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On approximation properties of a new construction of Baskakov operators

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Abstract

The purpose of this research is to construct sequences of Baskakov operators such that their construction consists of a function σ by use of two function sequences, ξ_n and η_n. In these operators, σ not only features the sequences of operators but also features the Korovkin function set $\{1, \sigma, \sigma^2\}$ in a weighted function space such that the operators fix exactly two functions from the set. Thereafter, weighted uniform approximation on an unbounded interval, the degree of approximation with regards to a weighted modulus of continuity, and an asymptotic formula of the new operators are presented. Finally, some illustrative results are provided in order to observe the approximation properties of the newly defined Baskakov operators. The results demonstrate that the introduced operators provide better results in terms of the rate of convergence according to the selection of σ.

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1 Introduction

Approximation theory, which is related to a number of branches of mathematics, is the common subject of applied mathematics and functional analysis. Basically, the purpose of approximation theory is to express functions by polynomials or rational functions with more useful structures. With the theorem given by the German mathematician Karl Weierstrass in 1885, the foundation of the problem of approximation to continuous functions was laid. Weierstrass [29] proved the existence of at least one polynomial sequence which converges uniformly to the function u which is continuous in a closed and finite interval $[a, b]$. In 1912, the Russian mathematician S. N. Bernstein [7] proved an easy and understandable proof of the Weierstrass theorem by determining this series of polynomials, which are known to converge properly to the function $u \in C[0, 1]$.

In 1951, Bohman [8] propounded a crucial theorem about the positive linear operator sequences being uniformly convergent to the continuous function u in the interval $[0, 1]$. In 1953, Korovkin [16] made a significant contribution to the use of linear positive operators in approximation theory by generalizing Bohman's result. According to Korovkin's theorem, the sequence of linear positive operators $(L_n)_{n\in\mathbb{N}}$ is uniformly convergent to the

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function $u \in C[a,b]$, which is bounded in the real line, if and only if the sequence $\{L_n(x^k)\}$ is uniformly convergent to the function x^k for $k = 0,1,2$. After proving this theorem, a number of positive linear operators that satisfy these three conditions mentioned in the theorem have been introduced, and the approximation properties of these operators have been studied.

In parallel with these developments, in 1957, Baskakov [6] introduced the following operator named after him:

$$R_n(u;x) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(x)^k}{(1+x)^{k+n}} u\left(\frac{k}{n}\right), \quad x \in [0,\infty), n \in \mathbb{N},$$

for $u \in [0,\infty)$ whenever the above sum converges and examined the approximation properties of this operator. Baskakov–Kantorovich operators, Baskakov–Durrmeyer operators and others have been introduced on the basis of Baskakov operators, and various properties of these operators have been examined, and studies are still ongoing. Some of the findings can be seen in the relevant direction in [2, 3, 11–13, 15, 17–21, 25].

On the other hand, Cardenes-Morales et al. [9] handled the Bernstein operators in a different way, obtained the new operator defined by $B_n(u \circ \tau^{-1}) \circ \tau$ with particular assumptions, and proved their approximation properties. However, the most significant feature of this newly introduced Bernstein operator is that it preserves the set of $\{1, \tau, \tau^2\}$ instead of the classical Korovkin's test functions $\{1, x, x^2\}$. Accordingly, in [1] Bernstein–Durrmeyer operators by Acar et al., in [4, 5] Szász–Mirakyan operators by Aral et al., in [26] Bernstein–Chlodowksi operators by Usta, in [27] Balázs operators by Usta, and in [24] Lupaş operators by Qasim et al. have been introduced. In this respect, Patel et al. made a similar study for the Baskakov operators and obtained the following operator:

$$R_n^{\sigma}(u;x) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(\sigma(x))^k}{(1+\sigma(x))^{k+n}} \left(u \circ \sigma^{-1}\right)\left(\frac{k}{n}\right), \quad x \in [0,\infty), n \in \mathbb{N},$$

for $u \in [0,\infty)$ such that $\sigma(x)$ is a continuous and infinite times differentiable map which satisfies:

C1 $\sigma(0) = 0$ and $\inf_{x\in[0,\infty)} \sigma'(x) \geq 1$,

C2 σ is a continuously differentiable function on positive real line.

Although this newly defined Baskakov operator is valuable, the use of the $\sigma(x)$ function in three different places in the operator has limited the use of the operator, since a change in $\sigma(x)$ will make the same change in three different places where this function is used. For this reason, using different types and characteristics functions in these three places will both cover the existing operators and provide more opportunities to obtain new Baskakov type operators. Therefore, in this study, we aim to obtain a more specific and more comprehensive new Baskakov operator by replacing the $\frac{(\sigma(x))^k}{(1+\sigma(x))^{k+n}}$ term in the operator with $\frac{(\xi_n(x))^k}{(1+\eta_n(x))^{k+n}}$. By determining the properties of $\xi_n(x)$ and $\eta_n(x)$, we will have obtained a more competitive Baskakov operator.

