Research Article

Muhammad Aamir Ali, Hüseyin Budak, Abdullah Akkurt, and Yu-Ming Chu*

Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus

https://doi.org/10.1515/math-2021-0020
received December 27, 2020; accepted January 29, 2021

Abstract: In this paper, we first prove an identity for twice quantum differentiable functions. Then, by utilizing the convexity of $|{}^{b}D_q^2 f|$ and $|{}_{a}D_q^2 f|$, we establish some quantum Ostrowski inequalities for twice quantum differentiable mappings involving q_a and q^b-quantum integrals. The results presented here are the generalization of already published ones.

Keywords: Ostrowski inequality, q-integral, quantum calculus, convex function

MSC 2020: 26-xx

1 Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested in both pure and applied mathematics. One of the many fundamental mathematical discoveries of Ostrowski [1] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information, and integral operator theory. During the past few years, a number of scientists have

*** Corresponding author: Yu-Ming Chu,** Department of Mathematics, Huzhou University, 313000, Huzhou, China, e-mail: chuyuming2005@126.com
Muhammad Aamir Ali: Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China, e-mail: mahr.muhammad.aamir@gmail.com
Hüseyin Budak: Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, e-mail: hsyn.budak@gmail.com
Abdullah Akkurt: Department of Mathematics, Faculty of Science and Arts, Kahramanmaraş Sütçü İmam University, Kahramanmaraş, Turkey, e-mail: abdullahmat@gmail.com

focused on Ostrowski-type inequalities for function of bounded variation, see, for example, [2–11], [12, pp. 468–484]. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

On the other hand, many studies have recently been carried out in the field of q-analysis, starting with Euler due to a high demand for mathematics that models quantum computing q-calculus appeared as a connection between mathematics and physics. It has several applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other sciences, quantum theory, mechanics, and the theory of relativity [13–16]. Apparently, Euler was the founder of this branch of mathematics, by using the parameter q in Newton's work on infinite series. Later, Jackson was the first to develop q-calculus that knows without limit calculus in a systematic way [13]. In 1908–1909, Jackson defined the general q-integral and q-difference operator [15]. In 1969, Agarwal described the q-fractional derivative for the first time [17]. In 1966–1967, Al-Salam introduced a q-analogue of the Riemann-Liouville fractional integral operator and the q-fractional integral operator [18]. In 2004, Rajkovic et al. [19] gave a definition of the Riemann-type q-integral that generalized the Jackson q-integral. In 2013, Tariboon and Ntouyas introduced the ${}_aD_q$-difference operator [20]. Recently, in 2020, Bermudo et al. introduced the notion of ${}^bD_q$ derivative and integral [21].

Many well-known integral inequalities such as Hölder inequality, Hermite-Hadamard inequalities, Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz inequality, Gruss inequality, Gruss-Cebyšev inequality, and other integral inequalities have been studied in the setup of q-calculus using the concept of classical convexity. For more results in this direction, the readers may refer to [8,13,14,22–38].

2 Preliminaries of q-calculus and some inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation (see [16]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0,1).$$

Jackson [15] defined the q-Jackson integral of a given function f from 0 to b as follows:

$$\int_0^b f(x)\,d_qx = (1-q)b\sum_{n=0}^{\infty} q^n f(bq^n), \quad \text{where } 0 < q < 1 \tag{2.1}$$

provided that the sum converges absolutely.

Jackson [15] defined the q-Jackson integral of a given function over the interval $[a,b]$ as follows:

$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx.$$

Definition 2.1. [20] Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The q_a-derivative of f at $x \in [a,b]$ is identified by the following expression:

$${}_aD_qf(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{2.2}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, we can define

$${}_aD_qf(a) = \lim_{x\to a} {}_aD_qf(x).$$

The function f is said to be q_a-differentiable on $[a, b]$ if ${}_aD_qf(x)$ exists for all $x \in [a, b]$. If we take $a = 0$ in (2.2), then we have ${}_0D_qf(x) = D_qf(x)$, where $D_qf(x)$ is a known q-derivative of f at $x \in [a, b]$ (see [16]) given by

$$D_qf(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0.$$

Definition 2.2. [21] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The q^b-derivative of f at $x \in [a, b]$ is given by

$${}^bD_qf(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

Definition 2.3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The second q^b-derivative of f at $x \in [a, b]$ is given by

$$\begin{aligned} {}^bD_q^2f(ta + (1 - t)b) &= {}^bD_q({}^bD_qf(ta + (1 - t)b)) \\ &= \frac{f(q^2ta + (1 - tq^2)b) - (1 + q)f(qta + (1 - qt)b) + qf(ta + (1 - t)b)}{(1 - q)^2q(b - a)^2t^2}. \end{aligned}$$

