



On the generalization of some integral inequalities and their applications

Mehmet Zeki Sarikaya*, Nesip Aktan

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

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ABSTRACT

In this paper, a general integral identity for convex functions is derived. Then, we establish some new inequalities of the Simpson and the Hermite–Hadamard type for functions whose absolute values of derivatives are convex. Some applications for special means of real numbers are also provided.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

is known in the literature as the Hadamard inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave [1].

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [2–13]) and the references therein.

In [12], Sarikaya et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° , $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f'' \in L_1[a, b]$, then

$$\frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2} \int_0^1 m(t) [f''(ta + (1-t)b) + f''(tb + (1-t)a)] dt,$$

where

$$m(t) := \begin{cases} t^2, & t \in \left[0, \frac{1}{2}\right) \\ (1-t)^2, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

* Corresponding author.

E-mail addresses: sarikayamz@gmail.com (M.Z. Sarikaya), nesipaktan@gmail.com (N. Aktan).

Also, the main inequalities in [12] are pointed out as follows:

Theorem 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a, b]$. If $|f''|$ is a convex on $[a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right]. \quad (1.2)$$

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I, a < b$. If $|f''|^q$ is a convex on $[a, b], q \geq 1$, then

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}} \quad (1.3)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [14], Sarikaya et al. prove some inequalities related to Simpson's inequality for functions whose derivatives in absolute value at certain powers are convex functions:

Theorem 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|^q$ is a convex on $[a, b]$ and $q \geq 1$, then the following inequality holds:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a)^2 \left(\frac{1}{162} \right)^{1-\frac{1}{q}} \left[\left(\frac{59|f''(a)|^q + 133|f''(b)|^q}{3^5 \times 2^7} \right)^{\frac{1}{q}} + \left(\frac{133|f''(a)|^q + 59|f''(b)|^q}{3^5 \times 2^7} \right)^{\frac{1}{q}} \right]$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [15,16,14,17,18].

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose absolute values of derivatives are convex, we will derive a general integral identity for convex functions. Finally, some applications for special means of real numbers are provided.

2. Main results

In order to prove our main theorems, we need the following Lemma.

Lemma 2. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$. If $f : I \rightarrow \mathbb{R}$ is a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. Then the following identity holds:

$$(\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b-a} \int_a^b f(x) dx = (b-a)^2 \int_0^1 k(t) f''(ta + (1-t)b) dt \quad (2.1)$$

where

$$k(t) = \begin{cases} \frac{1}{2}t(t-\lambda), & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2}(1-t)(1-\lambda-t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t) f''(ta + (1-t)b) dt \\ &= \frac{1}{2} \int_0^{\frac{1}{2}} t(t-\lambda) f''(ta + (1-t)b) dt + \frac{1}{2} \int_{\frac{1}{2}}^1 (1-t)(1-\lambda-t) f''(ta + (1-t)b) dt \\ &= I_1 + I_2. \end{aligned} \quad (2.2)$$

Integrating by parts twice, we can state:

$$\begin{aligned}
 I_1 &= \frac{1}{2} \int_0^{\frac{1}{2}} t(t - \lambda)f''(ta + (1 - t)b)dt \\
 &= \frac{1}{2}t(t - \lambda)\frac{f'(ta + (1 - t)b)}{a - b}\Big|_0^{\frac{1}{2}} - \frac{1}{2(a - b)} \int_0^{\frac{1}{2}} (2t - \lambda)f'(ta + (1 - t)b)dt \\
 &= \frac{1}{4(b - a)} \left(\lambda - \frac{1}{2} \right) f' \left(\frac{a + b}{2} \right) + \frac{(\lambda - 1)}{2(b - a)^2} f \left(\frac{a + b}{2} \right) \\
 &\quad - \frac{\lambda}{2(b - a)^2} f(b) + \frac{1}{(b - a)^2} \int_0^{\frac{1}{2}} f(ta + (1 - t)b)dt,
 \end{aligned} \tag{2.3}$$

