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A new extension of quantum Simpson's and quantum Newton's type inequalities for quantum differentiable convex functions

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In this paper, we prove two identities involving quantum derivatives, quantum integrals, and certain parameters. Using the newly proved identities, we prove new inequalities of Simpson's and Newton's type for quantum differentiable convex functions under certain assumptions. Moreover, we discuss the special cases of our main results and obtain some new and existing Simpson's type inequalities, Newton's type inequalities, midpoint type inequalities, and trapezoidal type inequalities.

KEYWORDS
convex functions, Newton's inequalities, quantum calculus, Simpson's inequalities

MSC CLASSIFICATION
26D10; 26D15; 26A51

1 | INTRODUCTION

Thomas Simpson has evolved essential techniques for the numerical integration and estimation of definite integrals taken into consideration as Simpson's rule during (1710–1761). Nevertheless, a comparable approximation became utilized by J. Kepler nearly earlier than 10 decades, so it's also called Kepler's rule. Simpson's rule consists of the 3-point Newton-Cotes quadrature rule, so estimation primarily based totally on 3-step quadratic kernel is every so often known as Newton-type results.

1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}(\kappa_2)\right].$$

2) Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right].$$

There are a huge variety of estimations associated with those quadrature rules inside the literature: certainly considered one among them is the subsequent estimation called Simpson's inequality:

Theorem 1. *Suppose that* $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *is a four times continuously differentiable mapping on* (κ_1, κ_2), *and let* $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{x\in(\kappa_1,\kappa_2)} \left|\mathcal{F}^{(4)}(x)\right| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}+2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx\right| \le \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

In recent years, many writers have focused on Simpson's type inequality in various categories of mappings. Specifically, some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map, because convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. For example, Dragomir et al[1] presented the new Simpson's inequalities and their applications in quadrature formulas for numerical integration. In addition, some inequalities of Simpson's type of s-convex functions were determined by Alomari et al.[2] Subsequently, Sarikaya et al. note the variance of Simpson's type inequality based on convexity in Sarikaya et al.[3] For the further studies of this area, one can consult previous works.[4–6]

On the other hand, several works in the field of q-analysis, beginning with Euler, have been implemented in order to master the mathematics that underpins quantum computing. The term q-calculus creates a link between mathematics and physics. It's used in combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other fields, as well as relativity theory, mechanics, and quantum theory.[7–9] In quantum information theory, it has many applications.[10,11] Euler used the q-parameter in Newton's work on infinite series that's why he is thought to be inventor of this important branch of mathematics. The concept of q-calculus that is known to be calculus without limits was given by Jackson[12,13] first time in a proper way. The notions about the q-fractional integral and q-Riemann–Liouville fractional integral were given by Al-Salam[14] in 1996. Since the research increase gradually in this field, therefore, Tariboon and Ntouyas[15] gave the idea about the ${}_{\kappa_1}D_q$-difference operator and q_{κ_1}-integral. The notions about the ${}^{\kappa_2}D_q$-difference operator and q^{κ_2} -integral were given by Bermudo et al[16] very recently in 2020. Sadjang[17] generalized the concept of q-calculus by introducing the concepts of (p, q)-calculus. In 2020, Soontharanon and Sitthiwirattham[18] introduced the concepts of fractional (p, q)-calculus. The (p, q)-variant of ${}_{\kappa_1}D_q$ -difference operator and q_{κ_1}-integral was introduced by Tunç and Göv.[19] Recently, in 2021, Chu et al. introduced the notions of ${}^{\kappa_2}D_{p,q}$ derivative and $(p, q)^{\kappa_2}$-integral in Chu et al.[20]

Many integral inequalities for many sorts of functions have indeed been investigated employing quantum as well as post-quantum integrals. For example, the H–H inequalities and their right-left estimates for convex and coordinated convex functions via ${}_{\kappa_1}D_q, {}^{\kappa_2}D_q$-derivatives and $q_{\kappa_1}, q^{\kappa_2}$-integrals were given by different authors in oter studies.[21–31] Noor et al[32] used the pre-invexity to prove H–H inequalities in the setup of q-calculus. Some parameterized q-integral inequalities for generalized quasi-convex functions were established by Nwaeze and Tameru.[33] Khan et al. used the notions of Green functions to establish some new inequalities of H–H type in their work.[34] Budak et al.,[35] Ali et al,[36,37] and Vivas-Cortez et al[38] proved some new boundries for Simpson's and Newton's type inequalities for convex and coordinated convex functions in the setting of q-calculus. One can consult other studies[39–42] for q-Ostrowski's inequalities for convex and coordinated convex functions.

Inspired by the ongoing studies, we prove some new parameterized Simpson's and Newton's type inequalities for q-differentiable convex functions in the setting of q-calculus. In the special cases of newly established parameterized inequalities, we also obtain some midpoint and trapezoidal type inequalities for q-differentiable convex functions.

The following is how this paper is organized: Section 2 provides a brief explanation of the concepts of q-calculus as well as some related works in this field. In Section 3, we prove two important quantum integral equalities. Sections 4 and 5 prove some new Simpson's and Newton's type inequalities for differentiable convex functions in q calculus. The relationship between the results presented here and comparable results in the literature is also considered. Section 6 presents some findings as well as future research directions.

2 | PRELIMINARIES OF *Q*-CALCULUS AND SOME INEQUALITIES

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation (see Kac and Cheung[9]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1},\ q \in (0,1).$$

Jackson[13] defined the q-Jackson integral of a given function $\mathcal{F}$ from 0 to κ_2 as follows:

$$\int_0^{\kappa_2} \mathcal{F}(x)\, d_q x = (1-q)\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{F}\left(\kappa_2 q^n\right), \text{ where } 0<q<1 \tag{2.1}$$

provided that the sum converges absolutely. Moreover, he defined the q-Jackson integral of a given function over the interval $[\kappa_1, \kappa_2]$ as follows:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\, d_q x = \int_0^{\kappa_2} \mathcal{F}(x)\, d_q x - \int_0^{\kappa_1} \mathcal{F}(x)\, d_q x.$$

Definition 1 (Tariboon and Ntouyas[15]). The q_{κ_1}-derivative of mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined as

$$_{\kappa_1}D_q \mathcal{F}(x) = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\kappa_1)}{(1-q)(x-\kappa_1)}, x \neq \kappa_1. \tag{2.2}$$

If $x = \kappa_1$, we define $_{\kappa_1}D_q \mathcal{F}(\kappa_1) = \lim_{x\to\kappa_1} {_{\kappa_1}D_q} \mathcal{F}(x)$ if it exists and it is finite.

