s[-g(Z, \varphi Y + hY) (\varphi^2 X + \varphi hX) + g(Z, \varphi X + hX) (\varphi^2 Y + \varphi hY) \\
&+ g(Z, \varphi^2 X + \varphi hX) (\varphi Y + hY) - g(Z, \varphi^2 Y + \varphi hY) (\varphi X + hX)] \\
&- \bar{\eta}(Z) [\kappa (\bar{\eta}(Y) \varphi X - \bar{\eta}(X) \varphi Y) + \mu (\bar{\eta}(Y) \varphi hX - \bar{\eta}(X) \varphi hY) \\
&- \nu (\bar{\eta}(Y) hX - \bar{\eta}(X) hY)].
\end{aligned}$$

Proof. We proceed fixing a point $x \in M$ and local vector fields X, Y, Z such that $\nabla X, \nabla Y$ and ∇Z vanish at x . Applying several times (27), using (8) and the symmetry of $\nabla \varphi^2$, we get in x

$$\begin{aligned}
&\nabla_X ((\nabla_Y \varphi)Z) - \nabla_Y ((\nabla_X \varphi)Z) \\
&= [g((\nabla_X \varphi)Y - (\nabla_Y \varphi)X, Z) + g((\nabla_X h)Y - (\nabla_Y h)X, Z)] \bar{\xi} s \\
&\times [g(Z, \varphi X + hX) (\varphi^2 Y + \varphi hY) - g(Z, \varphi Y + hY) (\varphi^2 X + \varphi hX) \\
&+ g(Z, \varphi^2 X + \varphi hX) (\varphi Y + hY) - g(Z, \varphi^2 Y + \varphi hY) (\varphi X + hX)] \\
&- \bar{\eta}(Z) [((\nabla_X \varphi)Y - (\nabla_Y \varphi)X) + ((\nabla_X h)Y - (\nabla_Y h)X)].
\end{aligned}$$

From the last identity, using $R_{XY}\varphi Z - \varphi R_{XY}Z = \nabla_X (\nabla_Y \varphi)Z - \nabla_Y (\nabla_X \varphi)Z$ and (28), we get the claimed identity. \square

Remark 4.5. *In particular, from Lemma 9 it follows that for a Kenmotsu f -manifold $(M, \varphi, \xi_i, \eta_j, g)$ the following formula holds, for all $X, Y, Z \in \Gamma(TM)$,*

$$\begin{aligned}
R_{XY}\varphi Z - \varphi R_{XY}Z &= (\bar{\eta}(X) g(\varphi Y, Z) - \bar{\eta}(Y) g(\varphi X, Z)) \\
&s[-g(Z, \varphi Y) \varphi^2 X + g(Z, \varphi X) \varphi^2 Y + g(Z, \varphi^2 X) \varphi Y - g(Z, \varphi^2 Y) \varphi X] \\
&- \bar{\eta}(Z) [(\bar{\eta}(Y) \varphi X - \bar{\eta}(X) \varphi Y)].
\end{aligned}$$

Theorem 2. *Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition with $\kappa < -1$. Then for each $X_+, Y_+, Z_+ \in \Gamma(\mathcal{D}_+)$, $X_-, Y_-, Z_- \in \Gamma(\mathcal{D}_-)$, we have*

$$\begin{aligned}
R_{X_- Y_-} Z_+ &= s(\kappa + 1) [g(\varphi Y_-, Z_+) \varphi X_- - g(\varphi X_-, Z_+) \varphi Y_-] \\
&+ s\lambda [g(\varphi X_-, Z_+) Y_- - g(\varphi Y_-, Z_+) X_-],
\end{aligned}$$

$$\begin{aligned} R_{X_+Y_+}Z_+ &= s [g(X_+, Z_+) Y_+ - g(Y_+, Z_+) X_+] \\ &\quad + s\lambda [g(Y_+, Z_+) \varphi X_+ - g(X_+, Z_+) \varphi Y_+], \end{aligned} \quad (38)$$