and similarly, we get

$$\begin{aligned}
 I_2 &= \frac{1}{2} \int_{\frac{1}{2}}^1 (1 - t)(1 - \lambda - t)f''(ta + (1 - t)b)dt \\
 &= -\frac{1}{4(b - a)} \left(\lambda - \frac{1}{2} \right) f' \left(\frac{a + b}{2} \right) + \frac{(\lambda - 1)}{2(b - a)^2} f \left(\frac{a + b}{2} \right) - \frac{\lambda}{2(b - a)^2} f(a) \\
 &\quad + \frac{1}{(b - a)^2} \int_{\frac{1}{2}}^1 f(ta + (1 - t)b)dt.
 \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), it follows that

$$I = \frac{1}{(b - a)^2} \left[(\lambda - 1)f \left(\frac{a + b}{2} \right) - \lambda \frac{f(a) + f(b)}{2} + \int_0^1 f(ta + (1 - t)b)dt \right].$$

Thus, using the change of the variable $x = ta + (1 - t)b$ for $t \in [0, 1]$ and by multiplying both sides by $(b - a)^2$, we have the conclusion (2.1). \square

Using this Lemma we can obtain the following general integral inequalities:

Theorem 4. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \rightarrow \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|$ is a convex on $[a, b]$, then the following inequalities hold:

$$\begin{aligned}
 &\left| (\lambda - 1)f \left(\frac{a + b}{2} \right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b - a} \int_a^b f(x)dx \right| \\
 &\leq \begin{cases} \frac{(b - a)^2}{12} \left[\left(\lambda^4 + (1 + \lambda)(1 - \lambda)^3 + \frac{5\lambda - 3}{4} \right) |f''(a)| + \left(\lambda^4 + (2 - \lambda)\lambda^3 + \frac{1 - 3\lambda}{4} \right) |f''(b)| \right], \\ \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(b - a)^2(3\lambda - 1)}{48} [|f''(a)| + |f''(b)|] \text{ for } \frac{1}{2} \leq \lambda \leq 1. \end{cases}
 \end{aligned} \tag{2.5}$$

Proof. From Lemma 2 and by definition of $k(t)$, we get

$$\begin{aligned}
 &\left| (\lambda - 1)f \left(\frac{a + b}{2} \right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b - a} \int_a^b f(x)dx \right| \leq (b - a)^2 \int_0^1 |k(t)| |f''(ta + (1 - t)b)| dt \\
 &= \frac{(b - a)^2}{2} \left\{ \int_0^{\frac{1}{2}} |t(t - \lambda)| |f''(ta + (1 - t)b)| dt + \int_{\frac{1}{2}}^1 |(1 - t)(1 - \lambda - t)| |f''(ta + (1 - t)b)| dt \right\} \\
 &= \frac{(b - a)^2}{2} \{J_1 + J_2\}.
 \end{aligned} \tag{2.6}$$

We assume that $0 \leq \lambda \leq \frac{1}{2}$, then using the convexity of $|f''|$, we get

$$J_1 \leq \int_0^{\frac{1}{2}} |t(t - \lambda)| [t|f''(a)| + (1 - t)|f''(b)|] dt$$

$$\begin{aligned}
 &= \int_0^\lambda t(\lambda - t)[t|f''(a)| + (1 - t)|f''(b)|]dt + \int_\lambda^{\frac{1}{2}} t(t - \lambda)[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \left[\frac{\lambda^4}{6} + \frac{3 - 8\lambda}{3 \times 2^6} \right] |f''(a)| + \left[\frac{(2 - \lambda)\lambda^3}{6} + \frac{5 - 16\lambda}{3 \times 2^6} \right] |f''(b)|,
 \end{aligned} \tag{2.7}$$

and similarly, we have

$$\begin{aligned}
 J_2 &\leq \int_{\frac{1}{2}}^{1-\lambda} (1 - t)(1 - \lambda - t)[t|f''(a)| + (1 - t)|f''(b)|]dt + \int_{1-\lambda}^1 (1 - t)(t + \lambda - 1)[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \left[\frac{1 + \lambda}{6}(1 - \lambda)^3 + \frac{48\lambda - 27}{3 \times 2^6} \right] |f''(a)| + \left[\frac{\lambda^4}{6} + \frac{3 - 8\lambda}{3 \times 2^6} \right] |f''(b)|.
 \end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), we see that the first inequality of (2.5) holds.