The whole structure of the paper consists of nine sections, including this one. The rest of this paper is structured as follows: In Sect. 2, some preliminaries are summarized. In Sect. 3, the new Baskakov operators are constructed with $\xi_n(x)$, $\eta_n(x)$, and $\sigma(x)$, while the fundamental features of the new definition are discussed in Sect. 4. In Sect. 5, direct

estimates in a weighted space of this newly defined operators are given in some spaces of continuous functions. Then, a quantitative type theorem is given in Sect. 6. In Sect. 7, we present the Voronovskaya type theorem for these operators, while some numerical examples are discussed in Sect. 8. In Sect. 9, we provide some conclusions and further directions of research.

2 Preliminaries

Here, we summarize the definitions of some of the fundamental facts that are vital in the approximation theory for use in the following sections.

Let u be a continuous function on a positive real line symbolized by $\mathbb{R}^+$ and $\rho(x)$ be a weight function such that

$$\rho(x) = 1 + \sigma^2(x),$$

and let $\mathcal{B}_\tau(\mathbb{R}^+)$ be the normed space by

$$\mathcal{B}_\rho\left(\mathbb{R}^+\right) = \left\{u : \mathbb{R}^+ \to \mathbb{R} : \left|u(x)\right| \leq \mathcal{M}_u \rho(x), x \in \{0\} \cup \mathbb{R}\right\},$$

endowed with the norm

$$\|u\|_\rho = \sup_{x \in \mathbb{R}^+} \frac{u(x)}{\rho(x)},$$

where $\mathcal{M}_u$ is a constant which depends on u. In addition to these, let us define the following function spaces:

$$\mathcal{C}_\rho\left(\mathbb{R}^+\right) = \left\{u \in \mathcal{B}_\rho\left(\mathbb{R}^+\right) : u \text{ is continuous on } \mathbb{R}^+\right\},$$
$$\mathcal{C}^*_\rho\left(\mathbb{R}^+\right) = \left\{u \in \mathcal{C}_\rho\left(\mathbb{R}^+\right) : \lim_{x \to \infty} u(x)/\rho(x) \text{ is a constant}\right\},$$
$$\mathcal{E}_\rho\left(\mathbb{R}^+\right) = \left\{u \in \mathcal{C}_\rho\left(\mathbb{R}^+\right) : u(x)/\rho(x) \text{ is uniformly continuous on } \mathbb{R}^+\right\}.$$

Another significant element in approximation theory is weighted modulus of continuity which needs to be reviewed. Let $\omega_\sigma(u;\delta)$ be the weighted modulus of continuity introduced by Holhoş [14] for $\delta > 0$, that is,

$$\omega_\sigma(u,\delta) = \sup_{t,x \in \mathbb{R}^+, |\sigma(t)-\sigma(x)| \leq \delta} \frac{|u(t)-u(x)|}{\rho(t)-\rho(x)}, \tag{2.1}$$

where $u \in \mathcal{C}_\rho(\mathbb{R}^+)$. It is worth noting that $\omega_\sigma(u,0) = 0$ for each $u \in \mathcal{C}_\rho(\mathbb{R}^+)$. Moreover, $\omega_\sigma(u,\delta)$ is nonnegative and nondecreasing with regard to δ for each $u \in \mathcal{C}_\rho(\mathbb{R}^+)$.

Lemma 1 $\lim_{\delta \to 0} \omega_\sigma(u,\delta) = 0$ *for every* $u \in \mathcal{E}_\rho(\mathbb{R}^+)$.

Proof See [14]. □

In the light of this information, we can now construct new Baskakov operators which depend on the function $\sigma(x)$ and two sequences of functions, $\xi_n(x)$ and $\eta_n(x)$.

3 Configuration of the operators

Let $\Omega \subset [0,\infty)$ be an interval, $\mathbb{N}^* = \{n \in \mathbb{N} : n \geq n^* \text{ for given } n^* \in \mathbb{N}\}$, and u be a continuous map on a positive real line symbolized by $\mathbb{R}^+$. Additionally, let $\xi, \eta : \Omega \to \mathbb{R}$ be positive maps on Ω for each $x \in \Omega$ and $n \in \mathbb{N}^*$. By taking into account all of these, the new Baskakov type operators can be introduced as follows.

Definition 1 For $x \in [0,\infty)$, the new construction of Baskakov operators is

$$\mathcal{R}_n^\sigma u(x) =: \mathcal{R}_n^\sigma(u;x) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(\xi_n(x))^k}{(1+\eta_n(x))^{k+n}} \left(u \circ \sigma^{-1}\right)\left(\frac{k}{n}\right), \tag{3.1}$$

where σ is a function satisfying conditions **C1** and **C2**.