$$\begin{aligned} R_{X_+Y_+}Z_- &= s\lambda [g(Z_-, \varphi Y_+) X_+ - g(Z_-, \varphi X_+) Y_+] \\ &\quad + s(\kappa + 1) [g(Z_-, \varphi Y_+) \varphi X_+ - g(Z_-, \varphi X_+) \varphi Y_+], \\ R_{X_+Y_-}Z_- &= -sg(Y_-, Z_-) X_+ + s(\kappa + 1) g(\varphi X_+, Z_-) \varphi Y_- \\ &\quad + s\lambda [g(Y_-, Z_-) \varphi X_+ - g(\varphi X_+, Z_-) Y_-], \end{aligned} \quad (39)$$

$$\begin{aligned} R_{X_+Y_-}Z_+ &= sg(X_+, Z_+) Y_- - s(\kappa + 1) g(\varphi Y_-, Z_+) \varphi X_+ \\ &\quad + s\lambda [g(X_+, Z_+) \varphi Y_- - g(\varphi Y_-, Z_+) X_+], \end{aligned} \quad (40)$$

$$\begin{aligned} R_{X_-Y_-}Z_- &= s [g(X_-, Z_-) Y_- - g(Y_-, Z_-) X_-] \\ &\quad - s\lambda [g(Y_-, Z_-) \varphi X_- - g(X_-, Z_-) \varphi Y_-]. \end{aligned} \quad (41)$$

Proof. First of all, for any $X_+, Y_+, Z_+ \in \mathcal{D}_+$, applying Lemma 7, we get

$$\lambda R_{X_+Y_+}Z_+ - hR_{X_+Y_+}Z_+ = 2s\lambda^2 (g(Z_+, Y_+) \varphi X_+ - g(Z_+, X_+) \varphi Y_+)$$

and by scalar multiplication with $W_- \in \mathcal{D}_-$, one has

$$2\lambda (R_{X_+Y_+}Z_+, W_-) = 2s\lambda^2 (g(Z_+, Y_+) g(\varphi X_+, W_-) - g(Z_+, X_+) g(\varphi Y_+, W_-))$$

from which, being $\lambda \neq 0$,

$$(R_{X_+Y_+}Z_+, W_-) = s\lambda (g(Z_+, Y_+) g(\varphi X_+, W_-) - g(Z_+, X_+) g(\varphi Y_+, W_-)). \quad (42)$$

With a similar argument, for any $X_+, W_+ \in \mathcal{D}_+$ and $Y_-, Z_- \in \mathcal{D}_-$, we also obtain

$$(R_{X_+Y_-}Z_-, W_+) = (\kappa + 1) s (g(Z_-, \varphi X_+) g(\varphi Y_-, W_+) - g(Z_-, Y_-) g(X_+, W_+)) \quad (43)$$

and, from (42), by symmetries of the tensor field R , for any $X_+, Y_+, W_+ \in \mathcal{D}_+$ and $Z_- \in \mathcal{D}_-$

$$(R_{X_+Y_+}Z_-, W_+) = s\lambda (g(Z_-, \varphi Y_+) g(X_+, W_+) - g(Z_-, \varphi X_+) g(Y_+, W_+)). \quad (44)$$

Next, fixed a local φ -basis $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$, with $e_i \in \mathcal{D}_+$ we compute $R_{X_+Y_+}Z_-$. The nullity condition implies $g(R_{X_+Y_+}Z_-, \xi_i) = 0$, while using the first Bianchi identity, (43) and (44), we get

$$\begin{aligned} g(R_{X_+Y_+}Z_-, e_i) &= \lambda s (g(Z_-, \varphi Y_+) g(X_+, e_i) - g(Z_-, \varphi X_+) g(Y_+, e_i)), \\ g(R_{X_+Y_+}Z_-, \varphi e_i) &= (\kappa + 1) s (g(\varphi Z_-, X_+) g(Y_+, e_i) - g(\varphi Z_-, Y_+) g(X_+, e_i)), \end{aligned}$$

so that, summing on i , the expression for $R_{X_+Y_+}Z_-$ follows.