On the other hand, let $\frac{1}{2} \leq \lambda \leq 1$, then, using the convexity of $|f''|$ and by simple computation we have

$$\begin{aligned}
 J_1 &\leq \int_0^{\frac{1}{2}} |t(t - \lambda)|[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \int_0^{\frac{1}{2}} t(\lambda - t)[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \frac{8\lambda - 3}{3 \times 2^6} |f''(a)| + \frac{16\lambda - 5}{3 \times 2^6} |f''(b)|,
 \end{aligned} \tag{2.9}$$

and similarly,

$$\begin{aligned}
 J_2 &\leq \int_{\frac{1}{2}}^1 |(1 - t)(1 - \lambda - t)|[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \int_{\frac{1}{2}}^1 (1 - t)(t + \lambda - 1)[t|f''(a)| + (1 - t)|f''(b)|]dt \\
 &= \frac{16\lambda - 5}{3 \times 2^6} |f''(a)| + \frac{8\lambda - 3}{3 \times 2^6} |f''(b)|.
 \end{aligned} \tag{2.10}$$

Thus, if we use (2.9) and (2.10) in (2.6), we obtain the second inequality of (2.5). This completes the proof. \square

Another similar result may be extended in the following theorem.

Theorem 5. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \rightarrow \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|^q$ is a convex on $[a, b]$, $q \geq 1$, then the following inequalities hold:

$$\begin{aligned}
 &\left| (\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(x)dx \right| \\
 &\leq \begin{cases} \frac{(b-a)^2}{2} \left(\frac{\lambda^3}{3} + \frac{1-3\lambda}{24} \right)^{1-\frac{1}{q}} \\ \quad \times \left\{ \left(\left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(a)|^q + \left[\frac{(2-\lambda)\lambda^3}{6} + \frac{5-16\lambda}{3 \times 2^6} \right] |f''(b)|^q \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\left[\frac{1+\lambda}{6}(1-\lambda)^3 + \frac{48\lambda-27}{3 \times 2^6} \right] |f''(a)|^q + \left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(b)|^q \right)^{\frac{1}{q}} \right\}, & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(b-a)^2}{2} \left(\frac{3\lambda-1}{24} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{8\lambda-3}{3 \times 2^6} |f''(a)|^q + \frac{16\lambda-5}{3 \times 2^6} |f''(b)|^q \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\frac{16\lambda-5}{3 \times 2^6} |f''(a)|^q + \frac{8\lambda-3}{3 \times 2^6} |f''(b)|^q \right)^{\frac{1}{q}} \right\} & \text{for } \frac{1}{2} \leq \lambda \leq 1, \end{cases} \tag{2.11}
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $q \geq 1$. From Lemma 2 and using the well known power mean inequality, we have

$$\begin{aligned} & \left| (\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq (b-a)^2 \int_0^1 |k(t)||f''(tb+(1-t)a)|dt \\ & \leq \frac{(b-a)^2}{2} \left\{ \int_0^{\frac{1}{2}} |t(t-\lambda)||f''(ta+(1-t)b)|dt + \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)||f''(ta+(1-t)b)|dt \right\} \\ & = \frac{(b-a)^2}{2} \left\{ \left(\int_0^{\frac{1}{2}} |t(t-\lambda)|dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} |t(t-\lambda)||f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)||f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{2.12}$$

Let $0 \leq \lambda \leq \frac{1}{2}$. Then, since $|f'|^q$ is convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(ta+(1-t)b)|^q \leq t|f'(a)|^q + (1-t)|f'(b)|^q,$$