Definition 2 (Bermudo et al.[16]). The q^{κ_2}-derivative of mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined as

$$^{\kappa_2}D_q \mathcal{F}(x) = \frac{\mathcal{F}(qx + (1-q)\kappa_2) - \mathcal{F}(x)}{(1-q)(\kappa_2 - x)}, x \neq \kappa_2.$$

If $x = \kappa_2$, we define $^{\kappa_2}D_q \mathcal{F}(\kappa_2) = \lim_{x\to\kappa_2} {^{\kappa_2}D_q} \mathcal{F}(x)$ if it exists and it is finite.

Definition 3 (Tariboon and Ntouyas[15]). The q_{κ_1}-definite integral of $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ on $[\kappa_1, \kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\, {_{\kappa_1}d_q x} = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n \kappa_2 + \left(1-q^n\right)\kappa_1\right) = (\kappa_2-\kappa_1)\int_0^1 \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2)\, d_q\tau.$$

Alp et al[23] proved the following q_{κ_1} -Hermite–Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2. *If $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0<q<1$. Then we have*

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \le \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\, {_{\kappa_1}d_q x} \le \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{[2]_q}. \tag{2.3}$$

In Alp et al[23] and Noor et al,[30] the authors established some bounds for the left and right hand sides of the inequality (2.3).

On the other hand, in their work,[16] Bermudo et al. gave the following definition and obtained the related Hermite–Hadamard type inequalities:

Definition 4 (Bermudo et al.[16]). The q^{κ_2}-definite integral of $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ on $[\kappa_1, \kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}^{\kappa_2}d_q x = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n \mathcal{F}\left(q^n\kappa_1+\left(1-q^n\right)\kappa_2\right) = (\kappa_2-\kappa_1)\int_0^1 \mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\ d_q\tau.$$

Theorem 3 (Bermudo et al.[16]). *If $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite–Hadamard inequalities are given as follows:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}^{\kappa_2}d_q x \leq \frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}. \tag{2.4}$$

From Theorems 2 and 3, one can obtain the following inequalities:

Corollary 1 (Bermudo et al.[16]). *For any convex function $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right) + \mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_q x + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}^{\kappa_2}d_q x\right\} \leq \mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2) \tag{2.5}$$

and

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_q x + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}^{\kappa_2}d_q x\right\} \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{2.6}$$

Budak[25] proved the left and right bounds of the inequality (2.4).

3 | CRUCIAL IDENTITIES

In this section, we prove three different identities to obtain the main results of this paper.

Let's start with the following useful Lemma.

Lemma 1. *If $\mathcal{F} : [\kappa_1, \kappa_2] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on (κ_1, κ_2) such that ${}_{\kappa_1}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$, then we have the following identity:*

$$\begin{aligned} &\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)_{\kappa_1}d_q x \\ &= (\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}(q\tau-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{2}}^{1}(q\tau-\mu)_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right] \end{aligned} \tag{3.1}$$

where $q \in (0, 1)$.

Proof. From Definition 1, we have

$${}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1) = \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}. \tag{3.2}$$

Using the fundamental properties of quantum integrals and from equality (3.2), we obtain that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} (q\tau-\lambda)\, {}_{\kappa_1}D_q\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)d_q\tau+\int_{\frac{1}{2}}^{1} (q\tau-\mu)\, {}_{\kappa_1}D_q\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)d_q\tau\\
&=\int_0^{\frac{1}{2}} (\mu-\lambda)\, {}_{\kappa_1}D_q\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)d_q\tau+\int_0^{1} (q\tau-\mu)\, {}_{\kappa_1}D_q\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)d_q\tau\\
&=(\mu-\lambda)\int_0^{\frac{1}{2}}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau\\
&\quad+q\int_0^{1}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)}d_q\tau\\
&\quad-\mu\int_0^{1}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau.
\end{aligned}
\tag{3.3}
$$

From Definition 3, we have the following relations:

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^n}{2}\kappa_2+\left(1-\frac{q^n}{2}\right)\kappa_1\right)-\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^{n+1}}{2}\kappa_2+\left(1-\frac{q^{n+1}}{2}\right)\kappa_1\right)\right]\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\mathcal{F}(\kappa_1)\right],
\end{aligned}
\tag{3.4}
$$

$$
\int_0^{1}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau=\frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)\right]
\tag{3.5}
$$

and

$$
\begin{aligned}
&\int_0^{1}\frac{\mathcal{F}\left(\tau\kappa_2+(1-\tau)\kappa_1\right)-\mathcal{F}\left(q\tau\kappa_2+(1-q\tau)\kappa_1\right)}{(1-q)(\kappa_2-\kappa_1)}d_q\tau\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^n\kappa_2+\left(1-q^n\right)\kappa_1\right)-\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^{n+1}\kappa_2+\left(1-q^{n+1}\right)\kappa_1\right)\right]\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^n\kappa_2+\left(1-q^n\right)\kappa_1\right)-\frac{1}{q}\sum_{n=1}^{\infty}q^n\mathcal{F}\left(q^n\kappa_2+\left(1-q^n\right)\kappa_1\right)\right]\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^n\kappa_2+\left(1-q^n\right)\kappa_1\right)-\frac{1}{q}\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^n\kappa_2+\left(1-q^n\right)\kappa_1\right)+\frac{1}{q}\mathcal{F}(\kappa_2)\right]\\
&=\frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\mathcal{F}(\kappa_2)-\frac{1}{q(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right].
\end{aligned}
\tag{3.6}
$$

By substituting the computed integrals (3.4)–(3.6) in (3.3), we obtain the required identity (3.1), and the proof is completed. □

Remark 1. In Lemma 1, if we choose $\lambda=\frac{1}{6}$ and $\mu=\frac{5}{6}$, then we obtain the following.[43, Lemma 3]

