



On generalized fractional integral inequalities for twice differentiable convex functions

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ABSTRACT

In this article, some new generalized fractional integral inequalities of midpoint and trapezoid type for twice differentiable convex functions are obtained. In view of this, we obtain new integral inequalities of midpoint and trapezoid type for twice differentiable convex functions in a form classical integral and Riemann–Liouville fractional integrals. Finally, we apply our new inequalities to construct inequalities involving moments of a continuous random variable.

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1. Introduction

A function $g : \mathcal{J} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on the interval $\mathcal{J}$, if the inequality

$$g(\zeta x + (1 - \zeta)y) \leq \zeta g(x) + (1 - \zeta)g(y) \quad (1)$$

holds for all $x, y \in \mathcal{J}$ and $\zeta \in [0, 1]$. We say that g is concave if $-g$ is convex.

For differentiable convex functions, many equalities or inequalities have been established by many authors; for example Hermite–Hadamard type inequality [1], Ostrowski type inequality [2], Hardy type inequality [3], Olsen type inequality [4] and Gagliardo–Nirenberg type inequality [5]. But for twice differentiable convex functions, the most important inequalities are midpoint type inequality [6] and trapezoidal type inequality [7] which are defined by:

$$\left| \int_u^v g(x) dx - (v - u)g\left(\frac{u + v}{2}\right) \right| \leq \frac{(v - u)^3}{24} \|g''\|_{\infty}, \quad (2)$$

$$\left| \int_u^v g(x) dx - \frac{v - u}{2} [g(u) + g(v)] \right| \leq \frac{(v - u)^3}{12} \|g''\|_{\infty}, \quad (3)$$

respectively, where $g : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be twice differentiable function on an open interval (u, v) with the second derivative bounded on (u, v) . That is, $\|g''\|_{\infty} := \sup_{x \in (u, v)} |g''(x)| < \infty$.

For more details and applications on inequalities (2) and (3), see [6–9].

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Now, we recall some necessary definitions and mathematical preliminaries of the generalized fractional integrals which are defined by Sarikaya and Ertugral [10].

Let $\vartheta : [0, \infty) \rightarrow [0, \infty)$ which satisfies the following conditions:

$$\int_0^1 \frac{\vartheta(\zeta)}{\zeta} d\zeta < \infty.$$

We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$${}_{u+}\mathbf{I}_{\vartheta}g(x) = \int_u^x \frac{\vartheta(x-\zeta)}{x-\zeta} g(\zeta) d\zeta, \quad x > u, \quad (4)$$

$${}_{v-}\mathbf{I}_{\vartheta}g(x) = \int_x^v \frac{\vartheta(\zeta-x)}{\zeta-x} g(\zeta) d\zeta, \quad x < v. \quad (5)$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral [1,11,12], k -Riemann–Liouville fractional integral [13], Katugampola fractional integrals [14,15], conformable fractional integral [16], Hadamard fractional integrals [6], and so on. These important special cases of the integral operators (4) and (5) are mentioned below:

(a) if $\vartheta(\zeta) = \zeta$, the operators (4) and (5) reduce to the Riemann integral:

$$I_{u+}g(x) = \int_u^x g(\zeta) d\zeta, \quad x > u,$$

$$I_{v-}g(x) = \int_x^v g(\zeta) d\zeta, \quad x < v.$$

(b) if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, the operators (4) and (5) reduce to the Riemann–Liouville fractional integral [6]:

$$I_{u+}g(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-\zeta)^{\alpha-1} g(\zeta) d\zeta, \quad x > u,$$

$$I_{v-}g(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (\zeta-x)^{\alpha-1} g(\zeta) d\zeta, \quad x < v.$$

(c) if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, the operators (4) and (5) reduce to the k -Riemann–Liouville fractional integral:

$$I_{u+,k}g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_u^x (x-\zeta)^{\frac{\alpha}{k}-1} g(\zeta) d\zeta, \quad x > u,$$

$$I_{v-,k}g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^v (\zeta-x)^{\frac{\alpha}{k}-1} g(\zeta) d\zeta, \quad x < v,$$

where

$$\Gamma_k(\alpha) = \int_0^\infty \zeta^{\alpha-1} e^{-\frac{\zeta^k}{k}} d\zeta, \quad \Re(\alpha) > 0,$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \Re(\alpha) > 0; k > 0$$

which are given by Mubeen and Habibullah in [17].