hence, by simple computation

$$\begin{aligned} \int_0^{\frac{1}{2}} |t(t-\lambda)||f''(ta+(1-t)b)|^q dt & \leq \int_0^\lambda t(\lambda-t)[t|f''(a)|^q \\ & \quad + (1-t)|f''(b)|^q]dt + \int_\lambda^{\frac{1}{2}} t(t-\lambda)[t|f''(a)|^q + (1-t)|f''(b)|^q]dt \\ & = \left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(a)|^q + \left[\frac{(2-\lambda)\lambda^3}{6} + \frac{5-16\lambda}{3 \times 2^6} \right] |f''(b)|^q, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)||f''(ta+(1-t)b)|^q dt & \leq \int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)[t|f''(a)|^q + (1-t)|f''(b)|^q]dt \\ & \quad + \int_{1-\lambda}^1 (1-t)(t+\lambda-1)[t|f''(a)|^q + (1-t)|f''(b)|^q]dt \\ & = \left[\frac{1+\lambda}{6}(1-\lambda)^3 + \frac{48\lambda-27}{3 \times 2^6} \right] |f''(a)|^q \\ & \quad + \left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(b)|^q, \end{aligned} \tag{2.14}$$

$$\int_0^{\frac{1}{2}} |t(t-\lambda)|dt = \int_0^\lambda t(\lambda-t)dt + \int_\lambda^{\frac{1}{2}} t(t-\lambda)dt = \frac{\lambda^3}{3} + \frac{1-3\lambda}{24}, \tag{2.15}$$

and

$$\int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|dt = \int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)dt + \int_{1-\lambda}^1 (1-t)(t+\lambda-1)dt = \frac{\lambda^3}{3} + \frac{1-3\lambda}{24}. \tag{2.16}$$

Thus, using (2.13)–(2.16) in (2.12), we obtain the first inequality of (2.11).

Now, let $\frac{1}{2} \leq \lambda \leq 1$, then, using the convexity of $|f''|^q$, we have

$$\begin{aligned} \int_0^{\frac{1}{2}} |t(t-\lambda)||f''(ta+(1-t)b)|^q dt & \leq \int_0^{\frac{1}{2}} |t(t-\lambda)|[t|f''(a)|^q + (1-t)|f''(b)|^q]dt \\ & = \int_0^{\frac{1}{2}} t(\lambda-t)[t|f''(a)|^q + (1-t)|f''(b)|^q]dt \\ & = \frac{8\lambda-3}{3 \times 2^6} |f''(a)|^q + \frac{16\lambda-5}{3 \times 2^6} |f''(b)|^q, \end{aligned} \tag{2.17}$$

and similarly,

$$\begin{aligned} \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|f''(ta+(1-t)b)|^q dt &\leq \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|[t|f''(a)|^q + (1-t)|f''(b)|^q] dt \\ &= \int_{\frac{1}{2}}^1 (1-t)(t+\lambda-1)[t|f''(a)|^q + (1-t)|f''(b)|^q] dt \\ &= \frac{16\lambda-5}{3.2^6} |f''(a)|^q + \frac{8\lambda-3}{3.2^6} |f''(b)|^q, \end{aligned} \quad (2.18)$$

and so,

$$\int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)| dt = \int_0^{\frac{1}{2}} |t(t-\lambda)| dt = \int_0^{\frac{1}{2}} t(\lambda-t) dt = \frac{3\lambda-1}{24}. \quad (2.19)$$

Thus, if we use (2.17)–(2.19) in (2.12), we obtain the second inequality of (2.11). This completes the proof. \square

3. Applications to quadrature formulas

In this section we point out some particular inequalities which generalize some classical results such as: trapezoid inequality, Simpson's inequality, midpoint inequality and others.

Proposition 1 (Midpoint Inequality). Under the assumptions Theorem 4 with $\lambda = 0$ in Theorem 4, then we get the following inequality,

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

Proposition 2 (Trapezoid Inequality). Under the assumptions Theorem 4 with $\lambda = 1$ in Theorem 4, then we have

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

Proposition 3 (Simpson Inequality). Under the assumptions Theorem 4 with $\lambda = \frac{1}{3}$ in Theorem 4, then we get

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{162} [|f''(a)| + |f''(b)|].$$

Proposition 4. Under the assumptions Theorem 4 with $\lambda = \frac{1}{2}$ in Theorem 4, then we get

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] \right| \leq \frac{(b-a)^2}{48} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

Proposition 5. Under assumptions Theorem 5 with $\lambda = 0$ in Theorem 5, then we get the following “midpoint inequality”,