Remark 2. In Lemma 1, if we choose $\lambda=\mu=\frac{q}{[2]_q}$, then we obtain the following.[44, Lemma 3.1]

Corollary 2. *In Lemma 1, if we choose $\lambda = 0$ and $\mu = 1$, then we obtain the following new identity:*

$$\begin{aligned}
&\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_qx \\
&= (\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}} q\tau\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{2}}^{1} (q\tau-1)\,{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right].
\end{aligned}$$

Remark 3. In Lemma 1, if we take the limit $q \to 1^-$, then we have the following,[45, Lemma 2.1] for $m = 1$

Lemma 2. *If $\mathcal{F} : [\kappa_1,\kappa_2] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on (κ_1,κ_2) such that ${}_{\kappa_1}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$, then we have the following identity:*

$$\begin{aligned}
&\lambda\mathcal{F}(\kappa_1) + (\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right) + (\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right) + (1-\nu)\mathcal{F}(\kappa_2) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_qx \\
&= (\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{3}} (q\tau-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{3}}^{\frac{2}{3}} (q\tau-\mu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right. \\
&\quad \left. + \int_{\frac{2}{3}}^{1} (q\tau-\nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right]
\end{aligned} \tag{3.7}$$

where $q \in (0,1)$.

Proof. From the fundamental properties of quantum integrals, we have

$$\begin{aligned}
&\int_0^{\frac{1}{3}} (q\tau-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{3}}^{\frac{2}{3}} (q\tau-\mu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau \\
&\quad + \int_{\frac{2}{3}}^{1} (q\tau-\nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau \\
&= \int_0^{\frac{1}{3}} (\mu-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau + \int_0^{\frac{2}{3}} (\nu-\mu)\,{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau \\
&\quad + \int_0^{1} (q\tau-\nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau.
\end{aligned}$$

If the same steps in the proof of Lemma 1 are applied for the rest of this proof, we can obtain the desired identity (3.7). □

Remark 4. If we take $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$ in Lemma 2, then we obtain the following.[4, Lemma 2]

Remark 5. If we take $\lambda = \mu = \nu = \frac{q}{[2]_q}$, in Lemma 2, then we obtain the following.[44, Lemma 3.1]

Corollary 3. *If we take the limit $q \to 1^-$ in Lemma 2, then we obtain the following new identity:*

$$\begin{aligned}
&\lambda\mathcal{F}(\kappa_1) + (\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right) + (\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right) + (1-\nu)\mathcal{F}(\kappa_2) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\,dx \\
&= (\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{3}} (\tau-\lambda)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau + \int_{\frac{1}{3}}^{\frac{2}{3}} (\tau-\mu)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau\right. \\
&\quad \left. + \int_{\frac{2}{3}}^{1} (\tau-\nu)\mathcal{F}'(\tau\kappa_2+(1-\tau)\kappa_1)\,d\tau\right]
\end{aligned}$$

For brevity, let us prove another lemma that will be used frequently in the main results.

Lemma 3. *The following quantum integrals holds for* $\lambda,\ \mu,\ \nu \geq 0$*:*

$$\Omega_{11} = \int_0^{\frac{1}{2}} |q\tau - \lambda|\, d_q\tau = \begin{cases} \frac{8\lambda^2+q}{4[2]_q} - \frac{\lambda}{2}, & q > 2\lambda, \\ \frac{\lambda}{2} - \frac{q}{4[2]_q}, & q \leq 2\lambda, \end{cases} \tag{3.8}$$

$$\Omega_{12} = \int_{\frac{1}{2}}^{1} |q\tau - \mu|\, d_q\tau = \begin{cases} \frac{\mu}{2} - \frac{3q}{4[2]_q}, & q < \mu, \\ \frac{8\mu^2+5q}{4[2]_q} - \frac{3\mu}{2}, & \mu \leq q \leq 2\mu, \\ \frac{3q}{4[2]_q} - \frac{\mu}{2}, & q > 2\mu, \end{cases} \tag{3.9}$$

$$\Omega_{13} = \int_0^{\frac{1}{3}} |q\tau - \lambda|\, d_q\tau = \begin{cases} \frac{2\lambda^2}{[2]_q} + \frac{q}{9[2]_q} - \frac{\lambda}{3}, & q > 3\lambda, \\ \frac{\lambda}{3} - \frac{q}{9[2]_q}, & q \leq 3\lambda, \end{cases} \tag{3.10}$$

$$\Omega_{14} = \int_{\frac{1}{3}}^{\frac{2}{3}} |q\tau - \mu|\, d_q\tau = \begin{cases} \frac{\mu}{3} - \frac{q}{3[2]_q}, & q < \frac{3\mu}{2}, \\ \frac{18\mu^2+5q}{9[2]_q} - \mu, & \frac{3\mu}{2} \leq q \leq 3\mu, \\ \frac{q}{3[2]_q} - \frac{\mu}{3}, & q > 3\mu, \end{cases} \tag{3.11}$$

$$\Omega_{15} = \int_{\frac{2}{3}}^{1} |q\tau - \nu|\, d_q\tau = \begin{cases} \frac{\nu}{3} - \frac{5q}{9[2]_q}, & q < \nu, \\ \frac{18\nu^2+13q}{9[2]_q} - \frac{5\nu}{3}, & \nu \leq q \leq \frac{3\nu}{2}, \\ \frac{5q}{9[2]_q} - \frac{\nu}{3}, & q > \frac{3\nu}{2}, \end{cases} \tag{3.12}$$

$$\Omega_1 = \int_0^{\frac{1}{2}} \tau\, |q\tau - \lambda|\, d_q\tau = \begin{cases} \frac{2\lambda^3}{[2]_q[3]_q} + \frac{q}{8[3]_q} - \frac{\lambda}{4[2]_q}, & q > 2\lambda, \\ \frac{\lambda}{4[2]_q} - \frac{q}{8[3]_q}, & q \leq 2\lambda, \end{cases} \tag{3.13}$$