Definition 2.4. [20] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, the q_a-definite integral on $[a, b]$ is defined by

$$\int_a^b f(x)\,{}_ad_qx = (1 - q)(b - a)\sum_{n=0}^{\infty} q^nf(q^nb + (1 - q^n)a) = (b - a)\int_0^1 f((1 - t)a + tb)d_qt.$$

Alp et al. [26] proved the following q_a-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2.5. *If $f : [a, b] \to \mathbb{R}$ is a convex differentiable function on $[a, b]$ and $0 < q < 1$, then we have*

$$f\left(\frac{qa + b}{[2]_q}\right) \leq \frac{1}{b - a}\int_a^b f(x)\,{}_ad_qx \leq \frac{qf(a) + f(b)}{[2]_q}. \tag{2.3}$$

In [26] and [39], the authors established some bounds for the left- and right-hand sides of the inequality (2.3).

On the other hand, in [21], Bermudo et al. gave the following definition and obtained the related Hermite-Hadamard-type inequalities:

Definition 2.6. [21] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, the q^b-definite integral on $[a, b]$ is given by

$$\int_a^b f(x)\,{}^bd_qx = (1 - q)(b - a)\sum_{n=0}^{\infty} q^nf(q^na + (1 - q^n)b) = (b - a)\int_0^1 f(ta + (1 - t)b)d_qt.$$

Theorem 2.7. [21] *If $f : [a, b] \to \mathbb{R}$ is a convex differentiable function on $[a, b]$ and $0 < q < 1$, then, q-Hermite-Hadamard inequalities are given as follows*:

$$f\left(\frac{a + qb}{[2]_q}\right) \leq \frac{1}{b - a}\int_a^b f(x)\,{}^bd_qx \leq \frac{f(a) + qf(b)}{[2]_q}. \tag{2.4}$$

From Theorems 2.5 and 2.7, one can obtain the following inequalities:

Corollary 2.8. [21] *For any convex function $f : [a, b] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$f\left(\frac{qa+b}{[2]_q}\right) + f\left(\frac{a+qb}{[2]_q}\right) \le \frac{1}{b-a}\left\{\int_a^b f(x)\,{}_a d_q x + \int_a^b f(x)\,{}^b d_q x\right\} \le f(a) + f(b) \tag{2.5}$$

and

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left\{\int_a^b f(x)\,{}_a d_q x + \int_a^b f(x)\,{}^b d_q x\right\} \le \frac{f(a)+f(b)}{2}. \tag{2.6}$$

Theorem 2.9. (Hölder's inequality, [40, p. 604]) *Suppose that $x > 0$, $0 < q < 1$, $p_1 > 1$. If $\frac{1}{p_1} + \frac{1}{r_1} = 1$, then*

$$\int_0^x |f(x)g(x)|d_q x \le \left(\int_0^x |f(x)|^{p_1} d_q x\right)^{\frac{1}{p_1}} \left(\int_0^x |g(x)|^{r_1} d_q x\right)^{\frac{1}{r_1}}.$$

In [34], Noor et al. proved the following lemma and related quantum Ostrowski inequality.

Lemma 2.10. *If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a q-differentiable function on (a, b) such that ${}_aD_q f$ is continuous and integrable on $[a, b]$, then we have:*

$$f(x) - \frac{1}{b-a}\int_a^x f(t)\,{}_a d_q t = q\frac{(x-a)^2}{b-a}\int_0^1 t\,{}_aD_q f(tx + (1-t)a)\,d_q t + q\frac{(b-x)^2}{b-a}\int_0^1 t\,{}_aD_q f(tx + (1-t)b)\,d_q t,$$

where $0 < q < 1$.

Theorem 2.11. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on (a, b), such that ${}_aD_q f$ is continuous and integrable on $[a, b]$. If $|{}_aD_q f|$ is convex on $[a, b]$ and $|{}_aD_q f| \le M$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^x f(t)\,{}_a d_q t \right| \le \frac{qM[(x-a)^2 + (b-x)^2]}{(b-a)(1+q)},$$

where $0 < q < 1$.

