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A New Class of f-Structures Satisfying $f^3 - f = 0$

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Abstract. In this study, we introduce a new class of pseudo f-structure, called hyperbolic f-structure. We give some classifications of this new structure. Further, we extend the notion of (κ, μ, ν)-nullity distribution to hyperbolic almost Kenmotsu f-manifolds. Finally, we construct some non-trivial examples of such manifolds.

1. Introduction

The notion of f-structure was introduced which satisfies

$$f^3 + f = 0 \tag{1}$$

by Yano in 1961 [14]. This is a generalization of some structure defined on different type differentiable manifolds. Almost complex structure J and almost contact structure (φ, ξ, η) are well-known f-structure. By virtue of the definitions of these structures, it is clear that they satisfies (1). While almost complex structure was defined by Weil in 1947 [13] as almost contact structure was introduced by Sasaki in 1960 [11]. Later, many author continued to study on f-structure. Goldberg and Yano defined and studied globally framed metric f-structure on $(2n + s)$-dimensional differentiable manifolds [6]. A globally framed metric f-structure is a generalization of an almost complex structure and an almost contact structure if $s = 0$ and $s = 1$, respectively, where s denotes the dimension of orthogonal distribution on globally framed metric f-manifolds. Then, Blair gave some classes of globally framed metric f-manifolds in 1970 [3]. Recently, Falcitelli and Pastore defined almost Kenmotsu f-manifold in [5] and Öztürk et al. introduced almost α-cosymplectic f-manifold in [8], which are new classes of globally framed metric f-manifolds.

In a similar way, Matsumoto introduced a pseudo f-structure satisfying

$$f^3 - f = 0 \tag{2}$$

which generalizes some different types of structures [7]. Many authors focused on this structure and made some different classifications (for instance, see [9], [10], [12]).

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By motivated these studies, in this paper first, we give some fundamental notations and we compute the normality condition of hyperbolic metric f-structure. Then we prove the existence of hyperbolic metric f-structure on a special hrpersurface of a pseudo almost complex manifold. Next, we focus on a special class of this new structure of Kenmotsu type. Then we compute some Riemannian curvature properties of hyperbolic almost Kenmotsu f-manifolds. Also, we obtain some conditions for hyperbolic almost Kenmotsu f-manifolds to be flat. Moreover, we extend the notion of $(\kappa,\ \mu,\ \nu)$-nullity distribution to hyperbolic almost Kenmotsu f-manifolds and we get its sectional curvature as 1 contrary to Kenmotsu case. Finally, we construct some non-trivial examples satisfying characteristic equations of this new structure.

2. Globally Framed Hyperbolic Metric f-Structure

Let M be a $(2n+s)$-dimensional manifold and φ is a non-null $(1,\ 1)$ tensor field on M. If φ satisfies

$$\varphi^3 - \varphi = 0, \tag{3}$$

then φ is called a pseudo f-structure and M is called f-manifold. If $rank\varphi = 2n$, namely $s = 0$, φ is called almost pseudo complex structure and if $rank\varphi = 2n+1$, namely $s = 1$, then φ reduces an almost pseudo contact structure. $rank\varphi$ is always constant [7].

On an pseudo f-manifold M, P_1 and P_2 operators are defined by

$$P_1 = \varphi^2, \quad P_2 = -\varphi^2 + I, \tag{4}$$

which satisfy

$$P_1 + P_2 = I,\ P_1^2 = P_1,\ P_2^2 = P_2,\ \varphi P_1 = P_1\varphi = \varphi,\ P_2\varphi = \varphi P_2 = 0. \tag{5}$$

These properties show that P_1 and P_2 are complementary projection operators. There are D and $D^{\perp}$ distributions with respect to P_1 and P_2 operators, respectively. Also, $\dim(D) = 2n$ and $\dim(D^{\perp}) = s$ [1].

Now, we give the definition of hyperbolic metric f-structure.

Definition 2.1. *Let M be a $(2n+s)$-dimensional f-manifold and φ is a $(1,\ 1)$ tensor field, ξ_i is vector field and η^i is 1-form for each $1 \le i \le s$ on M, respectively. If $\left(\varphi,\ \xi_i,\ \eta^i\right)$ satisfy*

$$\eta^j(\xi_i) = -\delta_i^j, \tag{6}$$

$$\varphi^2 = I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \tag{7}$$

then $\left(\varphi,\ \xi_i,\ \eta^i\right)$ is called globally framed hyperbolic f-structure or simply framed hyperbolic f-structure and M is called globally framed hyperbolic f-manifold or simply framed hyperbolic f-manifold.