$$\Omega_2 = \int_0^{\frac{1}{2}} (1-\tau)\, |q\tau - \lambda|\, d_q\tau = \Omega_{11} - \Omega_1 = \begin{cases} \frac{8\lambda^2+\lambda+q}{4[2]_q} - \frac{\lambda}{2} - \frac{q}{8[3]_q} - \frac{2\lambda^3}{[2]_q[3]_q}, & q > 2\lambda, \\ \frac{\lambda}{2} - \frac{\lambda+q}{4[2]_q} + \frac{q}{8[3]_q}, & q \leq 2\lambda, \end{cases} \tag{3.14}$$

$$\Omega_3 = \int_{\frac{1}{2}}^{1} \tau\,|q\tau - \mu|\, d_q\tau = \begin{cases} \frac{3\mu}{4[2]_q} - \frac{7q}{8[3]_q}, & q < \mu, \\ \frac{2\mu^3}{[2]_q[3]_q} - \frac{5\mu}{4[2]_q} + \frac{9q}{8[3]_q}, & \mu \leq q \leq 2\mu, \\ \frac{7q}{8[3]_q} - \frac{3\mu}{4[2]_q}, & q > 2\mu, \end{cases} \tag{3.15}$$

$$\Omega_4 = \int_{\frac{1}{2}}^{1} (1-\tau)\,|q\tau - \mu|\, d_q\tau = \Omega_{12} - \Omega_3 = \begin{cases} \frac{\mu}{2} - \frac{3(\mu+q)}{4[2]_q} + \frac{7q}{8[3]_q}, & q < \mu, \\ \frac{8\mu^2+5q+5\mu}{4[2]_q} - \frac{3\mu}{2} - \frac{9q}{8[3]_q} - \frac{2\mu^3}{[2]_q[3]_q}, & \mu \leq q \leq 2\mu, \\ \frac{3(\mu+q)}{4[2]_q} - \frac{\mu}{2} - \frac{7q}{8[3]_q}, & q > 2\mu, \end{cases} \tag{3.16}$$

$$\Omega_5 = \int_{0}^{\frac{1}{3}} \tau\,|q\tau - \lambda|\, d_q\tau = \begin{cases} \frac{2\lambda^3}{[2]_q[3]_q} + \frac{q}{27[3]_q} - \frac{\lambda}{9[2]_q}, & q > 3\lambda, \\ \frac{\lambda}{9[2]_q} - \frac{q}{27[3]_q}, & q \leq 3\lambda, \end{cases} \tag{3.17}$$

$$\Omega_6 = \int_{0}^{\frac{1}{3}} (1-\tau)\,|q\tau - \lambda|\, d_q\tau = \Omega_{13} - \Omega_5 = \begin{cases} \frac{18\lambda^2+\lambda+q}{9[2]_q} - \frac{\lambda}{3} - \frac{q}{27[3]_q} - \frac{2\lambda^3}{[2]_q[3]_q}, & q > 2\lambda, \\ \frac{\lambda}{3} - \frac{\lambda+q}{9[2]_q} + \frac{q}{27[3]_q}, & q \leq 2\lambda, \end{cases} \tag{3.18}$$

$$\Omega_7 = \int_{\frac{1}{3}}^{\frac{2}{3}} \tau\,|q\tau - \mu|\, d_q\tau = \begin{cases} \frac{\mu}{3[2]_q} - \frac{7q}{27[3]_q}, & q < \frac{3\mu}{2}, \\ \frac{2\mu^3}{[2]_q[3]_q} - \frac{5\mu}{9[2]_q} + \frac{q}{3[3]_q}, & \frac{3\mu}{2} \leq q \leq 3\mu, \\ \frac{7q}{27[3]_q} - \frac{\mu}{3[2]_q}, & q > 3\mu, \end{cases} \tag{3.19}$$

$$\Omega_8 = \int_{\frac{1}{3}}^{\frac{2}{3}} (1-\tau)\,|q\tau - \mu|\, d_q\tau = \Omega_{14} - \Omega_7 = \begin{cases} \frac{\mu}{3} - \frac{q+\mu}{3[2]_q} + \frac{7q}{27[3]_q}, & q < \frac{3\mu}{2}, \\ \frac{18\mu^2+5q+5\mu}{9[2]_q} - \mu - \frac{q}{3[3]_q} - \frac{2\mu^3}{[2]_q[3]_q}, & \frac{3\mu}{2} \leq q \leq 3\mu, \\ \frac{q+\mu}{3[2]_q} - \frac{\mu}{3} - \frac{7q}{27[3]_q}, & q > 3\mu, \end{cases} \tag{3.20}$$

$$\Omega_9 = \int_{\frac{2}{3}}^{1} \tau \left| q\tau - \nu \right| d_q\tau = \begin{cases} \frac{5\nu}{9[2]_q} - \frac{19q}{27[3]_q}, & q < \nu, \\ \frac{2\nu^3}{[2]_q[3]_q} - \frac{13\nu}{9[2]_q} + \frac{35q}{27[3]_q}, & \nu \leq q \leq \frac{3\nu}{2}, \\ \frac{19q}{27[3]_q} - \frac{5\nu}{9[2]_q}, & q > \frac{3\nu}{2}, \end{cases} \tag{3.21}$$

$$\Omega_{10} = \int_{\frac{2}{3}}^{1} (1-\tau) \left| q\tau - \nu \right| d_q\tau = \Omega_{15} - \Omega_9 = \begin{cases} \frac{\nu}{3} - \frac{5(q+\nu)}{9[2]_q} + \frac{19q}{27[3]_q}, & q < \nu, \\ \frac{18\nu^2+13q+13\nu}{9[2]_q} - \frac{5\nu}{3} - \frac{35q}{27[3]_q} - \frac{2\nu^3}{[2]_q[3]_q}, & \nu \leq q \leq \frac{3\nu}{2}, \\ \frac{5(q+\nu)}{9[2]_q} - \frac{\nu}{3} - \frac{19q}{27[3]_q}, & q > \frac{3\nu}{2}. \end{cases} \tag{3.22}$$

Proof. Case I: Let $q > 2\lambda$.