The new construction of Baskakov operators described above may not be an approximation process for all cases of $\xi_n(x)$ and $\eta_n(x)$. Then, we need to determine the properties of the $\xi_n(x)$ and $\eta_n(x)$ functions given in the definition in order for the new construction of Baskakov operators described above to meet the approximation conditions. Thus, by determining what $\xi_n(x)$ and $\eta_n(x)$ are, new Baskakov type operators can be defined. Now, accordingly, we are enforcing a couple of assumptions that can fulfil the requirements of the approximation procedure.

To begin with, the new construction of Baskakov operators must preserve the constant function 1 straight-forwardly or in the limit case. In other words, the following assumption must hold:

$$\mathcal{R}_n^\sigma 1(x) = 1 + \mu_n(x)$$

for $x \in \Omega$ such that $\mu_n : \Omega \to \mathbb{R}$ is a function. Through the instrumentality of newly defined Baskakov operators, we have

$$\begin{aligned}\mathcal{R}_n^\sigma 1(x) &= \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(\xi_n(x))^k}{(1+\eta_n(x))^{k+n}} \left(1 \circ \sigma^{-1}\right)\left(\frac{k}{n}\right) \\ &= \left(\eta_n(x) - \xi_n(x) + 1\right)^{-n}\end{aligned}$$

for every $x \in \Omega$ and $|\xi_n(x)| < |\eta_n(x) + 1|$. So we have

$$\left(\eta_n(x) - \xi_n(x) + 1\right)^{-n} = 1 + \mu_n(x). \tag{3.2}$$

Secondly, the new Baskakov operators must also preserve any given function σ directly or in the limit case similarly, that is,

$$\mathcal{R}_n^\sigma \sigma(x) = \sigma(x) + \nu_n(x)$$

for $x \in \Omega$ such that $\mu_n : \Omega \to \mathbb{R}$ is a map. Likewise, using the newly defined Baskakov operators, we deduce that

$$\mathcal{R}_n^\sigma \sigma(x) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(\xi_n(x))^k}{(1+\eta_n(x))^{k+n}} \left(\sigma \circ \sigma^{-1}\right)\left(\frac{k}{n}\right) = \xi_n(x)\left(\eta_n(x) - \xi_n(x) + 1\right)^{-n-1}$$

for every $x \in \Omega$ and $|\xi_n(x)| < |\eta_n(x) + 1|$, which yields

$$\xi_n(x)\big(\eta_n(x) - \xi_n(x) + 1\big)^{-n-1} = \sigma(x) + \nu_n(x). \tag{3.3}$$

From equations (3.2) and (3.3), we immediately deduce that

$$\xi_n(x) = \frac{\sigma(x) + \nu_n(x)}{(1 + \mu_n(x))^{1+1/n}}$$

and

$$\eta_n(x) = \frac{1 + \sigma(x) + \mu_n(x) + \nu_n(x)}{(1 + \mu_n(x))^{1+1/n}} - 1$$

for any $x \in \Omega$ and $n \in \mathbb{N}^*$ such that $\mu_n(x) > 1$. As a consequence, if all these deduced values are substituted in equation (3.1), we have

$$\mathcal{R}_n^{\sigma} u(x) = \big(1 + \mu_n(x)\big)^{1+n} \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{(\sigma(x) + \nu_n(x))^k}{(1 + \sigma(x) + \mu_n(x) + \nu_n(x))^{k+n}} \big(u \circ \sigma^{-1}\big)\left(\frac{k}{n}\right)$$

for each $x \in \Omega$ and each $n \in \mathbb{N}^*$.

Certainly, the characteristics of the $\mu_n(x)$ and $\nu_n(x)$ functions also play a crucial role in the operators given above, and these functions must fulfill some conditions in order that the new definition is an approximation process. In other words, $(\mathcal{R}_n^{\sigma})_{n \geq n^*}$ refers to an approximation procedure on Ω if the following conditions hold:

$$\lim_{n \to \infty} \mu_n(x) = 0 \quad \text{and} \quad \big|\mu_n(x)\big| \leq \mu_n \tag{3.4}$$

and

$$\lim_{n \to \infty} \nu_n(x) = 0 \quad \text{and} \quad \big|\nu_n(x)\big| \leq \nu_n \tag{3.5}$$

together with the relation of $\xi_n(x)$ and $\eta_n(x)$ according to the weighted Korovkin theorem.

3.1 Some special cases:

Now let us demonstrate that the newly constructed Baskakov operators produce some of the Baskakov operators that exist in the literature under appropriate selection of $\mu_n(x)$, $\nu_n(x)$, and $\sigma(x)$ functions.

(A) By choosing $\mu_n(x) = 0$ and $\nu_n(x) = 0$ in (3.1), we deduce the Baskakov operators introduced in [23].

(B) By choosing $\mu_n(x) = 0$, $\nu_n(x) = 0$, and $\sigma(x) = x$ in (3.1), we deduce the standard Baskakov operators given in [6].