Recently, in [10], Sarikaya and Ertugra established the following generalized fractional integral inequalities of trapezoid type for differentiable convex function:

Theorem 1.1. Let $g : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on (u, v) with $u < v$. If $|g'|$ is convex on $[u, v]$. Then

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Lambda(1)} [{}_{u+}\mathbf{I}_{\vartheta}g(v) + {}_{v-}\mathbf{I}_{\vartheta}g(u)] \right| \\ \leq \frac{v-u}{\Lambda(1)} \int_0^1 \zeta |\Lambda(1-\zeta) - \Lambda(\zeta)| d\zeta \left(\frac{|g'(u)| + |g'(v)|}{2} \right),$$

where

$$\Lambda(\zeta) = \int_0^\zeta \frac{\vartheta((v-u)z)}{z} dz < \infty.$$

Theorem 1.2. Let $g : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on (u, v) with $u < v$. If $|g'|^q$ is convex on $[u, v]$. Then

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Lambda(1)} [{}_{u^+}\mathbf{I}_\vartheta g(v) + {}_{v^-}\mathbf{I}_\vartheta g(u)] \right| \leq \frac{v-u}{2\Lambda(1)} \left(\int_0^1 |\Lambda(1-\zeta) - \Lambda(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\frac{|g'(u)|^q + |g'(v)|^q}{2} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

Furthermore, in [18], Ertugral and Sarikaya established the following generalized fractional integral inequalities of trapezoid type for differentiable convex function:

Theorem 1.3. Let $g : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous function on $[u, v]$ such that $g' \in L_1([u, v])$, where $u, v \in [u, v]$ with $u < v$. If the function $|g'|$ is convex on $[u, v]$. Then

$$\left| \frac{\nabla(0)g(v) + \Delta(0)g(u)}{v-u} - \frac{1}{v-u} [{}_{u^+}\mathbf{I}_\vartheta g(v) + {}_{v^-}\mathbf{I}_\vartheta g(u)] \right| \leq \frac{v-x}{v-u} |g'(x)| \int_0^1 |\nabla(\zeta)| \zeta d\zeta + \frac{x-u}{v-u} |g'(x)| \int_0^1 |\Delta(\zeta)| \zeta d\zeta + \frac{v-x}{v-u} |g'(v)| \int_0^1 |\nabla(\zeta)| (1-\zeta) d\zeta + \frac{x-u}{v-u} |g'(u)| \int_0^1 |\Delta(\zeta)| (1-\zeta) d\zeta,$$

where

$$\Delta(\zeta) = \int_\zeta^1 \frac{\vartheta((x-u)z)}{z} dz < \infty, \quad \nabla(\zeta) = \int_\zeta^1 \frac{\vartheta((v-x)z)}{z} dz < \infty.$$

Theorem 1.4. Let $g : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on (u, v) with $u < v$. If the function $|g'|^q$ is convex on $[u, v]$. Then

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Lambda(1)} [{}_{u^+}\mathbf{I}_\vartheta g(v) + {}_{v^-}\mathbf{I}_\vartheta g(u)] \right| \leq \frac{v-x}{v-u} \left(\int_0^1 |\nabla(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\frac{|g'(u)|^q + |g'(v)|^q}{2} \right)^{\frac{1}{q}} + \frac{v-x}{v-u} \left(\int_0^1 |\Delta(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\frac{|g'(u)|^q + |g'(v)|^q}{2} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

Tomar et al. [11] established the following Hermite–Hadamard type inequalities via Riemann–Liouville fractional integrals for twice differentiable functions:

Lemma 1.1. Let $g : \mathfrak{J} \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on the interior $\mathfrak{J}^\circ$ of an interval $\mathfrak{J}$ such that $g'' \in L([u, v])$, where $u, v \in \mathfrak{J}$ with $u < v$. Then for each $h \in (0, 1)$ and $\alpha > 0$ the following equality holds:

$$H_g(g; \alpha, u, v) = \frac{(v-u)^2}{8} \times \left[\int_0^1 \zeta^{\alpha+1} g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) d\zeta + \int_0^1 \zeta^{\alpha+1} g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) d\zeta \right], \quad (6)$$

where

$$H_g(g; \alpha, u, v) = \frac{2^{\alpha-1} \Gamma(\alpha+2)}{(v-u)^\alpha} \left[I_{\left(\frac{u+v}{2}\right)^+}^\alpha g(v) + I_{\left(\frac{u+v}{2}\right)^-}^\alpha g(u) \right] - (\alpha+1) g \left(\frac{u+v}{2} \right).$$

Theorem 1.5. With the similar assumptions of Lemma 1.1, if $|g''|$ is convex on $[u, v]$, then for each $h \in (0, 1)$. Then

$$|H_g(g; \alpha, u, v)| \leq \frac{(v-u)^2}{8(\alpha+2)} (|g''(u)| + |g''(v)|). \quad (7)$$

Theorem 1.6. With the similar assumptions of Lemma 1.1, if $|g''|^q$ is convex on $[u, v]$. Then

$$|H_g(g; \alpha, u, v)| \leq \frac{(v-u)^2}{8(\alpha+2)} \left(\frac{2}{(\alpha+1)p+1} \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|). \quad (8)$$

where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

Theorem 1.7. With the similar assumptions of Lemma 1.1, if $|g''|^q$ is convex on $[u, v]$ for $q > 1$. Then

$$|H_g(g; \alpha, u, v)| \leq \frac{(v-u)^2}{8} \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left[\left(\frac{1}{2(\alpha+3)} |g''(u)|^q + \left(\frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) |g''(v)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left(\left(\frac{1}{\alpha+2} - \frac{1}{2(\alpha+3)} \right) |g''(u)|^q + \frac{1}{2(\alpha+3)} |g''(v)|^q \right)^{\frac{1}{q}} \right].$$

There are many papers which study and give significant results for twice differentiable functions via fractional integrals; for more details see [12,19–23].

In this paper, we established some new midpoint and trapezoid type inequalities for twice differential convex functions via generalized fractional integrals.

2. Midpoint type inequalities for twice differentiable functions with generalized fractional integrals

Here we give midpoint type inequalities for twice differential convex functions via generalized fractional integral operator. To obtain these results, we need the following lemma:

Lemma 2.1. Let $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathcal{I}^0$ such that $g'' \in L([u, v])$, where $u, v \in \mathcal{I}^0$ with $u < v$. Then

$$\left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] - 2\Upsilon(1)g\left(\frac{u+v}{2}\right) \\ = \frac{(v-u)^2}{4} \left[\int_0^1 \Omega(\zeta)g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta + \int_0^1 \Omega(\zeta)g''\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v\right) d\zeta \right], \quad (9)$$

where

$$\Omega(\zeta) = \int_0^\zeta \Upsilon(s)ds, \quad \Upsilon(s) = \int_0^s \frac{\vartheta\left(\left(\frac{v-u}{2}\right)z\right)}{z} dz < \infty. \quad (10)$$

Proof. First, we let

$$\hbar_1 + \hbar_2 := \int_0^1 \Omega(\zeta)g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta + \int_0^1 \Omega(\zeta)g''\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v\right) d\zeta. \quad (11)$$

On integrating by parts twice one can have

$$\hbar_1 = \int_0^1 \Omega(\zeta)g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \\ = \frac{2}{v-u} \left[-\Omega(\zeta)g'\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) \Big|_0^1 + \int_0^1 \Upsilon(\zeta)g'\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \right] \\ = \frac{2}{v-u} \left[-\Omega(1)g'\left(\frac{u+v}{2}\right) + \int_0^1 \Upsilon(\zeta)g'\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \right] \\ = \frac{4}{(v-u)^2} \left[-\frac{v-u}{2} \Omega(1)g'\left(\frac{u+v}{2}\right) - \Upsilon(1)g\left(\frac{u+v}{2}\right) + \int_0^1 \frac{\vartheta\left(\left(\frac{v-u}{2}\right)t\right)}{\zeta} g\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \right].$$