The terms $R_{X_+Y_-}Z_+$ and $R_{X_+Y_-}Z_-$ are computed in a similar maner. Now, acting by φ on the formula just proved and using Lemma 10, we get

$$\begin{aligned} R_{X_+Y_+}\varphi Z_- &= s (g(\varphi Y_+, Z_-) X_+ - g(\varphi X_+, Z_-) Y_+) \\ &\quad - s\lambda (g(\varphi Y_+, Z_-) \varphi X_+ - g(\varphi X_+, Z_-) \varphi Y_+). \end{aligned}$$

Writing this formula for φZ_- , by the compatibility condition, we have the result for $R_{X_+Y_+}Z_+$. Similar computation yields $R_{X_-Y_-}Z_-$. Analogously, using the third formula and Lemma 10 we obtain $R_{X_+Y_-}Z_+$. \square

Now we are able to compute sectional curvature.

Theorem 3. *Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition with $\kappa < -1$. Then the sectional curvature K of M is determined by*

$$K(X, \xi_i) = \kappa g(X, X) + \mu g(hX, X) + \nu g(\varphi hX, X) = \begin{cases} \kappa + \mu\lambda & \text{if } X \in \mathcal{D}_+ \\ \kappa - \mu\lambda & \text{if } X \in \mathcal{D}_- \end{cases}, \quad (45)$$

$$K(X, Y) = \begin{cases} s & \text{if } X, Y \in \mathcal{D}_+ \\ s & \text{if } X, Y \in \mathcal{D}_- \\ -s - s(\kappa + 1)(g(X, \varphi Y)) & \text{if } X \in \mathcal{D}_+, Y \in \mathcal{D}_- \end{cases}. \quad (46)$$

Proof. Identities (45) follow directly from (5), while identities (46) consequence of (38), (41) and (39) respectively. \square

Corollary 4.1. *Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition with $\kappa < -1$. Then the Ricci operator verifies the following identities*

$$Q = s \left[(-2) \varphi^2 + \mu h + (2(n-1) + \nu)(\varphi \circ h) \right] + 2n\kappa \bar{\eta} \otimes \bar{\xi}, \quad (47)$$

$$Q \circ \varphi - \varphi \circ Q = 2s [\mu h \circ \varphi + ((n-1) + \nu) h]. \quad (48)$$

Proof. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ be a local φ -basis such that $\{e_1, \dots, e_n\}$ is a basis of \mathcal{D}_+ and let $X = X_+ + X_- \in \mathcal{D}_+ \oplus \mathcal{D}_-$. From (38) and (39) and (10) we get

$$QX_+ = s(-2 + \mu\lambda)X_+ + s(2\lambda(n-1) + \nu)\varphi X_+. \quad (49)$$

On the other hand from (40) and (41) we obtain

$$QX_+ = s(-2 - \mu\lambda)X_+ - s(2\lambda(n-1) + \nu)\varphi X_+. \quad (50)$$

Taking, (49), (50) and $Q\xi_i = 2n\kappa\bar{\xi}_i$ into account we get (47). Finally, identity (48) easily follows from (47). \square

Corollary 4.2. *Let M be an almost Kenmotsu f -manifold verifying the (κ, μ, ν) -nullity condition with $\kappa < -1$. Then the scalar curvature of (M, g) is constant and verifies the following identity*

$$S = 2ns(\kappa(2-n) - 2n). \quad (51)$$

Proof. Let $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi_1, \dots, \xi_s\}$ be a local φ -basis such that $\{e_1, \dots, e_n\}$ is a basis of \mathcal{D}_+ . Then from (38), (39) and (5) we have

$$g(Qe_i, e_i) = ksn + \mu\lambda sn - s(\kappa + 1)n^2 - sn^2. \quad (52)$$

Furthermore (40), (41) and (5)

$$g(Q\varphi e_i, \varphi e_i) = ksn - \mu\lambda sn - s(\kappa + 1)n^2 - sn^2. \quad (53)$$