For a framed hyperbolic f-manifold M, the following properties are satisfied :

$$\varphi\xi_i = 0, \tag{8}$$

$$\eta^i \circ \varphi = 0. \tag{9}$$

Definition 2.2. *If on a framed hyperbolic f-manifold M, there exists a Riemannian metric which satisfies*

$$\eta^i(X) = g(X,\ \xi_i), \tag{10}$$

and

$$g(\varphi X,\ \varphi Y) = -g(X,\ Y) - \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y), \tag{11}$$

for all vector fields X, Y on M, then M is called framed hyperbolic metric f-manifold. On a framed hyperbolic metric f-manifold, fundamental 2-form Φ defined by

$$\Phi(X,\ Y) = g(X,\ \varphi Y), \tag{12}$$

for all vector fields $X,\ Y \in \chi(M)$.

On a globally framed hyperbolic metric f-manifold the (1 1) tensor field φ is anti-symmetric, that is

$$g(X,\ \varphi Y) = -g(\varphi X,\ Y). \tag{13}$$

Now, we compute the normality condition for globally framed hyperbolic metric f-manifolds. In a similar way of previous studies for globally framed metric f-manifold, after easy calculations then we have four tensors $N^{(1)}$ $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ defined by

$$N^{(1)}(X,\ Y) = [\varphi,\ \varphi](X,\ Y) + 2\sum_{k=1}^{s} d\eta^k(X,\ Y)\,\xi_k, \quad N^{(2)}(X,\ Y) = \sum_{k=1}^{s}\left\{\left(\mathcal{L}_{\varphi X}\eta^k\right)(Y) - \left(\mathcal{L}_{\varphi Y}\eta^k\right)(X)\right\},$$

$$N^{(3)}(X) = \sum_{k=1}^{s}(\mathcal{L}_{\xi_k}\varphi)\,X, \quad N^{(4)}(X) = \sum_{k=1}^{s}\left(\mathcal{L}_{\xi_k}\eta^k\right)X,$$

where $\left(\mathcal{L}_{\varphi X}\eta^k\right)(Y) = \varphi X\eta^k(Y) - \eta^k([\varphi X,\ Y])$ for each $1 \le k \le s$. A globally framed hyperbolic metric f-manifold is normal if and only if these four tensors vanish. But we see that the vanishing of $N^{(1)}$ implies the vanishing of the other tensors. Thus the normality condition for globally framed hyperbolic metric f-manifold is

$$[\varphi,\ \varphi](X,\ Y) + 2\sum_{k=1}^{s} d\eta^k(X,\ Y)\,\xi_k = 0. \tag{14}$$

For a globally framed hyperbolic metric f-structure $\left(\varphi,\ \xi_i,\ \eta^i,\ g\right)$ the covariant derivative of φ is given by

$$\begin{aligned} 2g((\nabla_X\varphi)\,Y,\ Z) &= 3d\Phi(X,\ \varphi Y,\ \varphi Z) - 3d\Phi(X,\ Y,\ Z) - g\left(N^{(1)}(Y,\ Z),\ \varphi X\right) - N^{(2)}(Y,\ Z)\sum_{k=1}^{s}\eta^k(X) \\ &\quad - 2\sum_{k=1}^{s} d\eta^k(\varphi Y,\ X)\,\eta^k(Z) + 2\sum_{k=1}^{s} d\eta^k(\varphi Z,\ X)\,\eta^k(X). \end{aligned} \tag{15}$$

Now, we define a (1, 1) tensor field h_i for each $1 \le i \le s$ which plays an important role on the normality of a globally framed hyperbolic f-manifold as follows

$$h_i = \frac{1}{2}\mathcal{L}_{\xi_i}\varphi = \frac{1}{2}N^{(3)}, \tag{16}$$

where $\mathcal{L}$ denotes the Lie differentiation. If for each $1 \le i \le s, h_i's$ vanish identically zero, then the globally framed hyperbolic f-manifold is normal.

Proposition 2.3. *The tensor field h_i for each $1 \le i \le s$ is a symmetric operator and satisfies*

(i) $h_i\xi_j = 0$,
(ii) $h_i \circ \varphi = -\varphi \circ h_i$,
(iii) $trh_i = 0$,
(iv) $tr\varphi h_i = 0$.