In this paper, we establish some quantum Ostrowski-type inequalities for twice q-differentiable functions.

3 New Ostrowski-type inequalities for quantum integrals

In this section, we prove Ostrowski-type inequalities for twice quantum differentiable functions involving the quantum integrals.

Let's start with the following useful lemma.

Lemma 3.1. *If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a twice q-differentiable function on (a, b), such that ${}^bD_q^2 f$ and ${}_aD_q^2 f$ are continuous and integrable on $[a, b]$, then we have:*

$$(x-a)^2(b-x)^2\left[(a-x)\int_0^1 t^2\,{}_aD_q^2 f(tx + (1-t)a)\,d_q t + (x-b)\int_0^1 t^2\,{}^bD_q^2 f(tx + (1-t)b)\,d_q t\right] = {}_a^bL_q(x), \tag{3.1}$$

where

$$
\begin{aligned}
{}_a^bL_q(x) &= \frac{(x-a)(b-x)}{(1-q)q^3}[(x-a)qf(qx+(1-q)b)+(b-x)qf(qx+(1-q)a)-(q^2+q-1)(b-a)f(x)] \\
&\quad - \frac{[2]_q}{q^3}\left[(x-a)^2\int_x^b f(t)\,{}^bd_qt+(b-x)^2\int_a^x f(t)\,{}_ad_qt\right]
\end{aligned}
$$

and $0<q<1$.

Proof. From Definition 2.2, we have

$$
\begin{aligned}
&{}^bD_q^2f(ta+(1-t)b) \\
&= {}^bD_q({}^bD_q(f(ta+(1-t)b))) \\
&= {}^bD_q\left(\frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)(b-a)t}\right) \\
&= \frac{1}{(1-q)(b-a)t}\left[\frac{f(q^2ta+(1-tq^2)b)-f(qta+(1-qt)b)}{(1-q)q(b-a)t}\right. \\
&\quad\left.-\frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)(b-a)t}\right] \\
&= \frac{f(q^2ta+(1-tq^2)b)-f(qta+(1-qt)b)}{(1-q)^2q(b-a)^2t^2}-\frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)^2(b-a)^2t^2} \\
&= \frac{f(q^2ta+(1-tq^2)b)-(1+q)f(qta+(1-qt)b)+qf(ta+(1-t)b)}{(1-q)^2q(b-a)^2t^2}.
\end{aligned} \tag{3.2}
$$

Applying the notions of Definition 2.6, we obtain

$$
\begin{aligned}
&\int_0^1 t^2\,{}^bD_q^2f(tx+(1-t)b)\,d_qt \\
&= \int_0^1 \frac{f(q^2tx+(1-tq^2)b)-(1+q)f(qtx+(1-qt)b)+qf(tx+(1-t)b)}{(1-q)^2q(b-x)^2}d_qt \\
&= (1-q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+2}f(q^{n+2}x+(1-q^{n+2})b)}{(1-q)^2q^3(b-x)^3} \\
&\quad -(1-q)(1+q)(b-x)\sum_{n=0}^{\infty}\frac{q^{n+1}f(q^{n+1}x+(1-q^{n+1})b)}{(1-q)^2q^2(b-x)^3} + q(1-q)(b-x)\sum_{n=0}^{\infty}\frac{q^nf(q^nx+(1-q^n)b)}{(1-q)^2q(b-x)^3} \\
&= \frac{1}{(1-q)^2q^3(b-x)^3}\left(\int_x^b f(t)\,{}^bd_qt-(1-q)(b-x)f(x)-(1-q)(b-x)qf(qx+(1-q)b)\right) \\
&\quad -\frac{[2]_q}{(1-q)^2q^2(b-x)^3}\left(\int_x^b f(t)\,{}^bd_qt-(1-q)(1+q)(b-x)f(x)\right)+\frac{1}{(1-q)^2(b-x)^3}\int_x^b f(x)\,{}^bd_qx \\
&= \frac{1+q}{(b-x)^3q^3}\int_x^b f(t)\,{}^bd_qt+\frac{q^2+q-1}{(1-q)q^3(b-x)^2}f(x)-\frac{f(qx+(1-q)b)}{(1-q)q^2(b-x)^2}.
\end{aligned} \tag{3.3}
$$