(C) By choosing $\mu_n(x) = 0$, $\nu_n(x) = \frac{e^{-2ax/n}-1}{1-e^{2a/n}} - x$, and $\sigma(x) = x$ in (3.1), we deduce the Baskakov operators, which preserve $\{1, e^{2ax}\}$ where $a > 0$, given in [30].

(D) By choosing $\mu_n(x) = e^{ax}(1 - e^{-a(x+1)/n} + e^{-a/n})^{-n} - 1$, $\nu_n(x) = e^{ax-a/n}(1 - e^{-a(x+1)/n} + e^{-a/n})^{-n-1} \frac{e^{-ax/n}-1}{1-e^{a/n}} - x$, and $\sigma(x) = x$ in (3.1), we deduce the Baskakov operators, which preserve $\{e^{ax}, e^{2ax}\}$ where $a > 0$, given in [22].

It is left to the reader to investigate which values $\mu_n(x)$, $\nu_n(x)$, and $\sigma(x)$ should be chosen in order to obtain other Baskakov type operators in the literature.

4 The fundamental features of the operators

In this section, we provide some basic properties of the newly defined Baskakov operators such as moments and central moments that we will use while examining the approximation properties of them.

Lemma 2 *The following equalities are valid for the newly defined Baskakov operators for* $x \in \Omega$:

(1) $\mathcal{R}_n^{\sigma} 1(x) = 1 + \mu_n(x)$,

(2) $\mathcal{R}_n^{\sigma} \sigma(x) = \sigma + \nu_n(x)$.

(3) $\mathcal{R}_n^{\sigma} \sigma^2(x) = \left(\frac{n+1}{n}\right) \frac{(\sigma(x)+\nu_n(x))^2}{1+\mu_n(x)} + \frac{\sigma(x)+\nu_n(x)}{n}$,

(4) $\mathcal{R}_n^{\sigma} \sigma^3(x) = \left(\frac{n^2+3n+2}{n^2}\right) \frac{(\sigma(x)+\nu_n(x))^3}{(1+\mu_n(x))^2} + \left(\frac{3n+3}{n^2}\right) \frac{(\sigma(x)+\nu_n(x))^2}{(1+\mu_n(x))} + \frac{(\sigma(x)+\nu_n(x))}{n^2}$.

Proof As the equalities are comfortably deduced by direct computation, the proof is left to the reader. □

Now, the notation of the central moments of the newly defined Baskakov operators of degree q is $\mathcal{P}_{n,q}^{\sigma}$, i.e.,

$$\mathcal{P}_{n,q}^{\sigma} = \mathcal{R}_n^{\sigma} \big(\sigma(t) - \sigma(x)\big)^q (x).$$

Then the following lemma holds.

Lemma 3 *The following central moment values are valid for the newly defined Baskakov operators for* $x \in \Omega$:

(1) $\mathcal{P}_{n,0}^{\sigma} = 1 + \mu(x)$,

(2) $\mathcal{P}_{n,1}^{\sigma} = \nu_n(x) - \sigma\, \mu_n(x)$

(3) $\mathcal{P}_{n,2}^{\sigma} = \left(\frac{n+1}{n}\right) \frac{(\sigma(x)+\nu_n(x))^2}{1+\mu_n(x)} + \frac{\sigma(x)+\nu_n(x)}{n} + \sigma^2(x)(\mu_n(x) - 1) - 2\sigma(x)\nu_n(x)$.

Proof Similarly, the proofs of the above findings are left to the reader as all of them can be computed by basic calculation. □

With the results obtained above, we can now present the fundamental approximation features for the newly defined Baskakov operators in the next sections.

5 Direct results

In this section, we present the direct estimations for the newly constructed Baskakov operators in a weighted space reviewed in the previous sections. However, before we start, let us remember the following weighted Korovkin type theorem defined by Gadjiev [10].

Let $(\mathcal{S}_n)_{n\geq 1}$ be a sequence of linear positive operators acting from $\mathcal{C}_{\rho}(\mathbb{R}^+) \to \mathcal{B}_{\rho}(\mathbb{R}^+)$. Then the following lemma holds.

Lemma 4 ([10]) *Let* $\mathcal{H}_n > 0$ *be a constant which depends upon just* n. *Then* $(\mathcal{S}_n)_{n\geq 1} : \mathcal{C}_{\rho}(\mathbb{R}^+) \to \mathcal{B}_{\rho}(\mathbb{R}^+)$ *iff the inequality* $|\mathcal{S}_n(\rho; x)| \leq \mathcal{H}_n \rho(x)$ *holds.*

Proof See [10]. □

Theorem 1 *Let $u \in \mathcal{C}_\rho^*(\mathbb{R}^+)$ and $(\mathcal{S}_n)_{n\geq 1}$ be an approximation procedure defined above. In this circumstance, for any function u, the following identity*

$$\lim_{n\to\infty} \|\mathcal{S}_n u - u\|_\rho = 0,$$

holds if the following identities are satisfied

$$\lim_{n\to\infty} \left\|\mathcal{S}_n \rho^l - \rho^l\right\|_\rho = 0,$$

where $l = 0, 1, 2$.