By changing the variable x by $x = \frac{\zeta}{2}u + \frac{2-\zeta}{2}v$, we get

$$\begin{aligned} \hbar_1 &= \frac{4}{(v-u)^2} \left[-\frac{v-u}{2} \Omega(1)g' \left(\frac{u+v}{2} \right) - \gamma(1)g \left(\frac{u+v}{2} \right) + \int_{\frac{u+v}{2}}^v \frac{\vartheta(v-x)}{v-x} g(x) dx \right] \\ &= \frac{4}{(v-u)^2} \left[-\frac{v-u}{2} \Omega(1)g' \left(\frac{u+v}{2} \right) - \gamma(1)g \left(\frac{u+v}{2} \right) + \left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) \right]. \end{aligned}$$

Analogously, we have

$$\begin{aligned} \hbar_2 &= \int_0^1 \Omega(\zeta)g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) d\zeta \\ &= \frac{2}{v-u} \left[\Omega(1)g' \left(\frac{u+v}{2} \right) - \int_0^1 \gamma(\zeta)g' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) d\zeta \right] \\ &= \frac{4}{(v-u)^2} \left[\frac{v-u}{2} \Omega(1)g' \left(\frac{u+v}{2} \right) - \gamma(1)g \left(\frac{u+v}{2} \right) + \int_0^1 \frac{\vartheta \left(\left(\frac{v-u}{2} \right) t \right)}{\zeta} g \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) d\zeta \right] \\ &= \frac{4}{(v-u)^2} \left[\frac{v-u}{2} \Omega(1)g' \left(\frac{u+v}{2} \right) - \gamma(1)g \left(\frac{u+v}{2} \right) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right]. \end{aligned}$$

Using $\hbar_1$ and $\hbar_2$ in (11) and multiplying the result by $\frac{(v-u)^2}{4}$, we get equality (9). $\square$

Remark 2.1. With the similar assumptions of Lemma 2.1, if $\vartheta(\zeta) = \zeta$, then identity (9) reduces to the following identity:

$$\begin{aligned} &\frac{1}{v-u} \int_u^v g(x) dx - g \left(\frac{u+v}{2} \right) \\ &= \frac{(v-u)^2}{16} \left[\int_0^1 \zeta^2 g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) d\zeta + \int_0^1 \zeta^2 g'' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) d\zeta \right] \end{aligned}$$

which is proved by Sarikaya and Kiris in [24].

Remark 2.2. With the similar assumptions of Lemma 2.1, if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then identity (9) reduces to identity (6).

Corollary 2.1. With the similar assumptions of Lemma 2.1, if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we get the following identity:

$$\begin{aligned} &\frac{(v-u)^2}{8} \left[\int_0^1 \zeta^{\frac{\alpha}{k}+1} g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) d\zeta + \int_0^1 \zeta^{\frac{\alpha}{k}+1} g'' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) d\zeta \right] \\ &= \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + 2k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{\left(\frac{u+v}{2} \right)^+, k}^\alpha g(v) + I_{\left(\frac{u+v}{2} \right)^-, k}^\alpha g(u) \right] - (\alpha + k)g \left(\frac{u+v}{2} \right). \end{aligned}$$

Theorem 2.1. Let $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathcal{I}^0$ such that $g'' \in L([u, v])$, where $u, v \in \mathcal{I}^0$ with $u < v$. If the function $|g''|$ is convex on $[u, v]$. Then

$$\begin{aligned} &\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] - 2\gamma(1)g \left(\frac{u+v}{2} \right) \right| \\ &\leq \frac{(v-u)^2}{4} \left(|g''(u)| + |g''(v)| \right) \int_0^1 |\Omega(\zeta)| d\zeta. \end{aligned} \tag{12}$$