Proof. The proof can be easily derived in a similar way of [3], thus we omit it. □

3. Existence of Globally Framed Hyperbolic Metric f-Structure

Let $\left(\overline{N},\ J,\ g\right)$ be a pseudo Kähler manifold and let M be a hypersurface of $\overline{N}$ with dimension $2n+s$. It is well-known that the almost complex structure J on $\overline{N}$ satisfies

$$J^2 = I, \tag{17}$$

where I denotes the identity map. Furthermore, since M is a hypersurface of $\overline{N}$, we have

$$JX = \varphi X + \sum_{k=1}^{s} \eta^k (X) N, \ \ N = -\sum_{k=1}^{s} J(\xi_k), \tag{18}$$

for any vector field X on M. Now, by applying φ on both sides of (18) and using (17), we obtain

$$\varphi^2 X = X + \sum_{k=1}^{s} \eta^k (X) \xi_k, \tag{19}$$

which means that $\left(\varphi,\ \xi_k,\ \eta^k\right)$ is a globally framed hyperbolic f-structure. Now for any vector fields X, Y on M, we have

$$g(JX,\ JY) = g\left(\varphi X + \sum_{k=1}^{s} \eta^k (X) N,\ \varphi Y + \sum_{k=1}^{s} \eta^k (Y) N\right). \tag{20}$$

By using (17) in (20) and since N is an orthonormal vector field, then we derive

$$-g(X,\ Y) = g(\varphi X,\ \varphi Y) + \sum_{k=1}^{s} \eta^k (X) \eta^k (Y) \tag{21}$$

and for any ξ_i, we obtain

$$g(X,\ \xi_i) = \eta^i (X). \tag{22}$$

From (21) and (22), it is clear that $\left(\varphi,\ \xi_k,\ \eta^k,\ g\right)$ is an f-structure.

4. Hyperbolic Almost Kenmotsu f-Manifolds

Definition 4.1. *Let M be a globally framed hyperbolic metric f-manifold with hyperbolic f-structure $\left(\varphi,\ \xi_k,\ \eta^k,\ g\right)$. If for each $k = 1,\ \ldots,\ s$ the 1-forms are closed, that is $d\eta^k = 0$ and $d\Phi = 2\overline{\eta} \wedge \Phi$ where $\overline{\eta} = \sum_{k=1}^{s} \eta^k$, then M is called hyperbolic almost Kenmotsu f-manifold. Furthermore, if M is normal then it is a hyperbolic Kenmotsu f-manifold.*

Theorem 4.2. *On a hyperbolic almost Kenmotsu f-manifold M the following characteristic equations hold*

$$(\nabla_X \varphi)(Y) = \sum_{k=1}^{s} \left\{ g(\varphi X + h_k X,\ Y) \xi_k - \eta^k (Y)(\varphi X + h_k X) \right\}, \tag{23}$$

$$\nabla_X \xi_i = \varphi^2 X + \varphi h_i X, \tag{24}$$

$$(\nabla_{\xi_i} \varphi) X = 0 \tag{25}$$

and

$$\nabla_{\xi_i} \xi_j = 0 \tag{26}$$

for any $X,\ Y$ on M.

Proof. By using (16) in (15) and since M is a hyperbolic almost Kenmotsu f-manifold, then we get (23). For the second part, by taking $Y = \xi_i$ and using (7), it yields the desired result. (25) and (26) can be easily seen from (23) and (25), respectively. □

Lemma 4.3. *Let M be a hyperbolic almost Kenmotsu f-manifold. Then for each $i,\ j,\ k \in \{1,\ \ldots,\ s\}$, we have*

$$\left(\nabla_{\xi_i} h_j\right) X = \varphi R\left(\xi_i,\ X\right)\xi_j - \varphi X - \left(h_i + h_j\right) X + \left(\varphi \circ h_i \circ h_j\right) X, \tag{27}$$

$$\left(\nabla_{\xi_i} h_i\right) X = \varphi R\left(\xi_i,\ X\right)\xi_i - \varphi X - 2h_i X + \left(\varphi \circ h_i^2\right) X, \tag{28}$$

$$\varphi R\left(\xi_i,\ \varphi X\right)\xi_j + R\left(\xi_i,\ X\right)\xi_j = 2\left(\varphi^2 - h_i \circ h_j\right) X, \tag{29}$$

$$\varphi R\left(\xi_i,\ \varphi X\right)\xi_i + R\left(\xi_i,\ X\right)\xi_i = 2\left(\varphi^2 - h_i^2\right) X, \tag{30}$$

$$\eta^k\left(R\left(\xi_i,\ X\right)\xi_j\right) = 0, \tag{31}$$

$$R\left(\xi_i,\ \xi_k\right)\xi_j = 0, \tag{32}$$

for any vector field X on M.