By the definition q-integral, we have

$$\begin{aligned} \Omega_1 &= \int_0^{\frac{1}{2}} \tau \left| q\tau - \lambda \right| d_q\tau \\ &= \int_0^{\frac{\lambda}{q}} \tau (\lambda - q\tau)\, d_q\tau + \int_{\frac{\lambda}{q}}^{\frac{1}{2}} \tau (\lambda - q\tau)\, d_q\tau \\ &= 2\int_0^{\frac{\lambda}{q}} \tau (\lambda - q\tau)\, d_q\tau + \int_0^{\frac{1}{2}} \tau (\lambda - q\tau)\, d_q\tau \\ &= \frac{2\lambda^3}{[2]_q[3]_q} + \frac{q}{8[3]_q} - \frac{\lambda}{4[2]_q}. \end{aligned}$$

Case I: Let $q \leq 2\lambda$.

From definition quantum integral, we get

$$\Omega_1 = \int_0^{\frac{1}{2}} \tau \left| q\tau - \lambda \right| d_q\tau = \int_0^{\frac{1}{2}} \tau (\lambda - q\tau)\, d_q\tau = \frac{\lambda}{4[2]_q} - \frac{q}{8[3]_q}.$$

This gives the proof of the equality (3.13). The others can be calculated in similar way. □

4 | SIMPSON'S TYPE INEQUALITIES FOR QUANTUM INTEGRALS

In this section, we prove a new generalization of quantum Simpson's inequalities for quantum differentiable convex functions via quantum integrals.

Theorem 4. *We assume that the given conditions of Lemma 1 hold. If the mapping $\left|{}_{\kappa_1}D_q\mathcal{F}\right|$ is convex on $[\kappa_1, \kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned} &\left| \lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\, {}_{\kappa_1}d_qx \right| \\ &\leq (\kappa_2-\kappa_1)\left[(\Omega_1+\Omega_3)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right| + (\Omega_2+\Omega_4)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right| \right] \end{aligned} \tag{4.1}$$

where Ω_1–Ω_4 are given in (3.13)–(3.16), respectively.

Proof. Taking the modulus in Lemma 1 and using the convexity of $\left|{}_{\kappa_1}D_q\mathcal{F}\right|$, we obtain

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\
&\leq(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}|q\tau-\lambda|\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|d_q\tau+\int_{\frac{1}{2}}^{1}|q\tau-\mu|\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|d_q\tau\right]\\
&\leq(\kappa_2-\kappa_1)\left[\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|\left\{\int_0^{\frac{1}{2}}\tau|q\tau-\lambda|\,d_q\tau+\int_{\frac{1}{2}}^{1}\tau|q\tau-\mu|\,d_q\tau\right\}\right.\\
&\quad\left.+\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\left\{\int_0^{\frac{1}{2}}(1-\tau)|q\tau-\lambda|\,d_q\tau+\int_{\frac{1}{2}}^{1}(1-\tau)|q\tau-\mu|\,d_q\tau\right\}\right]\\
&=(\kappa_2-\kappa_1)\left[(\Omega_1+\Omega_3)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Omega_2+\Omega_4)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]
\end{aligned}$$

which completes the proof. □

Remark 6. If we take the limit $q\to1^-$ in Theorem 4, then we have the following,[45, Theorem 2.1] for $s=m=1$

Remark 7. If we assume $\lambda=\mu=\frac{q}{[2]_q}$ in Theorem 4, then we obtain the following.[44, Theorem 4.1]

Corollary 4. *In Theorem 4, if we choose $\lambda=0$ and $\mu=1$, then we obtain the following midpoint type inequality:*

$$\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\leq(\kappa_2-\kappa_1)\left[\frac{3}{4[2]_q[3]_q}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+\frac{2q^2+2q-1}{4[2]_q[3]_q}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right].$$

Corollary 5. *If we assume $\lambda=\frac{1}{6}$ and $\mu=\frac{5}{6}$ in Theorem 4, then we obtain the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\
&\leq(\kappa_2-\kappa_1)\left[\left(\Omega_1^*+\Omega_3^*\right)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+\left(\Omega_2^*+\Omega_4^*\right)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]
\end{aligned}$$

where

$$\Omega_1^*=\int_0^{\frac{1}{2}}\tau\left|q\tau-\frac{1}{6}\right|d_q\tau=\begin{cases}\frac{2}{216.[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{1}{24.[2]_q}, & \frac{1}{3}<q<1,\\ \frac{1}{24.[2]_q}-\frac{q}{8[3]_q}, & 0<q\leq\frac{1}{3},\end{cases}$$

$$\Omega_2^*=\int_0^{\frac{1}{2}}(1-\tau)\left|q\tau-\frac{1}{6}\right|d_q\tau=\begin{cases}\frac{7}{72.[2]_q}+\frac{q}{4[2]_q}-\frac{1}{12}-\frac{q}{8[3]_q}-\frac{2}{216.[2]_q[3]_q}, & \frac{1}{3}<q<1,\\ \frac{1}{12}-\frac{1}{24[2]_q}-\frac{q}{4[2]_q}+\frac{q}{8[3]_q}, & 0<q\leq\frac{1}{3},\end{cases}$$

$$\Omega_3^*=\int_{\frac{1}{2}}^{1}\tau\left|q\tau-\frac{5}{6}\right|d_q\tau=\begin{cases}\frac{15}{24.[2]_q}-\frac{7q}{8[3]_q}, & 0<q<\frac{5}{6},\\ \frac{250}{216.[2]_q[3]_q}-\frac{25}{24.[2]_q}+\frac{9q}{8[3]_q}, & \frac{5}{6}\leq q<1,\end{cases}$$

$$\Omega_4^*=\int_{\frac{1}{2}}^{1}(1-\tau)\left|q\tau-\frac{5}{6}\right|d_q\tau=\begin{cases}\frac{5}{12}-\frac{15}{24.[2]_q}-\frac{3q}{4[2]_q}+\frac{7q}{8[3]_q}, & 0<q<\frac{5}{6},\\ \frac{50}{216.[2]_q}+\frac{5q}{4[2]_q}-\frac{25}{24.[2]_q}-\frac{15}{12}-\frac{9q}{8[3]_q}-\frac{250}{216.[2]_q[3]_q}, & \frac{5}{6}\leq q<1\end{cases}$$

which is given by Tunç et al.[43, Theorem 1] *the coefficients of* $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|$ *and* $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|$ *in this inequality are more modified than the inequality of Tunç et al.*