Similarly, from Definitions 2.1 and 2.4, we have

$$
\int_0^1 t^2\,{}_aD_q^2f(tx+(1-t)a)\,d_qt = \frac{[2]_q}{(x-a)^3q^3}\int_a^x f(t)\,{}_ad_qt+\frac{q^2+q-1}{(1-q)q^3(x-a)^2}f(x)-\frac{f(qx+(1-q)a)}{(1-q)q^2(x-a)^2}. \tag{3.4}
$$

By multiplying the equalities (3.3) and (3.4) by $(b-x)^3(x-a)^2$ and $(b-x)^2(x-a)^3$, respectively, and adding the resultant equalities we get the required identity (3.1). Thus, the proof is completed. □

Remark 3.2. If we take the limit $q \to 1^-$ in Lemma 3.1, then we obtain [41, Lemma 2.1 for $\alpha = 1$].

Theorem 3.3. *If the assumptions of* Lemma 3.1 *hold, then we have the following inequality provided that* $|{}^bD_q^2 f|$ *and* $|{}_aD_q^2 f|$ *are convex on* $[a, b]$

$$\begin{aligned} |{}_a^bL_q(x)| \le (x-a)^2(b-x)^2\Bigg[&(x-a)\left(\frac{1}{[4]_q}|{}_aD_q^2f(x)| + \frac{q^3}{[3]_q[4]_q}|{}_aD_q^2f(a)|\right) \\ &+ (b-x)\left(\frac{1}{[4]_q}|{}^bD_q^2f(x)| + \frac{q^3}{[3]_q[4]_q}|{}^bD_q^2f(b)|\right)\Bigg], \end{aligned} \tag{3.5}$$

where $0 < q < 1$.

Proof. On taking the modulus in Lemma 3.1, applying the convexity of $|{}^bD_q^2 f|$ and $|{}_aD_q^2 f|$, we obtain that

$$\begin{aligned} |{}_a^bL_q(x)| &\le (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2|{}_aD_q^2f(tx+(1-t)a)|d_qt + (b-x)\int_0^1 t^2|{}^bD_q^2f(tx+(1-t)b)|d_qt\right] \\ &\le (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2(t|{}_aD_q^2f(x)| + (1-t)|{}_aD_q^2f(a)|)d_qt\right] \\ &\quad + (x-a)^2(b-x)^2\left[(b-x)\int_0^1 t^2(t|{}^bD_q^2f(x)| + (1-t)|{}^bD_q^2f(b)|)d_qt\right] \\ &= (x-a)^2(b-x)^2\Bigg[(x-a)\left(\frac{1}{[4]_q}|{}_aD_q^2f(x)| + \frac{q^3}{[3]_q[4]_q}|{}_aD_q^2f(a)|\right) \\ &\quad + (b-x)\left(\frac{1}{[4]_q}|{}^bD_q^2f(x)| + \frac{q^3}{[3]_q[4]_q}|{}^bD_q^2f(b)|\right)\Bigg], \end{aligned}$$

which completes the proof. □

Corollary 3.4. (Quantum Ostrowski inequality) *In* Theorem 3.3, *if we set* $|{}^bD_q^2 f|, |{}_aD_q^2 f| \le M$, *then we obtain the following quantum Ostrowski-type inequality for twice quantum differentiable functions*:

$$|{}_a^bL_q(x)| \le \frac{M(x-a)^2(b-x)^2(b-a)}{[3]_q}. \tag{3.6}$$

Remark 3.5. In Corollary 3.4, if we take the limit $q \to 1^-$, then we obtain [41, Theorem 2.1 for $\alpha = 1$].

Remark 3.6. In Corollary 3.4, if we assume the limit $q \to 1^-$ and $x = \frac{a+b}{2}$, then we obtain the following inequality:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \frac{M(b-a)^2}{24},$$

which is offered by Meftah et al. in [41, Corollary 2.1] and it can be found in $\left[42,\ \text{Theorem 2.2 for } x = \frac{a+b}{2}\right]$.