Proof. For any vector field X on M, we have

$$R\left(\xi_i,\ X\right)\xi_j = \nabla_{\xi_i}\nabla_X \xi_j - \nabla_X \nabla_{\xi_i}\xi_j - \nabla_{[\xi_i,\ X]}\xi_j. \tag{33}$$

By using (24) and (26) in (33), we derive

$$R\left(\xi_i,\ X\right)\xi_j = \varphi\left(\left(\nabla_{\xi_i} h_j\right) X\right) + \varphi^2 X + \left(\varphi \circ h_i\right) X + \left(\varphi \circ h_j\right) X - \left(h_i \circ h_j\right) X. \tag{34}$$

Applying φ on both sides of (34) and by virtue of (3), we find (27) and considering $i = j$ in (27) we get (28). Applying φ both sides of (34) and replacing X by φX in (34), we obtain

$$\varphi R\left(\xi_i,\ \varphi X\right)\xi_j = -\varphi\left(\left(\nabla_{\xi_i} h_j\right) X\right) + \varphi^2 X - \left(\varphi \circ h_i\right) X - \left(\varphi \circ h_j\right) X - \left(h_i \circ h_j\right) X. \tag{35}$$

By taking summation (34) and (35) side by side, we get (29). From (29) we have (30). The last two identities of the lemma are clear. □

Corollary 4.4. *If a hyperbolic almost Kenmotsu f-manifold is flat then we have*

$$h_i \circ h_j = \varphi^2$$

for each $i,\ j \in \{1,\ \ldots,\ s\}$.

Corollary 4.5. *For a hyperbolic almost Kenmotsu f-manifold, if $R\left(\xi_i,\ X\right)\xi_i = 0$ for $i \in \{1,\ \ldots,\ s\}$ and $X \in \Gamma(D)$, then it follows that*

$$h_i^2 = \varphi^2.$$

Lemma 4.6. *Let M be a hyperbolic almost Kenmotsu f-manifold. Then the Riemannian curvature satisfies*

$$g(R(\xi_i, X)Y, Z) = \sum_{k=1}^{s} \eta^k(Y) g\left(\varphi^2 Z + (\varphi \circ h_k) Z, X\right) - \sum_{k=1}^{s} \eta^k(Z) g\left(\varphi^2 Y + (\varphi \circ h_k)^2 Y, X\right) + g((\nabla_Y(\varphi \circ h_i)) Z - (\nabla_Z(\varphi \circ h_i)) Y, X) \tag{36}$$

and

$$g(R(\xi_i, X)Y, Z) + g(R(\xi_i, X)\varphi Y, \varphi Z) - g(R(\xi_i, \varphi X)Y, \varphi Z) - g(R(\xi_i, \varphi X)\varphi Y, Z) = 2\sum_{j=1}^{s} \left\{\eta^j(Z) g(h_i X + \varphi X, \varphi Y) - \eta^j(Y) g(h_i X + \varphi X, \varphi Z)\right\} \tag{37}$$

for any $X, Y, Z \in \Gamma(TM)$

Proof. For any $X, Y, Z \in \Gamma(TM)$ we have

$$g(R(\xi_i, X)Y, Z) = g(R(Y, Z)\xi_i, X) = \nabla_Y \nabla_Z \xi_i - \nabla_Z \nabla_Y \xi_i - \nabla_{[Y, Z]}\xi_i. \tag{38}$$