Then (52), (53) and (21) yield (51). \square

5 EXAMPLES

Example 1. Let R^{2n+s} be $(2n+s)$ -dimensional real vector space with standard coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$ and

$$M = \{(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s) \mid z_i \neq 0, 1 \leq i \leq s, n \in N, n \geq 1\}$$

be a $(2n+s)$ -dimensional manifold. For each $i = 1, \dots, n$ and $k = 1, \dots, s$,

$$\begin{aligned} X_i &= \left(-(z_i + 1) \pm \sqrt{(z_i + 1)^2 + e^{2z_i}} \right) \frac{\partial}{\partial x_i} + e^{z_i} \frac{\partial}{\partial z_i}, \\ Y_i &= \left(z_i + 1 \pm \sqrt{(z_i + 1)^2 + e^{2z_i}} \right) \frac{\partial}{\partial y_i}, \\ \zeta_i &= \frac{\partial}{\partial z_i}, \end{aligned}$$

be a basis of M .

Then, for each $i, j = 1, \dots, n$ and $k = 1, \dots, s$, we obtain

$$\begin{aligned} [X_i, Y_j] &= e^{z_i} (2z_i + 3 + 2e^{2z_i}) \frac{\partial}{\partial y_i}, \quad [Y_i, Y_j] = 0, \\ [X_i, \zeta_i] &= (2z_i + 3 + 2e^{2z_i}) \frac{\partial}{\partial x_i} - e^{z_i} \frac{\partial}{\partial z_i}, \quad [Y_i, \zeta_i] = (2z_i + 1 + 2e^{2z_i}) \frac{\partial}{\partial y_i}, \\ [X_i, X_j] &= -e^{z_i} (2z_i + 3 - 2e^{2z_i}) \frac{\partial}{\partial x_j} + e^{z_i} (2z_i + 3 - 2e^{2z_i}) \frac{\partial}{\partial x_i}. \end{aligned}$$

If we take $\eta_i = \frac{\partial}{\partial z_i}$, we get

$$\begin{aligned} g &= \sum_{i=1}^n \left(\frac{-1}{(z_i + 1) + \sqrt{(z_i + 1)^2 + e^{2z_i}}} dx_i^2 + \frac{1}{(z_i + 1) + \sqrt{(z_i + 1)^2 + e^{2z_i}}} dy_i^2 \right) + \sum_{j=1}^s dz_j^2, \\ \varphi \zeta_i &= 0, \quad \varphi \left(\frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_i}, \\ \varphi \left(\frac{\partial}{\partial y_i} \right) &= \frac{\partial}{\partial x_i} - \frac{e^{z_i}}{2(z_i + 1) \pm \sqrt{(2z_i + 2)^2 + 4e^{2z_i}}} \frac{\partial}{\partial z_i}. \end{aligned}$$

Then, we have an almost metric f -structure $(\varphi, \zeta_j, \eta_i, g)$ on M . On the other hand, for each $i = 1, \dots, s$ we obtain $d\eta_i = 0$. Moreover

$$\Phi_{ii} := g \left(\frac{\partial}{\partial x_i}, \varphi \frac{\partial}{\partial y_i} \right) = -\frac{1}{\left(-(z_i + 1) \pm \sqrt{(z_i + 1)^2 + e^{2z_i}} \right) \left((z_i + 1) + \sqrt{(z_i + 1)^2 + e^{2z_i}} \right)},$$

and for each $i, j = 1, \dots, s$ $\Phi_{ij} = 0$. Then we get

$$\begin{aligned} \Phi_{ii} &:= g \left(\frac{\partial}{\partial x_i}, \varphi \frac{\partial}{\partial y_i} \right) = \\ &= -\frac{1}{\left(-(z_i + 1) \pm \sqrt{(z_i + 1)^2 + e^{2z_i}} \right) \left((z_i + 1) + \sqrt{(z_i + 1)^2 + e^{2z_i}} \right)} dx_i \wedge dy_i, \end{aligned}$$

and

$$d\Phi = 2 \sum_{j=1}^s dz_j \wedge \left(\sum_{i=1}^n dx_i \wedge dy_i \right) = 2\bar{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is not equal zero and in view of this expression we get an almost Kenmotsu f -manifold.