Theorem 3.7. *We assume that the conditions of* Lemma 3.1 *hold. If the mappings* $|^{b}D_q^2 f|^{p_1}$ *and* $|_{a}D_q^2|^{p_1}$ $(p_1 \geq 1)$ *are convex, then, the following inequality holds*:

$$\begin{aligned}|{}_a^bL_q(x)| \leq (x-a)^2(b-x)^2\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{p_1}}\Bigg[(x-a)\left(\frac{1}{[4]_q}|{}_aD_q^2f(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}|{}_aD_q^2f(a)|^{p_1}\right)^{\frac{1}{p_1}}\\ +(b-x)\left(\frac{1}{[4]_q}|{}^bD_q^2f(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}|{}^bD_q^2f(b)|^{p_1}\right)^{\frac{1}{p_1}}\Bigg],\end{aligned}\tag{3.7}$$

where $0 < q < 1$.

Proof. Taking the modulus in Lemma 3.1 and applying the well-known power mean inequality, we have

$$\begin{aligned}|{}_a^bL_q(x)| &\leq (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2|{}_aD_q^2f(tx+(1-t)a)|d_qt+(b-x)\int_0^1 t^2|{}^bD_q^2f(tx+(1-t)b)|d_qt\right]\\ &\leq (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^2d_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t^2\,|{}_aD_q^2f(tx+(1-t)a)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right]\\ &\quad+(x-a)^2(b-x)^2\left[(b-x)\left(\int_0^1 t^2d_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t^2\,|{}^bD_q^2f(tx+(1-t)b)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

By the convexity of $|^{b}D_q^2 f|^{p_1}$ and $|_{a}D_q^2|^{p_1}$, we have

$$\begin{aligned}|{}_a^bL_q(x)| &\leq (x-a)^2(b-x)^2\left[(x-a)\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[4]_q}\,|{}_aD_q^2f(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}\,|{}_aD_q^2f(a)|^{p_1}\right)^{\frac{1}{p_1}}\right]\\ &\quad+(x-a)^2(b-x)^2\left[(b-x)\left(\frac{1}{[3]_q}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[4]_q}\,|{}^bD_q^2f(x)|^{p_1}+\frac{q^3}{[3]_q[4]_q}\,|{}^bD_q^2f(b)|^{p_1}\right)^{\frac{1}{p_1}}\right],\end{aligned}$$

which completes the proof. □

Theorem 3.8. *We assume that the conditions of* Lemma 3.1 *hold. If* $|^{b}D_q^2 f|^{p_1}$ *and* $|_{a}D_q^2 f|^{p_1}$ *are convex on* $[a, b]$ *for some* $p_1 > 1$ *and* $\frac{1}{r_1}+\frac{1}{p_1}=1$, *then we have,*

$$\begin{aligned}|{}_a^bL_q(x)| &\leq (x-a)^2(b-x)^2\left(\frac{1}{[2r_1+1]_q}\right)^{\frac{1}{r_1}}\\ &\quad\times\left[(x-a)\left(\frac{|{}_aD_q^2f(x)|^{p_1}+q\,|{}_aD_q^2f(a)|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}+(b-x)\left(\frac{|{}^bD_q^2f(x)|^{p_1}+q\,|{}^bD_q^2f(b)|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right],\end{aligned}\tag{3.8}$$

where $0 < q < 1$.

Proof. Taking the modulus in Lemma 3.1 and applying well-known Hölder's inequality, we obtain

$$|{}_a^bL_q(x)| \leq (x-a)^2(b-x)^2\left[(x-a)\int_0^1 t^2|{}_aD_q^2f(tx+(1-t)a)|d_qt+(b-x)\int_0^1 t^2|{}^bD_q^2f(tx+(1-t)b)|d_qt\right]$$

$$\leq (x-a)^2(b-x)^2\left[(x-a)\left(\int_0^1 t^{2r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1 |{}_aD_q^2 f(tx+(1-t)a)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right.$$
$$\left.+(b-x)\left(\int_0^1 t^{2r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1 |{}_aD_q^2 f(tx+(1-t)b)|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right].$$

Using the fact that $|^bD_q^2 f|^{p_1}$ and $|_aD_q^2 f|^{p_1}$ are convex, we have

$$|{}_a^bL_q(x)| \leq (x-a)^2(b-x)^2\left(\frac{1}{[2r_1+1]_q}\right)^{\frac{1}{r_1}}$$
$$\times\left[(x-a)\left(\frac{|{}_aD_q^2 f(x)|^{p_1}+q\,|{}_aD_q^2 f(a)|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}+(b-x)\left(\frac{|^bD_q^2 f(x)|^{p_1}+q\,|^bD_q^2 f(b)|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right],$$

which completes the proof. □

Corollary 3.9. *In* Theorem 3.8, *if we set* $|^bD_q^2 f(t)|, |_aD_q^2 f| \leq M$, *then we obtain the following quantum Ostrowski-type inequality for twice quantum differentiable functions,*

$$|{}_a^bL_q(x)| \leq M(x-a)^2(b-x)^2(b-a)\left(\frac{1}{[2r_1+1]_q}\right)^{\frac{1}{r_1}}.$$