By using (24) in (38), we find (36). For the second part of the lemma, let us introduce the operators A and B_i, $i \in \{1, ..., s\}$ defined by

$$A(X, Y, Z) := 2\sum_{j=1}^{s} \left\{\eta^j(Z) g(\varphi X, \varphi Y) - \eta^j(Y) g(\varphi X, \varphi Z)\right\} \tag{39}$$

and

$$B_i(X, Y, Z) := -g(\varphi X, (\nabla_Y(\varphi \circ h_i))\varphi Z) - g\left(\varphi X, \left(\nabla_{\varphi Y}(\varphi \circ h_i)\right) Z\right) + g(X, (\nabla_Y(\varphi \circ h_i)) Z) + g\left(X, \left(\nabla_{\varphi Y}(\varphi \circ h_i)\right)\varphi Z\right) \tag{40}$$

for each $X, Y, Z \in \Gamma(TM)$. By a direct computation and using (36) we obtain that the left hand side of (37) is equal to $A(X, Y, Z) + B_i(X, Y, Z) - B_i(X, Z, Y)$. Since

$$\eta_j\left(\left(\nabla_{\varphi Y} h_i\right) Z\right) = \eta_j\left(\nabla_{\varphi Y}(h_i Z)\right)$$

we can write

$$\begin{aligned} B_i(X,Y,Z) &= g(X, \nabla_Y((\varphi \circ h_i) Z)) - g(X, (\varphi \circ h_i) \nabla_Y Z) + g\left(X, \nabla_{\varphi Y}((\varphi \circ h_i \circ \varphi) Z)\right) \\ &- g\left(X, (\varphi \circ h_i)\left(\nabla_{\varphi Y}\varphi Z\right)\right) - g(\varphi X, \nabla_Y((\varphi \circ h_i \circ \varphi) Z)) + g(\varphi X, (\varphi \circ h_i)(\nabla_Y \varphi Z)) \\ &- g\left(\varphi X, \nabla_{\varphi Y}((\varphi \circ h_i) Z)\right) + g\left(\varphi X, (\varphi \circ h_i)\left(\nabla_{\varphi Y} Z\right)\right) \\ &= g(X, (\nabla_Y \varphi) h_i Z) - g(X, h_i((\nabla_Y \varphi) Z)) + g\left(X, (h_i \circ \varphi)\left(\left(\nabla_{\varphi Y}\varphi\right) Z\right)\right) \\ &+ g\left(X, \varphi\left(\left(\nabla_{\varphi Y}\varphi\right) h_i Z\right)\right) + \sum_{k=1}^{s} \eta^k\left(\left(\nabla_{\varphi Y} h_i\right) Z\right) \eta^k(X). \end{aligned} \tag{41}$$

Moreover, from (23), (24) and Proposition 2.3 it follows that

$$\left(\varphi \circ \left(\nabla_{\varphi X}\varphi\right)\right) Y = \left(\nabla_{\varphi X}\varphi^2\right) Y - \left(\nabla_{\varphi X}\varphi\right)(\varphi Y) = \sum_{j=1}^{s} \left(\left(\nabla_{\varphi X}\eta_j\right) Y \xi_j\right) + \sum_{j=1}^{s} \left(\eta_j(Y) \nabla_{\varphi X}\xi_j\right)$$

or

$$-\left(\nabla_{\varphi X}\varphi\right)(\varphi Y)=\sum_{j=1}^{s}\nabla_{\varphi X}\left(g\left(\xi_j,\ Y\right)\right)\xi-g\left(\nabla_{\varphi X}Y,\ \xi_j\right)\xi_j)+\sum_{j=1}^{s}\eta_j(Y)\left(\varphi X-h_jX\right)$$
$$-\sum_{j=1}^{s}\left\{\eta_j(Y)\left[h_jX+\varphi X\right]-2g(X,\ \varphi Y)\xi_j\right\}-(\nabla_X\varphi)Y.$$

Hence, we find

$$\left(\varphi\circ\left(\nabla_{\varphi X}\varphi\right)\right)Y=-3\sum_{j=1}^{s}g(X,\ \varphi Y)\xi_j+\sum_{j=1}^{s}g\left(Y,\ h_jX\right)\xi_j+2\sum_{j=1}^{s}\eta_j(Y)\varphi X-(\nabla_X\varphi)Y.$$

Taking into account of (23), then for each $i,\ j\in\{1,\cdots,s\}$ we have

$$\eta_i\left(\left(\nabla_{\varphi Y}h_j\right)Z\right)=\eta_i\left(\nabla_{\varphi Y}\left(h_jZ\right)\right)=\left(\nabla_{\varphi Y}\eta_i\right)\left(h_jZ\right)=-g\left(h_jZ,\ \nabla_{\varphi Y}\xi_i\right)=g\left(h_jZ,\ -h_iY+\varphi Y\right). \tag{42}$$