Proof. Using equality (9) and the convexity of $|g''|$ on $[u, v]$, we have

$$\begin{aligned} &\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] - 2\gamma(1)g \left(\frac{u+v}{2} \right) \right| \\ &\leq \frac{(v-u)^2}{4} \left| \int_0^1 \Omega(\zeta)g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) d\zeta + \int_0^1 \Omega(\zeta)g'' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) d\zeta \right| \\ &\leq \frac{(v-u)^2}{4} \left[\int_0^1 |\Omega(\zeta)| \left| g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) \right| d\zeta + \int_0^1 |\Omega(\zeta)| \left| g'' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) \right| d\zeta \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(v-u)^2}{4} \left[\int_0^1 |\Omega(\zeta)| \left(\frac{\zeta}{2} |g''(u)| + \frac{2-\zeta}{2} |g''(v)| \right) d\zeta + \int_0^1 |\Omega(\zeta)| \left(\frac{2-\zeta}{2} |g''(u)| + \frac{\zeta}{2} |g''(v)| \right) d\zeta \right] \\
&\leq \frac{(v-u)^2}{8} \left[|g''(u)| \int_0^1 |\Omega(\zeta)| t d\zeta + |g''(v)| \int_0^1 |\Omega(\zeta)| (2-\zeta) d\zeta \right. \\
&\quad \left. + |g''(u)| \int_0^1 |\Omega(\zeta)| (2-\zeta) d\zeta + |g''(v)| \int_0^1 |\Omega(\zeta)| t d\zeta \right] \\
&= \frac{(v-u)^2}{4} (|g''(u)| + |g''(v)|) \int_0^1 |\Omega(\zeta)| d\zeta.
\end{aligned}$$

Hence the proof is completed. $\square$

Remark 2.3. With the similar assumptions of Theorem 2.1, if $\vartheta(\zeta) = \zeta$, then inequality (12) reduces to the following inequality:

$$\left| \frac{1}{v-u} \int_u^v g(x) dx - \left(\frac{u+v}{2} \right) \right| \leq \frac{(v-u)^2}{24} \left[\frac{|g''(u)| + |g''(v)|}{2} \right]$$

which is proved by Sarikaya and Kiris in [24].

Remark 2.4. With the similar assumptions of Theorem 2.1, if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then inequality (12) reduces to the inequality (7).

Corollary 2.2. With the similar assumptions of Theorem 2.1, if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$. Then

$$\begin{aligned}
&\left| \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + 2k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{(\frac{u+v}{2})^+, k}^\alpha g(v) + I_{(\frac{u+v}{2})^-, k}^\alpha g(u) \right] - (\alpha + k) g\left(\frac{u+v}{2}\right) \right| \\
&\leq \frac{(v-u)^2}{8(\alpha + 2k)} (|g''(u)| + |g''(v)|).
\end{aligned}$$

Theorem 2.2. Let $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathcal{I}^o$ such that $g'' \in L([u, v])$, where $u, v \in \mathcal{I}^o$ with $u < v$. If the function $|g''|^q$ is convex on $[u, v]$. Then

$$\begin{aligned}
&\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_\vartheta g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_\vartheta g(u) \right] - 2\Upsilon(1) g\left(\frac{u+v}{2}\right) \right| \\
&\leq \frac{(v-u)^2}{4} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\
&\leq \frac{(v-u)^2}{2^{\frac{2}{q}}} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|),
\end{aligned} \tag{13}$$

where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

Proof. Using equality (9), the convexity of $|g''|^q$ on $[u, v]$ and Hölder's inequality, we have

$$\begin{aligned}
&\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_\vartheta g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_\vartheta g(u) \right] - 2\Upsilon(1) g\left(\frac{u+v}{2}\right) \right| \\
&\leq \frac{(v-u)^2}{4} \left\{ \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) \right|^q d\zeta \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) \right|^q d\zeta \right)^{\frac{1}{q}} \right\} \\
&\leq \frac{(v-u)^2}{4} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\int_0^1 \left(\frac{\zeta}{2} |g''(u)|^q + \frac{2-\zeta}{2} |g''(v)|^q \right) d\zeta \right)^{\frac{1}{q}} \right.
\end{aligned}$$

$$+ \left(\int_0^1 \left(\frac{2-\zeta}{2} |g''(u)|^q + \frac{\zeta}{2} |g''(v)|^q \right) d\zeta \right)^{\frac{1}{q}} \Bigg\} \\ \leq \frac{(v-u)^2}{4} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\}.$$