Proof See [10]. □

In the light of this information, we can now express the following theorem, which constitutes the main idea of this section.

Theorem 2 *Let $u \in C_\rho^*(\mathbb{R}^+)$ and $(\mathcal{R}_n^\sigma)_{n\geq n_0}$ be an approximation process defined in (3.1). If the inequalities given in (3.4)–(3.5) are valid, then the following limit*

$$\lim_{n\to\infty} \sup_{x\in\Omega} \frac{|\mathcal{R}_n^\sigma(u;x) - u(x)|}{\rho(x)} = 0$$

holds, where $\rho(x)$ is a weight function given in the previous sections.

Proof As we know, $|u(x)| \leq \mathcal{H}_u \rho(x)$ as $u \in C_\rho^*(\mathbb{R}^+)$ for $x \in [0, \infty)$. Under these conditions, the equality

$$\mathcal{R}_n^\sigma(\rho;x) = 1 + \mu_n(x) + \left(\frac{n+1}{n}\right)\frac{(\sigma(x) + \nu_n(x))^2}{1+\mu_n(x)} + \frac{\sigma(x)+\nu_n(x)}{n}$$

holds and leads to $(\mathcal{R}_n^\sigma)_{n\geq 1} : \mathcal{C}_\rho(\mathbb{R}^+) \to \mathcal{B}_\rho(\mathbb{R}^+)$. In addition to this, $(\mathcal{R}_n^\sigma)_{n\geq 1}$ being linear and positive is monotone.

Moreover, we have

$$\lim_{n\to\infty} \sup_{x\in\Omega} \frac{|\mathcal{R}_n^\sigma(\sigma^l) - \sigma^l|}{\rho(x)} = 0$$

for $l = 0, 1, 2$, under favor of the results of Lemma 2 and conditions (3.4)–(3.5).

Besides, the newly constructed Baskakov operators $(\mathcal{R}_n^\sigma)_{n\geq 1}$ are defined on Ω. In order to enlarge it on all the positive real line, let us define the following sequence of operators:

$$\mathcal{R}_n^\sigma(u;x) = \begin{cases} \mathcal{R}_n^\sigma(u;x), & x \in \Omega, \\ u(x), & x \in \mathbb{R}^+ \setminus \Omega, \end{cases}$$

which yields

$$\lim_{n\to\infty} \left\|\mathcal{R}_n^\sigma(u) - u\right\|_\rho = \sup_{x\in\Omega} \frac{\mathcal{R}_n^\sigma(u;x) - u(x)}{\rho(x)}. \tag{5.1}$$

Then, we immediately deduce that

$$\lim_{n\to\infty} \left\| \mathcal{R}_n^{\sigma}(u) - u \right\|_{\rho} = 0$$

as

$$\lim_{n\to\infty} \left\| \mathcal{R}_n^{\sigma}\left(\sigma^l\right) - \sigma^l \right\|_{\rho} = 0$$

for $l = 0, 1, 2$, such that

$$\left\| \mathcal{R}_n^{\sigma}\left(\sigma^l\right) - \sigma^l \right\|_{\rho} = \sup_{x\in\Omega} \frac{|\mathcal{R}_n^{\sigma}(\sigma^l) - \sigma^l)|}{\rho(x)},$$

with the help of Theorem 1, with $\mathcal{S}_n = \mathcal{R}_n^{\sigma}$. Consequently, the proof is finalized by considering (5.1). Thus the proof is completed. □

6 Order of approximation

Now it is time to deliver the quantitative type theorem for the newly defined Baskakov operator based on weighted modulus of continuity summarized in the previous sections. We will make use of the technique developed by Patel et al. [14] in presenting this theorem. So, let us briefly remember this theorem.

Theorem 3 *Let $\mathcal{F}_n : \mathcal{C}_{\tau}(\mathbb{R}^+) \to \mathcal{B}_{\tau}(\mathbb{R}^+)$ be a sequence of positive and linear operators with*

$$\left\| \mathcal{W}_n\left(\sigma^0\right) - \sigma^0 \right\|_{\rho^0} = \mathtt{a}_\mathtt{n},$$
$$\left\| \mathcal{W}_n\left(\sigma^1\right) - \sigma^1 \right\|_{\rho^{\frac{1}{2}}} = \mathtt{b}_\mathtt{n},$$
$$\left\| \mathcal{W}_n\left(\sigma^2\right) - \sigma^2 \right\|_{\rho^1} = \mathtt{c}_\mathtt{n},$$
$$\left\| \mathcal{W}_n\left(\sigma^3\right) - \sigma^3 \right\|_{\rho^{\frac{3}{2}}} = \mathtt{d}_\mathtt{n},$$