By virtue of (41) and (42), we deduce that

$$B_i(X,\ Y,\ Z)=g(X,\ (\nabla_Y\varphi)h_iZ)-g(X,\ h_i((\nabla_Y\varphi)Z))+2\sum_{j=1}^{s}\eta^j(Z)g(h_iX,\ \varphi Y)+g(h_iX,\ (\nabla_Y\varphi)Z)$$
$$-3\sum_{j=1}^{s}\eta^j(X)g(Y,\ \varphi h_iZ)-\sum_{j=1}^{s}\eta^j(X)g\left(h_iZ,\ h_jY\right)+\sum_{j=1}^{s}\eta_j(X)g(h_kZ,\ h_iY)$$
$$+\sum_{j=1}^{s}\eta^j(X)g\left(\varphi Y,\ h_jZ\right)-g(X,\ (\nabla_Y\varphi)h_iZ)$$
$$=2\sum_{j=1}^{s}\left(\eta^j(Z)g(h_iX,\ \varphi Y)+2\eta^j(X)g(\varphi Y,\ h_iZ)\right).$$

Therefore, we obtain

$$A(X,\ Y,\ Z)+B_i(X,\ Y,\ Z)-B_i(X,\ Z,\ Y)=2\sum_{j=1}^{s}\left\{\eta^j(Z)g(h_iX+\varphi X,\ \varphi Y)-2\eta^j(Y)g(h_iX+\varphi X,\ \varphi Z)\right\},$$

which gives (37). □

5. Hyperbolic Almost Kenmotsu f-Manifolds with $(\kappa,\ \mu,\ \nu)$-Nullity Distribution

In this section we generalize the $(\kappa,\ \mu)$-nullity distribution introduced by Blair et al. [4] for the hyperbolic almost Kenmotsu f-manifolds.

Definition 5.1. *Let M be a hyperbolic almost Kenmotsu f-manifold and $\kappa,\ \mu$ and ν are real constants. If for each $1\le i\le s$ and for any $X,\ Y\in\Gamma(TM)$, the characteristic vector fields $\xi_i's$ satisfy*

$$R(X,\ Y)\xi_i=\kappa\left\{\overline{\eta}(X)\varphi^2(Y)-\overline{\eta}(Y)\varphi^2(X)\right\}+\mu\left\{\overline{\eta}(Y)h_i(X)-\overline{\eta}(X)h_i(Y)\right\}$$
$$+\nu\left\{\overline{\eta}(Y)(\varphi\circ h_i)(X)-\overline{\eta}(X)(\varphi\circ h_i)(Y)\right\}. \tag{43}$$

then M verifies the $(\kappa,\ \mu,\ \nu)$-nullity condition.

Theorem 5.2. *Let M be a hyperbolic almost Kenmotsu f-manifold satisfying the $(\kappa,\ \mu,\ \nu)$-nullity condition. For each $1 \leq i,\ j \leq s$, we have*

(i) $h_i \circ h_j = h_j \circ h_i$,
(ii) $\kappa \leq 1$,
(iii) *if* $\kappa \leq 1$, *then* h_i *has eigenvalues* 0 *or* $\pm\sqrt{1-\kappa}$.

Proof. From (43), it follows that

$$\varphi R(\xi_i,\ \varphi X)\,\xi_j + R(\xi_i,\ X)\,\xi_j = 2\kappa\varphi^2 X. \tag{44}$$

By virtue of (29) and (44), we obtain

$$\left(h_i \circ h_j\right) X = (1-\kappa)\,\varphi^2 X = \left(h_j \circ h_i\right) X \tag{45}$$

which implies (i). Taking into account of (45), for any $X \in \Gamma(D)$, where D is $(\kappa,\ \mu,\ \nu)$-nullity distribution. Then, we derive

$$h_i^2 X = (1-\kappa)\,X \tag{46}$$

In view of Proposition 2.3 and (46), it is clear that the eigenvalues of h_i^2 are 0 or $(1-\kappa)$. Furthermore, h_i is symmetric and $\|h_i(X)\|^2 = (1-\kappa)\,\|X\|^2$. Thus $\kappa \leq 1$. Additionally, let t be a real eigenvalue of h_i and let X be eigenvector corresponding to t. Then it follows that $t^2\,\|X\|^2 = (1-\kappa)\,\|X\|^2$ and $t = \pm\sqrt{1-\kappa}$. From Proposition 2.3 and the above fact, we arrive at (iii). □