Theorem 5. *We assume that the given conditions of Lemma 1 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1 \geq 1$ *is convex on* $[\kappa_1, \kappa_2]$, *then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right| \\
&\leq (\kappa_2-\kappa_1)\left[\Omega_{11}^{1-\frac{1}{p_1}}\left(\Omega_1\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1} + \Omega_2\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\quad \left. + \Omega_{12}^{1-\frac{1}{p_1}}\left(\Omega_3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1} + \Omega_4\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned} \tag{4.2}$$

where Ω_{11}, Ω_{12} *and* Ω_1–Ω_4 *are given in (3.8), (3.9), and (3.13)–(3.16), respectively.*

Proof. Taking the modulus in Lemma 1 and using the power mean inequality, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right| \\
&\leq (\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By using the convexity of $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right| \\
&\leq (\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\,d_q\tau\right)^{1-\frac{1}{p_1}}\right. \\
&\quad \times\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{\frac{1}{2}}\tau\,|q\tau-\lambda|\,d_q\tau + \left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)\,|q\tau-\lambda|\,d_q\tau\right)^{\frac{1}{p_1}} \\
&\quad + \left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|\,d_q\tau\right)^{1-\frac{1}{p_1}} \\
&\quad \left.\times\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{\frac{1}{2}}^{1}\tau\,|q\tau-\mu|\,d_q\tau + \left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{\frac{1}{2}}^{1}(1-\tau)\,|q\tau-\mu|\,d_q\tau\right)^{\frac{1}{p_1}}\right] \\
&= \left[\Omega_{11}^{1-\frac{1}{p_1}}\left(\Omega_1\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1} + \Omega_2\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\quad \left. + \Omega_{12}^{1-\frac{1}{p_1}}\left(\Omega_3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1} + \Omega_4\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

and the proof is completed. ☐

Remark 8. If we take the limit $q \to 1^-$ in Theorem 5, then we have the following,[45, Theorem 2.3] for $s = m = 1$

Remark 9. If we assume $\lambda = \mu = \frac{q}{[2]_q}$ in Theorem 5, then we obtain the following.[44, Theorem 4.2]

Corollary 6. *If we assume $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$ in Theorem 5, then we obtain the following inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\ &\leq(\kappa_2-\kappa_1)\left[\Theta_1^{1-\frac{1}{p_1}}\left(\Omega_1^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_2^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\ &\left.+\Theta_2^{1-\frac{1}{p_1}}\left(\Omega_3^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_4^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]\end{aligned}$$

where $\Omega_1^* - \Omega_4^*$ are given in Corollary 5 and

$$\Theta_1=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{6}\right|d_q\tau=\begin{cases}\frac{2}{36[2]_q}+\frac{q}{4[2]_q}-\frac{1}{12}, & \frac{1}{3}<q<1,\\ \frac{1}{12}-\frac{q}{4[2]_q}, & 0<q\leq\frac{1}{3},\end{cases}$$

$$\Theta_2=\int_{\frac{1}{2}}^{1}\left|q\tau-\frac{5}{6}\right|d_q\tau=\begin{cases}\frac{5}{12}-\frac{3q}{4[2]_q}, & 0<q<\frac{5}{6},\\ \frac{50}{36[2]_q}+\frac{5q}{4[2]_q}-\frac{15}{12}, & \frac{5}{6}\leq q<1,\end{cases}$$

which is given by Tunç et al;[43, Theorem 3] the values of Θ_1, Θ_2 and the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}$ and $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}$ in this inequality are more modified than the inequality of Tunç et al.

Corollary 7. *In Theorem 5, if we choose $\lambda = 0$ and $\mu = 1$, then we obtain the following midpoint type inequality:*

$$\begin{aligned}&\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\leq(\kappa_2-\kappa_1)\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q}{8[3]_q}+\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q\left([3]_q+q^2\right)}{8[2]_q[3]_q}\right)^{\frac{1}{p_1}}\\ &+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}+\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\left(\frac{1}{2}-\frac{3q}{4[2]_q}-\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}\right)\right)^{\frac{1}{p_1}}.\end{aligned}$$

Theorem 6. *We assume that the given conditions of Lemma 1 hold. If the mapping $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1 > 1$ is convex on $[\kappa_1, \kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\ &\leq(\kappa_2-\kappa_1)\left[\Omega_{16}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}+\Omega_{17}^{\frac{1}{r_1}}\left(\frac{3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]\end{aligned} \tag{4.3}$$

where $p_1^{-1} + r_1^{-1} = 1$ and

$$\Omega_{16}=\int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}d_q\tau,\ \Omega_{17}=\int_{\frac{1}{2}}^{1}|q\tau-\mu|^{r_1}d_q\tau.$$

Proof. Taking the modulus in Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\
&\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{2}}\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{2}}^{1}\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Applying the convexity of $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\
&\leq(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{\frac{1}{2}}\tau d_q\tau+\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)\,d_q\tau\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{\frac{1}{2}}^{1}\tau d_q\tau+\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{\frac{1}{2}}^{1}(1-\tau)\,d_q\tau\right)^{\frac{1}{p_1}}\right]\\
&=(\kappa_2-\kappa_1)\left[\Omega_{16}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\Omega_{17}^{\frac{1}{r_1}}\left(\frac{3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right],
\end{aligned}$$

and the proof is finished. ☐

Remark 10. If we take the limit $q\to1^-$ in Theorem 6, then Theorem 6 becomes the following,[45, Theorem 2.2] for $s=m=1$

Corollary 8. *If we assume* $\lambda=\frac{1}{6}$ *and* $\mu=\frac{5}{6}$ *in Theorem 6, then we obtain the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\
&\leq(\kappa_2-\kappa_1)\left[\Theta_3^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q+1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\Theta_4^{\frac{1}{r_1}}\left(\frac{3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q}+\frac{(2q-1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where

$$\Theta_3=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{6}\right|^{r_1}d_q\tau,\ \Theta_4=\int_{\frac{1}{2}}^{1}\left|q\tau-\frac{5}{6}\right|^{r_1}d_q\tau.$$

5 | NEWTON'S TYPE INEQUALITIES FOR QUANTUM INTEGRALS

Some new generalized versions of quantum Newton's inequalities for quantum differentiable convex functions are offered in this section.