Remark 3.10. In Corollary 3.9, if we take the limit $q \to 1^-$, then we obtain [41, Corollary 2.3 for $\alpha = 1$].

4 Conclusion

We conclude our work by mentioning that here, we proved some new quantum integral inequalities of Ostrowski-type for twice quantum differentiable functions by using the notions of quantum derivatives and quantum integrals. It is important to mention that our results transformed into some new and known results by considering the limit $q \to 1^-$ in our main results. We strongly believe that it is an interesting and new problem for the upcoming researchers who can obtain similar inequalities via quantum fractional calculus in their future work.

Acknowledgments: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding information: This work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485, and 11971241).

Author contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of interest: The authors declare that they have no competing interests.

References

[1] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. **10** (1938), 226–227.

[2] H. Budak, M. Z. Sarikaya, and S. S. Dragomir, *Some perturbed Ostrowski-type inequalities for twice differentiable functions*, in: P. Agarwal, S. Dragomir, M. Jleli, B. Samet (eds.), Advances in Mathematical Inequalities and Applications, Trends in Mathematics, Birkhäuser, Singapore, 2018, pp. 279–294.

[3] Y.-M. Chu, M. U. Awan, S. Talib, M. A. Noor, and K. I. Noor, *New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right* (p,q)*-derivatives and definite integrals*, Adv. Difference Equ. **2020** (2020), 634, DOI: https://doi.org/10.1186/s13662-020-03094-x.

[4] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. Appl. **4** (2001), no. 1, 59–66

[5] S. S. Dragomir, *The Ostrowski's integral inequality for Lipschitzian mappings and applications*, Comput. Math. Appl. **38** (1999), no. 11–12, 33–37.

[6] P. Cerone, S. S. Dragomir, and J. Roumeliotis, *An inequality of Ostrowski-type for mappings whose second derivatives are bounded and applications*, East Asian Math. J. **15** (1999), 1–9.

[7] M. A. Latif and S. Hussain, *New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals*, J. Fractional Calc. Appl. **2** (2012), no. 9, 1–15.

[8] W. Liu and H. Zhuang, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, J. Appl. Anal. Comput. **7** (2017), no. 2, 501–522, DOI: https://doi.org/10.11948/2017031.

[9] M. Z. Sarikaya and H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc. **145** (2017), 1527–1538, DOI: https://doi.org/10.1090/proc/13488.

[10] M. Z. Sarikaya and H. Ogunmez, *On new inequalities via Riemann-Liouville fractional integration*, Abstr. Appl. Anal. **2012** (2012), 428983, DOI: https://doi.org/10.1155/2012/428983.

[11] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl. **63** (2012), no. 7, 1147–1154.

[12] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[13] T. Ernst, *The History of q-Calculus and New Method*, Department of Mathematics, Uppsala University, Sweden, 2000.

[14] H. Gauchman, *Integral inequalities in q-calculus*, Comput. Math. Appl. **47** (2004), 281–300.

[15] F. H. Jackson, *On a q-definite integrals*, Quarterly J. Pure Appl. Math. **41** (1910), 193–203.

[16] V. Kac and P. Cheung, *Quantum Calculus*, Springer, Springer-Verlag New York, 2002.

[17] R. Agarwal, *A propos d'une note de M. Pierre Humbert*, Comptes rendus de la Academie des Sciences **236** (1953), no. 21, 203–2032.

[18] W. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proc. Edinburgh Math. Soc. **15** (1966/1967), no. 2, 135–140.

[19] P. M. Rajkovic, M. S. Stankovic, and S. D. Marinkovic, *The zeros of polynomials orthogonal with respect to q-integral on several intervals in the complex plane*, in: I. M. Mladenov and A. C. Hirshfeld (eds.), Proceedings of The Fifth International Conference on Geometry, Integrability and Quantization, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences, Varna, Bulgaria, vol. 5, 2004, pp. 178–188.