Theorem 5.3. *Let M be a hyperbolic almost Kenmotsu f-manifold satisfying the $(\kappa,\ \mu,\ \nu)$-nullity condition. Then the following holds*

$$h_1 = \ldots = h_s. \tag{47}$$

Proof. If $\kappa = 1$, then by virtue of (46), we have $h_1 = \ldots = h_s = 0$. Now we assume that $\kappa \leq 1$. For any $p \in M$ and $1 \leq i \leq s$, we can write

$$D_p = (D_+)_p \oplus (D_-)_p\,,$$

where $(D_+)_p$ is the eigenspace of h_i corresponding p to the eigenvalue $\lambda = \sqrt{1-\kappa}$ and $(D_-)_p$ denotes the eigenspace of h_i corresponding p to the eigenvalue $-\lambda$. If $X \in D_p$, we have

$$X = X_+ + X_-,$$

where X_+ and X_- denote the components of X in the eigenspaces $(D_+)_p$ and $(D_-)_p$, respectively. Hence we deduce

$$h_i(X) = \lambda\,(X_+ + X_-)\,.$$

On the other hand, for $i \neq j$

$$h_j(X) = h_j(X_+ + X_-) = h_j\left(\frac{1}{\lambda}h_i(X_+) - \frac{1}{\lambda}h_i(X_-)\right) = \frac{1}{\lambda}\left(h_i \circ h_j\right)(X_+ + X_-) = \lambda\,(X_+ + X_-) = h_i(X)$$

which implies (47). □

Corollary 5.4. *Let M be a hyperbolic Kenmotsu f-manifold satisfying the $(\kappa,\ \mu,\ \nu)$-nullity condition. Then its sectional curvature $\kappa = 1$. In other words, Kenmotsu f-manifold is a manifold of positive curvature.*

Remark 5.5. Throughout this paper whenever (43), we put $h = h_1 = \ldots = h_s$ and therefore (43) takes the form

$$R(X, Y)\xi_i = \kappa\left\{\overline{\eta}(X)\varphi^2(Y) - \overline{\eta}(Y)\varphi^2(X)\right\} + \mu\{\overline{\eta}(Y)h(X) - \overline{\eta}(X)h(Y)\} + \nu\{\overline{\eta}(Y)(\varphi\circ h)(X) - \overline{\eta}(X)(\varphi\circ h)(Y)\}. \tag{48}$$

By using (48) and the symmetric properties of the curvature tensor, φ^2 and h, we conclude that

$$R(\xi_i, X)Y = \kappa\left\{\overline{\eta}(Y)\varphi^2X - g\left(X, \varphi^2Y\right)\overline{\xi}\right\} + \mu\left\{g(hX, Y)\overline{\xi} - \overline{\eta}(Y)hX\right\} + \nu\left\{g((\varphi\circ h)X, Y)\overline{\xi} - \overline{\eta}(Y)(\varphi\circ h)X\right\} \tag{49}$$

where $\overline{\xi} = \sum_{k=1}^{s}\xi_k$.

Remark 5.6. Let M be a hyperbolic almost Kenmotsu f-manifold satisfying the (κ, μ, ν)-nullity condition. Let us denote by D_+ and D_- the n-dimensional distributions of the eigenspaces of $\lambda = \sqrt{1-\kappa}$ and $-\lambda$, respectively. We can easily see that D_+ and D_- are mutually orthogonal. Furthermore, since φ anti-commuts with h, we derive $\varphi(D_+) = D_-$ and $\varphi(D_-) = D_+$. In other words, D_+ is a Legendrian distribution and D_- is the conjugate Legendrian distribution of D_+.

Proposition 5.7. *Let M be a hyperbolic almost Kenmotsu f-manifold satisfying the (κ, μ, ν)-nullity condition. Then M is a hyperbolic Kenmotsu f-manifold if and only if $\kappa = 1$.*

Proof. The result follows from (46) and by virtue of the definition of $(1, 1)$ tension field h. □

Remark 5.8. *Under the above proposition, we can consider a hyperbolic Kenmotsu f-manifold as a class of $(1, \mu, \nu)$-space.*

Remark 5.9. *Let M be a hyperbolic almost Kenmotsu f-manifold satisfying the (κ, μ, ν)-nullity condition. Then, we have*

$$R(\xi_i, X)\xi_j = \kappa\varphi^2X - \mu hX - \nu(\varphi\circ h)X \tag{50}$$

for any vector field X on M.