Example 2. Let R^{2n+s} be $(2n+s)$ -dimensional real vector space with standard coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s)$ and

$$M = \{(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_s) \mid z_i \neq 0, 1 \leq i \leq s, n \in \mathbb{N}, n \geq 1\}$$

be a $(2n+s)$ -dimensional manifold. For each $i = 1, \dots, n$ and $k = 1, \dots, s$,

$$\begin{aligned} X_i &= \left(-1 \pm \sqrt{1 + e^{2z_i}} \right) \frac{\partial}{\partial x_i} + z_i^2 \frac{\partial}{\partial z_i}, \\ Y_i &= \left(1 \pm \sqrt{1 + e^{2z_i}} \right) \frac{\partial}{\partial y_i}, \\ \xi_i &= \frac{\partial}{\partial z_i}, \end{aligned}$$

be a basis of M .

Then, for each $i, j = 1, \dots, n$ and $k = 1, \dots, s$, we obtain

$$\begin{aligned} [X_i, Y_j] &= 2z_i^2 e^{2z_i} \frac{\partial}{\partial y_i}, \quad [Y_i, Y_j] = 0, \\ [X_i, \xi_i] &= -2e^{2z_i} \frac{\partial}{\partial x_i} - z_i^2 \frac{\partial}{\partial z_i}, \quad [Y_i, \xi_i] = \pm 2e^{2z_i} \frac{\partial}{\partial y_i}, \\ [X_i, X_j] &= 2z_i^2 e^{2z_i} \frac{\partial}{\partial x_j} - 2z_i^2 e^{2z_i} \frac{\partial}{\partial x_i}. \end{aligned}$$

If we take $\eta_i = \frac{\partial}{\partial z_i}$, we get

$$\begin{aligned} g &= \sum_{i=1}^n \left(\frac{-1}{1 \pm \sqrt{1 + e^{2z_i}}} dx_i^2 + \frac{1}{1 \pm \sqrt{1 + e^{2z_i}}} dy_i^2 \right) + \sum_{j=1}^s dz_j^2, \\ \varphi \xi_i &= 0, \quad \varphi \left(\frac{\partial}{\partial x_i} \right) = -\frac{\partial}{\partial y_i}, \\ \varphi \left(\frac{\partial}{\partial y_i} \right) &= \frac{\partial}{\partial x_i} - \frac{z_i^2}{2 \pm \sqrt{4 + 4e^{2z_i}}} \frac{\partial}{\partial z_i}. \end{aligned}$$

Then, we have a metric f -structure $(\varphi, \xi_j, \eta_i, g)$ on M . On the other hand, for each $i = 1, \dots, s$ we obtain $d\eta_i = 0$. Moreover

$$\Phi_{ii} := g \left(\frac{\partial}{\partial x_i}, \varphi \frac{\partial}{\partial y_i} \right) = -\frac{1}{\left(-1 \pm \sqrt{1 + e^{2z_i}} \right) \left(1 \pm \sqrt{1 + e^{2z_i}} \right)},$$

and for each $i, j = 1, \dots, s$ $\Phi_{ij} = 0$. Then we get

$$\Phi_{ii} := g \left(\frac{\partial}{\partial x_i}, \varphi \frac{\partial}{\partial y_i} \right) = -\frac{1}{\left(-1 \pm \sqrt{1 + e^{2z_i}} \right) \left(1 \pm \sqrt{1 + e^{2z_i}} \right)} dx_i \wedge dy_i,$$

and

$$d\Phi = 2 \sum_{j=1}^s dz_j \wedge \left(\sum_{i=1}^n dx_i \wedge dy_i \right) = 2\bar{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is equal 0 and in view of these expressions we get a Kenmotsu f -manifold.

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В статті розглядаються узагальнені майже Кенмотсу f -многовиди. Отримано основні властивості Ріманової кривизни, секційних кривин і скалярної кривизни для таких типів многовидів. На сам кінець, надано два приклади.

Ключові слова і фрази: f -структура, майже Кенмотсу f -многовиди.

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