Theorem 7. *We assume that the given conditions of Lemma 2 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|$ *is convex on* $[\kappa_1, \kappa_2]$*, then the following Newton's type inequality holds:*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\&\leq(\kappa_2-\kappa_1)\left[(\Omega_5+\Omega_7+\Omega_9)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Omega_6+\Omega_8+\Omega_{10})\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned} \tag{5.1}$$

where Ω_5*–*Ω_{10} *are given in (3.17)–(3.22), respectively.*

Proof. If we consider Lemma 2 and apply the same method that used in the proof of Theorem 4, then we can obtain the desired inequality (5.1). ☐

Remark 11. If we assume $\lambda=\mu=\nu=\frac{q}{[2]_q}$ in Theorem 7, then we obtain the following.[44, Theorem 4.1]

Corollary 9. *If we take the limit* $q\to1^-$ *in Theorem 7, then we obtain the following Newton's type inequality:*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right|\\&\leq(\kappa_2-\kappa_1)\left[\left(\Omega_5^*+\Omega_7^*+\Omega_9^*\right)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+\left(\Omega_6^*+\Omega_8^*+\Omega_{10}^*\right)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned}$$

where

$$\Omega_5^*=\int_0^{\frac{1}{3}}\tau\left|\tau-\lambda\right|d\tau=\frac{\lambda^3}{3}+\frac{1}{81}-\frac{\lambda}{18},$$

$$\Omega_6^*=\int_0^{\frac{1}{3}}(1-\tau)\left|\tau-\lambda\right|d\tau=\frac{18\lambda^2+\lambda+1}{18}-\frac{28}{81}-\frac{\lambda^3}{3},$$

$$\Omega_7^*=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\mu\right|d\tau=\frac{\mu^3}{3}-\frac{5\mu}{18}+\frac{1}{9}$$

$$\Omega_8^*=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|\tau-\mu\right|d\tau=\frac{18\mu^2+5+5\mu}{18}-\mu-\frac{1}{9}-\frac{\mu^3}{3},$$

$$\Omega_9^*=\int_{\frac{2}{3}}^{1}\tau\left|\tau-\nu\right|d\tau=\frac{\nu^3}{3}-\frac{13\nu}{18}+\frac{35}{81},$$

$$\Omega_{10}^*=\int_{\frac{2}{3}}^{1}(1-\tau)\left|\tau-\nu\right|d\tau=\frac{18\nu^2+13+13\nu}{18}-\frac{5\nu}{3}-\frac{35}{81}-\frac{\nu^3}{3}.$$

Remark 12. If we take $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$ in Theorem 7, then we obtain the following inequality:

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\&\leq(\kappa_2-\kappa_1)\left[(\Theta_5+\Theta_7+\Theta_9)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Theta_6+\Theta_8+\Theta_{10})\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned}$$

where

$$\Theta_5=\int_0^{\frac{1}{3}}\tau\left|q\tau-\frac{1}{8}\right|d_q\tau=\begin{cases}\frac{1}{256.[2]_q[3]_q}+\frac{q}{27[3]_q}-\frac{1}{72[2]_q}, & \frac{3}{8}<q<1,\\ \frac{1}{72[2]_q}-\frac{q}{27[3]_q}, & 0<q\leq\frac{3}{8},\end{cases}$$

$$\Theta_6=\int_0^{\frac{1}{3}}(1-\tau)\left|q\tau-\frac{1}{8}\right|d_q\tau=\begin{cases}\frac{1}{32[2]_q}+\frac{1}{72[2]_q}+\frac{q}{9[2]_q}-\frac{1}{24}-\frac{q}{27[3]_q}-\frac{1}{256[2]_q[3]_q}, & \frac{1}{4}<q<1,\\ \frac{1}{24}-\frac{1}{72[2]_q}-\frac{q}{9[2]_q}+\frac{q}{27[3]_q}, & 0<q\le\frac{1}{4},\end{cases}$$

$$\Theta_7=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{1}{6[2]_q}-\frac{7q}{27[3]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{4[2]_q[3]_q}-\frac{5}{18[2]_q}+\frac{q}{3[3]_q}, & \frac{3}{4}\le q<1,\end{cases}$$

$$\Theta_8=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{1}{6}-\frac{q}{3[2]_q}-\frac{1}{6[2]_q}+\frac{7q}{27[3]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{2[2]_q}+\frac{5q}{9[2]_q}+\frac{5}{18[2]_q}-\frac{1}{2}-\frac{q}{3[3]_q}-\frac{1}{4[2]_q[3]_q}, & \frac{3}{5}\le q<1,\end{cases}$$

$$\Theta_9=\int_{\frac{2}{3}}^{1}\tau\left|q\tau-\frac{7}{8}\right|d_q\tau=\begin{cases}\frac{35}{72[2]_q}-\frac{19q}{27[3]_q}, & 0<q<\frac{7}{8},\\ \frac{343}{256[2]_q[3]_q}-\frac{91}{72[2]_q}+\frac{35q}{27[3]_q}, & \frac{7}{8}\le q<1,\end{cases}$$

$$\Theta_{10}=\int_{\frac{2}{3}}^{1}(1-\tau)\left|q\tau-\frac{7}{8}\right|d_q\tau=\begin{cases}\frac{7}{24}-\frac{5q}{9[2]_q}-\frac{7}{72[2]_q}+\frac{19q}{27[3]_q}, & 0<q<\frac{7}{8},\\ \frac{49}{32[2]_q}+\frac{13q}{9[2]_q}+\frac{91}{72[2]_q}-\frac{35}{24}-\frac{35q}{27[3]_q}-\frac{343}{256[2]_q[3]_q}, & \frac{7}{8}\le q<1\end{cases}$$

which is given by Erden et al.,[4, Theorem 1] but in our inequality, the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|$ and $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|$ are in modified form.

Theorem 8. *We assume that the given conditions of Lemma 2 hold. If the mapping $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1\geq 1$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|\\&\le(\kappa_2-\kappa_1)\left[\Omega_{13}^{1-\frac{1}{p_1}}\left(\Omega_5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_6\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\quad+\Omega_{14}^{1-\frac{1}{p_1}}\left(\left(\Omega_7\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_8\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right)\\&\quad\left.+\Omega_{15}^{1-\frac{1}{p_1}}\left(\Omega_9\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_{10}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]\end{aligned}\tag{5.2}$$

where Ω_5–Ω_{10} and Ω_{13}–Ω_{15} are given in (3.17)–(3.22) and (3.10)–(3.12), respectively.