Proposition 5.10. *Let M be a hyperbolic almost Kenmotsu f-manifold verifying the (κ, μ, ν)-nullity distribution. Then we have*

$$\nabla_{\xi_i}hX = -\mu(\varphi\circ h)X - (\nu+2)hX, \tag{51}$$

$$R(\xi_i, \varphi X)\xi_j - \varphi R(\xi_i, X)\xi_j = 2\mu(\varphi\circ h)X + 2\nu hX, \tag{52}$$

$$R(\xi_i, \varphi X)\xi_j + \varphi R(\xi_i, X)\xi_j = 2\kappa\varphi X, \tag{53}$$

$$Q\xi_i = 2n\kappa\overline{\xi}. \tag{54}$$

Proof. From (28) and (50), we get (51). By using (50), we derive (52) and (53). The last part can be proved in a similar fashion of [2]. □

6. Examples

In this section, we construct non-trivial examples of hyperbolic Kenmotsu f-manifolds.

Example 6.1. Let N be a 6-dimensional pseudo Kähler manifold and let $\mathcal{V}$ be a 2-dimensional non-degenerate vector space with the signature $(-, -)$. Denoting f the positive differentiable function, let us consider the warped product $M = N \times_f \mathcal{V}$ with the warping function f. Since N is a pseudo Kähler manifold, M satisfies (8)-(11), (23) and (24). Then we find a $(6+2)$-dimensional hyperbolic Kenmotsu f-manifold.

Example 6.2. Let us consider $(4+2)$-dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z_1, z_2) : (x_1, x_2, y_1, y_2, z_1, z_2) \neq (0, 0, 0, 0, 0, 0)\}$, where $(x_1, x_2, y_1, y_2, z_1, z_2)$ are the standart coordinates in R^6. The vector fields

$$e_1 = z_1 \frac{\partial}{\partial x_1}, \quad e_2 = z_2 \frac{\partial}{\partial x_2}, \quad e_3 = -z_1 \frac{\partial}{\partial y_1},$$
$$e_4 = -z_2 \frac{\partial}{\partial y_2}, \quad e_5 = -z_1 \frac{\partial}{\partial z_1} \quad e_6 = -z_2 \frac{\partial}{\partial z_2},$$

are linearly independent at each point of M. Let g be the nondegenerate semi-Riemannian metric defined by

$$g\left(e_i, e_j\right) = 0, \ i, j = 1, 2, 3, 4, 5, 6; \ i \neq j$$
$$g(e_k, e_k) = 1, \ k = 1, 2, 3, 4$$
$$g(e_l, e_l) = -1, \ l = 5, 6$$

Let η^1 and η^2 be 1 forms defined by $\eta^1(Z) = g(Z, e_5)$ and $\eta^2(Z) = g(Z, e_6)$ for each vector field $Z \in \chi(M)$. Let φ be the $(1, 1)$ tensor field defined by

$$\varphi e_1 = -e_3, \quad \varphi e_2 = -e_4, \quad \varphi e_5 = 0, \quad \varphi e_6 = 0.$$

By using the linearity of φ and g, we obtain

$$\eta^1(e_5) = -1, \quad \eta^2(e_6) = -1, \quad \varphi^2 Z = Z + \eta^1(Z) e_5 + \eta^2(Z) e_6$$
$$g(\varphi Z, \varphi W) = -g(Z, W) - \left\{\eta^1(Z)\eta^1(W) + \eta^2(Z)\eta^2(W)\right\}$$

for any $Z, W \in \chi(M)$. Thus $\left(\varphi, \xi_i, \eta^i, g\right)$ defines a globally framed hyperbolic f-structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_3] = [e_2, e_4] = 0, \ [e_1, e_5] = e_1, \ [e_1, e_4] = 0,$$
$$[e_2, e_6] = e_2, \ [e_2, e_5] = 0, \ [e_4, e_6] = e_4, \ [e_5, e_6] = 0,$$
$$[e_3, e_5] = e_3, \ [e_2, e_3] = 0, \ [e_1, e_6] = [e_1, e_2] = 0,$$
$$[e_3, e_4] = 0, \ [e_4, e_5] = 0, \ [e_3, e_6] = 0.$$

By using the Koszul's formula, we deduce

$$\nabla_X \xi_i = \varphi^2 X, \ i = 1, 2$$

for any X on M, which implies that M is a hyperbolic Kenmotsu f-manifold.

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