where $\mathtt{a}_\mathtt{n}$, $\mathtt{b}_\mathtt{n}$, $\mathtt{c}_\mathtt{n}$, and $\mathtt{d}_\mathtt{n}$ approach zero in case n goes to the infinity. So, we have

$$\left\| \mathcal{W}_n(u) - u \right\|_{\rho^{\frac{3}{2}}} \leq (7 + 4\mathtt{a}_\mathtt{n} + 2\mathtt{c}_\mathtt{n})\omega_{\sigma}(u;\delta_n) + \|u\|_{\rho}\mathtt{a}_\mathtt{n},$$

given these circumstances for all $u \in \mathcal{C}_{\rho}(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(1 + \mathtt{a}_\mathtt{n})(\mathtt{a}_\mathtt{n} + 2\mathtt{b}_\mathtt{n} + \mathtt{c}_\mathtt{n})} + \mathtt{a}_\mathtt{n} + 3\mathtt{b}_\mathtt{n} + 3\mathtt{c}_\mathtt{n} + \mathtt{d}_\mathtt{n}.$$

Proof See [14]. □

Now we can present the essential theorem of this section.

Theorem 4 *For every $u \in \mathcal{C}_{\rho}(\mathbb{R}^+)$, we have the following inequality*:

$$\left\| \mathcal{R}_n^{\sigma}(u) - u \right\|_{\rho^{\frac{3}{2}}} \leq \left(7 + 4\mu_n + 2\left(\frac{n+1}{n}\right)\left(2\nu_n + \nu_n^2\right) + \frac{4}{n} + \frac{4\nu_n}{n}\right)\omega_{\sigma}(u;\delta_n),$$

where

$$\begin{aligned}\delta_n = 2\sqrt{(1+\mu_n)\left(\mu_n + 2\nu_n + \left(\frac{n+1}{n}\right)\left(2\nu_n+\nu_n^2\right) + \frac{2}{n} + \frac{2\nu_n}{n}\right)} \\ + \mu_n + 3\nu_n + 3\left(\frac{n+1}{n}\right)\left(2\nu_n+\nu_n^2\right) + \frac{6}{n} + \frac{6\nu_n}{n} \\ + \frac{4}{n^2}(1+\nu_n) + 6\left(\frac{n+1}{n^2}\right)\left(1+\nu_n^2\right) + \left(\frac{n^2+3n+2}{n^2}\right)\left(3\nu_n+3\nu_n^2+\nu_n^3\right).\end{aligned}$$

Proof In order to prove this theorem, we first need to calculate $\mathtt{a_n}$, $\mathtt{b_n}$, $\mathtt{c_n}$, and $\mathtt{d_n}$. Using the results of Lemma 2, we can instantaneously obtain the following results:

$$\begin{aligned}&\left\|\mathcal{R}_n^{\sigma}\left(\sigma^0\right)-\sigma^0\right\|_{\rho^0} \le \mu_n = \mathtt{a_n},\\ &\left\|\mathcal{R}_n^{\sigma}\left(\sigma^1\right)-\sigma^1\right\|_{\rho^{\frac{1}{2}}} \le \nu_n = \mathtt{b_n},\\ &\left\|\mathcal{R}_n^{\sigma}\left(\sigma^2\right)-\sigma^2\right\|_{\rho^1} \le \left(\frac{n+1}{n}\right)\left(2\nu_n+\nu_n^2\right) + \frac{2}{n} + \frac{2\nu_n}{n} = \mathtt{c_n},\\ &\left\|\mathcal{R}_n^{\sigma}\left(\sigma^3\right)-\sigma^3\right\|_{\rho^{\frac{3}{2}}} \le \frac{4}{n^2}(1+\nu_n) + 6\left(\frac{n+1}{n^2}\right)\left(1+\nu_n^2\right)\\ &\qquad\qquad + \left(\frac{n^2+3n+2}{n^2}\right)\left(3\nu_n+3\nu_n^2+\nu_n^3\right) = \mathtt{d_n}.\end{aligned}$$

It has been observed that the conditions of Theorem 3 are satisfied, which yields the desired result. So, the proof is completed. □

Theorem 5 *For all $u \in \mathcal{E}_{\rho}(\mathbb{R}^+)$, we get*

$$\lim_{n\to\infty}\left\|\mathcal{R}_n^{\sigma}(u)-u\right\|_{\rho^{\frac{3}{2}}} = 0.$$

Proof The proof is immediately deduced by using Theorem 4 and Lemma 1. □

7 Transferring to Voronovskaya type theorem

Another significant subject of the approximation theory is the determination of the convergence rate. In 1932, Voronovskaya [28] expressed and proved his theorem which is called asymptotic approximation for Bernstein polynomials for the function $u(x)$, which is bounded in the interval of $[0,1]$ and has a second derivative at a certain point. Thereafter, Voronovskaya type theorems are provided for the derivative of Bernstein and many other operators. Hence, in this section Voronovskaya type theorem will be given to examine the asymptotic behavior of the new Baskakov operators. However, first let us summarize the following lemma expressed by Holhoș [14].