Proof. If we apply the steps used in the proof of Theorem 5 and taking into account Lemma 2, we can obtain the required inequality (5.2). □

Corollary 10. *If we take the limit $q \to 1^-$ in Theorem 8, then we obtain the following Newton's type inequality:*

$$\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right|$$
$$\leq(\kappa_2-\kappa_1)\left[\Theta_{11}^{1-\frac{1}{p_1}}\left(\Omega_5^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_6^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$+\Theta_{12}^{1-\frac{1}{p_1}}\left(\left(\Omega_7^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_8^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right)$$
$$\left.+\Theta_{13}^{1-\frac{1}{p_1}}\left(\Omega_9^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_{10}^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]$$

where Ω_5^*–Ω_{10}^* are defined in Corollary 9 and

$$\Theta_{11}=\int_0^{\frac{1}{3}}|\tau-\lambda|\,d\tau=\lambda^2+\frac{1}{9[2]_q}-\frac{\lambda}{3},$$

$$\Theta_{12}=\int_{\frac{1}{3}}^{\frac{2}{3}}|\tau-\mu|\,d\tau=\frac{18\mu^2+5}{18}-\mu,$$
$$\Theta_{13}=\int_{\frac{2}{3}}^{1}|\tau-\nu|\,d\tau=\frac{18\nu^2+13}{18}-\frac{5\nu}{3}.$$

Remark 13. If we take $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$ in Theorem 8, then we obtain the following inequality:

$$\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right|$$
$$\leq(\kappa_2-\kappa_1)\left[\Theta_{14}^{1-\frac{1}{p_1}}\left(\Theta_5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_6\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.$$
$$+\Theta_{15}^{1-\frac{1}{p_1}}\left(\left(\Theta_7\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_8\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right)$$
$$\left.+\Theta_{16}^{1-\frac{1}{p_1}}\left(\Theta_9\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_{10}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]$$

where Θ_5–Θ_{10} are given in Remark 12 and

$$\Omega_{14}=\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|d_q\tau=\begin{cases}\frac{1}{32[2]_q}+\frac{q}{9[2]_q}-\frac{1}{24}, & \frac{3}{8}<q<1,\\ \frac{1}{24}-\frac{q}{9[2]_q}, & 0<q\leq\frac{3}{8},\end{cases}$$

$$\Omega_{15}=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{1}{6}-\frac{q}{3[2]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{2[2]_q}+\frac{5q}{9[2]_q}-\frac{1}{2}, & \frac{3}{4}\leq q<1,\end{cases}$$

$$\Omega_{16}=\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|d_q\tau=\begin{cases}\frac{7}{24}-\frac{5q}{9[2]_q}, & 0<q<\frac{7}{8},\\ \frac{49}{4[2]_q}+\frac{13q}{9[2]_q}-\frac{35}{24}, & \frac{7}{8}\leq q<1\end{cases}$$

which is given by Erden et al.,[4, Theorem 4] but the values of Θ_{14}–Θ_{15} and the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}$, $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}$ are in more modified form.

Remark 14. If we assume $\lambda = \mu = \nu = \frac{q}{[2]_q}$ in Theorem 8, then we obtain the following.[44, Theorem 4.2]

Theorem 9. *We assume that the given conditions of Lemma 2 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}, p_1 > 1$ *is convex on* $[\kappa_1, \kappa_2]$*, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1) + (\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right) + (\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right) + (1-\nu)\mathcal{F}(\kappa_2) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right| \\
&\leq (\kappa_2-\kappa_1)\left[\Omega_{18}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q+2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left. + \Omega_{19}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{3[2]_q} + \frac{q\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}} + \Omega_{20}^{\frac{1}{r_1}}\left(\frac{5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q-2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned} \tag{5.3}$$

where $p_1^{-1} + r_1^{-1} = 1$ *and*

$$\Omega_{18} = \int_0^{\frac{1}{3}} |q\tau - \lambda|^{r_1} d_q\tau,\ \Omega_{19} = \int_{\frac{1}{3}}^{\frac{2}{3}} |q\tau - \mu|^{r_1} d_q\tau,\ \Omega_{20} = \int_{\frac{2}{3}}^{1} |q\tau - \nu|^{r_1} d_q\tau.$$

Proof. If we apply the steps used in the proof of Theorem 6 and taking into account Lemma 2, we can obtain the required inequality (5.3). ☐

Remark 15. If we take $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$ in Theorem 9, then we obtain the following inequality:

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1) + 3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right) + 3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right) + \mathcal{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,{}_{\kappa_1}d_qx\right| \\
&\leq (\kappa_2-\kappa_1)\left[\Theta_{17}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q+2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+ \Theta_{18}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{3[2]_q} + \frac{q\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}} \\
&\left. + \Theta_{19}^{\frac{1}{r_1}}\left(\frac{5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q-2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where

$$\Theta_{17} = \int_0^{\frac{1}{3}} \left|q\tau - \frac{1}{8}\right|^{r_1} d_q\tau,\ \Theta_{18} = \int_{\frac{1}{3}}^{\frac{2}{3}} \left|q\tau - \frac{1}{2}\right|^{r_1} d_q\tau,\ \Theta_{19} = \int_{\frac{2}{3}}^{1} \left|q\tau - \frac{7}{8}\right|^{r_1} d_q\tau$$

which is given by Erden et al.,[4, Theorem 2] but the values of Θ_{17}–Θ_{19} are in more modified form.

6 | CONCLUSIONS

We conclude our work by mentioning that here, we gave the extension of quantum Simpson's and quantum Newton's inequalities for quantum differentiable convex functions under certain parameters in the setting of quantum calculus. It is important to mention that our results transformed into some new and known results by considering the limit $q \to 1^-$ and by different variations of the involved parameters in our main results. We strongly believe that it is an interesting and new problem for the upcoming researchers who can obtain similar inequalities for other kinds of convexity and quantum integrals.

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AUTHOR CONTRIBUTIONS

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

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