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{48} \left[\left(\frac{3|f''(a)|^q + 5|f''(b)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{5|f''(a)|^q + 3|f''(b)|^q}{8} \right)^{\frac{1}{q}} \right].$$

Proposition 6. Under assumptions Theorem 5 with $\lambda = 1$ in Theorem 5, then we get a “trapezoid inequality”

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^2}{24} \left[\left(\frac{5|f''(a)|^q + 11|f''(b)|^q}{16} \right)^{\frac{1}{q}} + \left(\frac{11|f''(a)|^q + 5|f''(b)|^q}{16} \right)^{\frac{1}{q}} \right].$$

Proposition 7. Under assumptions Theorem 5 with $\lambda = \frac{1}{3}$ in Theorem 5, then we get a “Simpson inequality”

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{162} \left[\left(\frac{59|f''(a)|^q + 133|f''(b)|^q}{3 \times 2^6} \right)^{\frac{1}{q}} + \left(\frac{133|f''(a)|^q + 59|f''(b)|^q}{3 \times 2^6} \right)^{\frac{1}{q}} \right].$$

4. Applications to special means

We shall consider the following special means:

- (a) The arithmetic mean: $A = A(a, b) := \frac{a+b}{2}, a, b \geq 0,$
- (b) The geometric mean: $G = G(a, b) := \sqrt{ab}, a, b \geq 0,$
- (c) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0,$$

- (d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a, b > 0.$$

- (e) The Identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \quad a, b > 0.$$

- (f) The p -logarithmic mean

$$L_p = L_p(a, b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b, \\ a & \text{if } a = b \end{cases} \quad p \in \mathbb{R} \setminus \{-1, 0\}; a, b > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 3, some new inequalities are derived for the above means.

Proposition 8. Let $a, b \in \mathbb{R}, 0 < a < b$ and $n \in \mathbb{N}, n > 2$. Then, we have

$$\left| \frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a, b) - L_n^n(a, b) \right| \leq n(n-1) \frac{(b-a)^2}{162} [a^{n-2} + b^{n-2}].$$

Proof. The assertion follows from Proposition 3 applied to convex mapping $f(x) = x^n, x \in \mathbb{R}$. \square

Proposition 9. Let $a, b \in \mathbb{R}, 0 < a < b$. Then, for all $q \geq 1$, we have

$$\left| L^{-1}(a, b) - A^{-1}(a, b) \right| \leq n(n-1) \frac{(b-a)^2}{48} \left\{ \left(\frac{3a^{(n-2)q} + 5b^{(n-2)q}}{8} \right)^{\frac{1}{q}} + \left(\frac{5a^{(n-2)q} + 3b^{(n-2)q}}{8} \right)^{\frac{1}{q}} \right\}$$

and

$$\left| L^{-1}(a, b) - H^{-1}(a, b) \right| \leq n(n-1) \frac{(b-a)^2}{24} \left\{ \left(\frac{5a^{(n-2)q} + 11b^{(n-2)q}}{16} \right)^{\frac{1}{q}} + \left(\frac{11a^{(n-2)q} + 5b^{(n-2)q}}{16} \right)^{\frac{1}{q}} \right\}.$$

Proof. The assertion follows from Propositions 5 and 6 applied to the convex mapping $f(x) = 1/x, x \in [a, b]$, respectively. \square

Proposition 10. Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, we have

$$\left| \frac{1}{3}H^{-1}(a, b) + \frac{2}{3}A^{-1}(a, b) - L^{-1}(a, b) \right| \leq \frac{(b-a)^2}{162} \left\{ \left(\frac{59}{3 \times 2^6} \left| \frac{2}{b^3} \right|^q + \frac{133}{3 \times 2^6} \left| \frac{2}{a^3} \right|^q \right)^{\frac{1}{q}} + \left(\frac{133}{3 \times 2^6} \left| \frac{2}{b^3} \right|^q + \frac{59}{3 \times 2^6} \left| \frac{2}{a^3} \right|^q \right)^{\frac{1}{q}} \right\}.$$

Proof. The assertion follows from Proposition 7 applied to the convex mapping $f(x) = 1/x$, $x \in [a, b]$. \square

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