Lemma 5 *Let $a, x \ge 0$, $\delta > 0$, and $u \in \mathcal{C}_{\rho}(\mathbb{R}^+)$. Then we have*

$$\left|u(a)-u(x)\right| \le \left(\rho(a)+\rho(x)\right)\left[2+\frac{|\sigma(c)-\sigma(x)|}{\delta}\right]\omega_{\sigma}(u,\delta).$$

Proof See [14]. □

Theorem 6 *Let $u \in \mathcal{C}_\rho(\mathbb{R}^+)$, $x \in \Omega$, and assume that $(u \circ \sigma^{-1})'$ and $(u \circ \sigma^{-1})''$ exist at $\sigma(x)$. If $(u \circ \sigma^{-1})''$ is bounded on $\mathbb{R}^+$, the following equality*

$$\lim_{n\to\infty} n\big[\mathcal{R}_n^\sigma(u)(x) - u(x)\big] = u(x)\alpha + \big(\beta - \sigma(x)\alpha\big)\big(u \circ \sigma^{-1}\big)'\big(\rho(x)\big) + \frac{1}{2}\sigma(x)\big(1 + \sigma(x)\big)\big(u \circ \sigma^{-1}\big)''\big(\sigma(x)\big)$$

holds, where

$$\lim_{n\to\infty} n\mu_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} n\nu_n = \beta.$$

Proof Through the instrument of the well-known Taylor expansion of $(u \circ \sigma^{-1})$ at the point of $\sigma(x)$ such that there exists ε lying between t and x, we have

$$\begin{aligned}&\big(u \circ \sigma^{-1}\big)\big(\sigma(t)\big) - \big(u \circ \sigma^{-1}\big)\big(\sigma(x)\big)\\ &\quad = \big(\sigma(t) - \sigma(x)\big)\big(u \circ \sigma^{-1}\big)'\big(\sigma(x)\big) + \frac{1}{2}\big(\sigma(t) - \sigma(x)\big)^2\big(u \circ \sigma^{-1}\big)''\big(\sigma(x)\big)\\ &\qquad + \Delta_x(t)\big(\sigma(t) - \sigma(x)\big)^2,\end{aligned}$$

where

$$\Delta_x(t) = \frac{1}{2}\big[\big(u \circ \sigma^{-1}\big)''\big(\sigma(\varepsilon)\big) - \big(u \circ \sigma^{-1}\big)''\big(\sigma(x)\big)\big]. \tag{7.1}$$

It is obvious that

$$\lim_{t\to x} \Delta_x(t) = 0, \tag{7.2}$$

and $\Delta_x(t)$ is bounded such that $|\Delta_x(t)| \le H$, where H is a constant, because of the nature of function u and (7.1) for all t. Then, implementing the newly defined Baskakov operators given in (3.1) and multiplying by n, we immediately deduce that

$$\begin{aligned}&n\big[\mathcal{R}_n^\sigma u(x) - u(x)\mathcal{R}_n^\sigma 1(x)\big]\\ &\quad = n\mathcal{R}_n^\sigma\big(\sigma(t) - \sigma(x)\big)(x)\big(u \circ \sigma^{-1}\big)'\big(\sigma(x)\big)\\ &\qquad + \frac{1}{2}n\mathcal{R}_n^\sigma\big(\sigma(t) - \sigma(x)\big)^2(x)\big(u \circ \sigma^{-1}\big)''\big(\sigma(x)\big) + n\mathcal{R}_n^\rho\big(\Delta_x(t)\big(\sigma(t) - \sigma(x)\big)^2\big)(x).\end{aligned}$$

Here, thanks to Lemma 2 and Lemma 3, the following identities can be conveniently deduced:

$$\begin{aligned}&\lim_{n\to\infty} n\mathcal{R}_n^\sigma\big(\sigma(t) - \sigma(x)\big)(x) = \lim_{n\to\infty} n\big(\nu_n(x) - \sigma(x)\mu_n(x)\big) = \beta - \sigma(x)\alpha,\\ &\lim_{n\to\infty} n\mathcal{R}_n^\sigma\big(\big(\rho(t) - \rho(x)\big)^2\big)(x) = \sigma(x)\big(1 + \sigma(x)\big).\end{aligned}$$

So, we have

$$\begin{aligned}\lim_{n\to\infty} n\left[\mathcal{R}_n^{\sigma}u(x)-u(x)\right] &= u(x)\alpha+\left(\beta-\sigma(x)\alpha\right)\left(u\circ\sigma^{-1}\right)'\left(\rho(x)\right)\\ &\quad+\frac{1}{2}\sigma(x)\left(1+\sigma(x)\right)\left(u\circ\sigma^{-1}\right)''\left(\sigma(x)\right)\\ &\quad+\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left|\Delta_x(t)\right|\left(\sigma(t)-\sigma(x)\right)^2\right)(x). \end{aligned} \tag{7.3}$$