[20] J. Tariboon and S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ. **2013** (2013), 282, DOI: https://doi.org/10.1186/1687-1847-2013-282.

[21] S. Bermudo, P. Kórus, and J. N. Valdés, *On q-Hermite-Hadamard inequalities for general convex functions*, Acta Math. Hungar. **162** (2020), 364–374, DOI: https://doi.org/10.1007/s10474-020-01025-6.

[22] M. A. Ali, H. Budak, Z. Zhang, and H. Yıldırım, *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Math. Methods Appl. Sci. **44** (2020), no. 6, 4515–4540, DOI: https://doi.org/10.1002/mma.7048.

[23] M. A. Ali, H. Budak, M. Abbas, and Y.-M. Chu *Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second* q^b*-derivatives*, Adv. Differ. Equ. **2021** (2021), 7, DOI: https://doi.org/10.1186/s13662-020-03163-1.

[24] M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, and Y.-M. Chu, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Differ. Equ. **2021** (2021), 64, DOI: https://doi.org/10.1186/s13662-021-03226-x.

[25] M. A. Ali, Y.-M. Chu, H. Budak, A. Akkurt, H. Yıldırım, and M. A. Zahid, *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv. Differ. Equ. **2021** (2021), 25, DOI: https://doi.org/10.1186/s13662-020-03195-7.

[26] N. Alp, M. Z. Sarıkaya, M. Kunt, and I. Işcan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, *J. King Saud Univ. Sci.* **30** (2018), no. 2, 193–203.

[27] N. Alp and M. Z. Sarikaya, *Quantum Hermite-Hadamard's type inequalities for co-ordinated convex functions*, Appl. Math. E-Notes **20** (2020), 341–356.

[28] H. Budak, M. A. Ali, and M. Tarhanaci, *Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions*, J. Optim. Theory Appl. **186** (2020), 899–910, DOI: https://doi.org/10.1007/s10957-020-01726-6.

[29] H. Budak, S. Erden, and M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Math. Meth. Appl. Sci. **44** (2021), 378–390, DOI: https://doi.org/10.1002/mma.6742.

[30] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Springer, Basel, 2012.

[31] S. Jhanthanam, J. Tariboon, S. K. Ntouyas, and K. Nonlaopon, *On q-Hermite-Hadamard inequalities for differentiable convex functions*, Mathematics **7** (2019), no. 7, 632, DOI: https://doi.org/10.3390/math7070632.

[32] M. A. Khan, M. Noor, E. R. Nwaeze, and Y.-M. Chu, *Quantum Hermite-Hadamard inequality by means of a Green function*, Adv. Difference Equ. **2020** (2020), 99, DOI: https://doi.org/10.1186/s13662-020-02559-3.

[33] M. A. Noor, K. I. Noor, and M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput. **269** (2015), 242–251.

[34] M. A. Noor, M. U. Awan, and K. I. Noor, *Quantum Ostrowski inequalities for q-differentiable convex functions*, J. Math. Inequal. **10** (2016), no. 4, 1013–1018, DOI: http://doi.org/10.7153/jmi-10-81.

[35] E. R. Nwaeze and A. M. Tameru, *New parameterized quantum integral inequalities via η-quasiconvexity*, Adv. Difference Equ. **2019** (2019), 425, DOI: https://doi.org/10.1186/s13662-019-2358-z.

[36] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal. **9** (2015), no. 3, 781–793.

[37] M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, and Z. Zhang, *Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus*, Symmetry **12** (2020), no. 9, 1476, DOI: https://doi.org/10.3390/sym12091476.

[38] H. Zhuang, W. Liu, and J. Park, *Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions*, Mathematics **7** (2019), no. 2, 152, DOI: http://doi.org/10.3390/math7020152.

[39] M. A. Noor, K. I. Noor, and M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput. **251** (2015), 675–679.

[40] G. A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, New York, 2011.

[41] B. Meftah and K. Boukerrioua, *Some new Ostrowski-type inequalities for functions whose second derivative are h-convex via Riemann-Liouville fractional*, Malaya J. Matematik **2** (2014), no. 4, 445–459.

[42] Z. Liu, *Some companions of an Ostrowski type inequality and applications*, J. Inequal. Pure Appl. Math. **10** (2009), no. 2, 52.