Finally, in order to complete the proof, we need to show that the value of the last term of equation (7.3) is zero. To do this, we will make use of a well-recognized Cauchy–Schwarz inequality. In accordance with this purpose, let $\delta > 0$ such that $|\sigma_x(t) < \varepsilon|$ for each $t \geq 0$ because of (7.2) for each $\varepsilon \geq 0$. Then we obtain that

$$\begin{aligned}&\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left|\Delta_x(t)\right|\left(\sigma(t)-\sigma(x)\right)^2\right)(x)\\ &\quad\leq \varepsilon\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left(\sigma(t)-\sigma(x)\right)^2\right)(x)+\frac{M}{\delta^2}\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left(\sigma(t)-\sigma(x)\right)^4\right)(x).\end{aligned}$$

It is quite apparent

$$\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left|\Delta_x(t)\right|\left(\sigma(t)-\sigma(x)\right)^2\right)(x)=0,$$

in connection with

$$\lim_{n\to\infty} n\mathcal{R}_n^{\sigma}\left(\left(\sigma(t)-\sigma(x)\right)^4\right)(x)=0,$$

thus the proof is completed. □

8 Illustrative examples

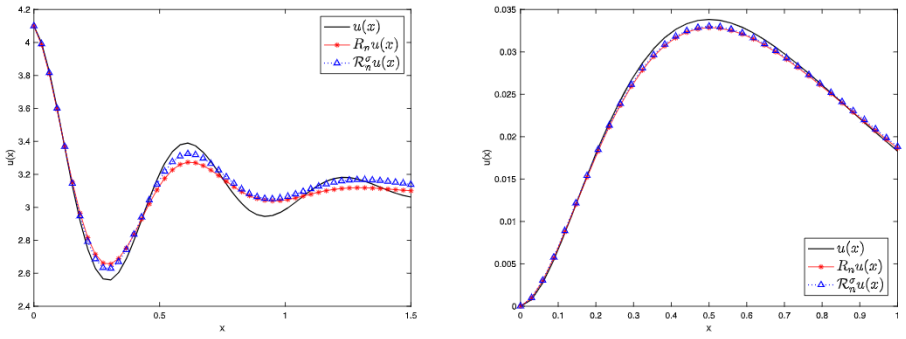
In this section, the Baskakov operators presented in this paper are applied to provide the approximation of three illustrative examples. Thus, we will have the opportunity to observe the relevance of the theoretical and graphical results or the new construction of Baskakov operators. All of the computational processes are performed on an Intel Core i5 personal laptop by running a code implemented in MATLAB 9.7.0.1190202 (R2019b) software. To clarify the accuracy and efficiency of the introduced operators, the values of approximation are compared with the values of a test function by plotting them on the same figure.

8.1 Example 1

As the first example, we consider the following function:

$$u(x)=\cos(10x)e^{-2x}+3.1$$

as a test function. In this example, we select $\mu_n(x)=\frac{x}{n-1}$, $\nu_n(x)=\frac{x}{n-1}$, and $\sigma(x)=2x$ for $n=100$.



(A) $\mu_n(x) = x/(n-1)$, $\nu_n(x) = x/(n-1)$, and $\sigma(x) = 2x$

(B) $\mu_n(x) = x/(n-1)$, $\nu_n(x) = x/n$, and $\sigma(x) = x$

Figure 1 Test function (black line), standard Baskakov operator (red-star), and newly constructed Baskakov operator (blue-upward pointing triangle) with the given parameters on equally spaced grid

8.2 Example 2

For the second example, we take

$$u(x) = e^{-4x}x^2$$

as a test function. In this example, we select $\mu_n(x) = \frac{x}{n-1}$, $\nu_n(x) = \frac{x}{n}$, and $\sigma(x) = x$ for $n = 100$.

Figure 1 indicates that the newly defined Baskakov operators show markedly better approximation behavior for the test functions $u(x) = \cos(10x)e^{-2x} + 3.1$ and $u(x) = e^{-4x}x^2$ in comparison to the classical Baskakov operators. Of course, the selection of the functions $\xi_n(x)$, $\eta_n(x)$, and $\sigma(x)$ plays a significant role in getting good approximation performance. These two experiments confirm that the newly defined Baskakov operators provide noteworthy performance with the suitable selection of $\xi_n(x)$, $\eta_n(x)$, and $\sigma(x)$.

9 Concluding remarks

The aim of the present study was to construct new Baskakov operators such that their construction depends on a function. This newly defined Baskakov operators have an important place in terms of covering the existing Baskakov operators and giving the opportunity to define new Baskakov operators that will give better approximation results under appropriate conditions. In order to demonstrate that these newly defined operators are an approximation process, some fundamental properties of them are also presented. At the end, some numerical examples are given.

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