Let $c_1 = |g''(u)|^q$, $c_2 = 3|g''(u)|^q$, $d_1 = 3|g''(v)|^q$, $d_2 = |g''(v)|^q$, then using the fact that

$$\sum_{i=1}^n (c_i + d_i)^m \leq \sum_{i=1}^n c_i^m + \sum_{i=1}^n d_i^m, \quad 0 \leq m < 1,$$

and $1 + 3^{\frac{1}{q}} \leq 4$, we have

$$\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] - 2\gamma(1)g\left(\frac{u+v}{2}\right) \right| \\ \leq \frac{(v-u)^2}{2^{\frac{2}{q}}} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|).$$

This completes the proof of (13). $\square$

Remark 2.5. With the similar assumptions of Theorem 2.2, if $\vartheta(\zeta) = \zeta$, then inequalities (13) reduce to the following inequalities:

$$\left| \frac{1}{v-u} \int_u^v g(x) dx g\left(\frac{u+v}{2}\right) \right| \\ \leq \frac{(v-u)^2}{16(2p+1)^{\frac{1}{p}}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ \leq \frac{(v-u)^2}{2^{2+\frac{1}{q}}(2p+1)^{\frac{1}{p}}} (|g''(u)| + |g''(v)|)$$

which are proved by Sarikaya and Kiris in [24].

Remark 2.6. With the similar assumptions of Theorem 2.2, if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then inequalities (13) reduce to the inequalities (8).

Corollary 2.3. With the similar assumptions of Theorem 2.2, if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$. Then

$$\left| \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha+2k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{\left(\frac{u+v}{2}\right)^+, k}^{\alpha} g(v) + I_{\left(\frac{u+v}{2}\right)^-, k}^{\alpha} g(u) \right] - (\alpha+k)g\left(\frac{u+v}{2}\right) \right| \\ \leq \frac{(v-u)^2}{8} \left(\frac{k}{(\alpha+k)p+k} \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ \leq \frac{(v-u)^2}{2^{\frac{3}{q}}} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|).$$

3. Trapezoid type inequalities for twice differentiable functions with generalized fractional integrals

Here we give trapezoid type inequalities for twice differential convex functions via generalized fractional integral operator. To obtain these results, we need the following lemma:

Lemma 3.1. Let $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathcal{I}^o$ such that $g'' \in L([u, v])$, where $u, v \in \mathcal{I}^o$ with $u < v$. Then

$$\frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] \\ = \frac{(v-u)^2}{8\Phi(0)} \left[\int_0^1 \Omega(\zeta) g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) d\zeta + \int_0^1 \Omega(\zeta) g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) d\zeta \right]. \quad (14)$$

where the functions $\Omega(\zeta)$ and $\Phi(\zeta)$ are defined by

$$\Omega(\zeta) = \int_0^\zeta \Phi(s) ds, \quad \Phi(s) = \int_s^1 \frac{\vartheta\left(\frac{v-u}{2}z\right)}{z} dz < \infty. \quad (15)$$

Proof. On integrating by parts twice one can have

$$\begin{aligned} \mathcal{J}_1 &= \int_0^1 \Omega(\zeta) g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \\ &= -\frac{2}{v-u} \Omega(\zeta) g'\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) \Big|_0^1 + \frac{2}{v-u} \int_0^1 \Phi(\zeta) g'\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \\ &= -\frac{2}{v-u} \Omega(1) g'\left(\frac{u+v}{2}\right) + \frac{2}{v-u} \\ &\quad \times \left[-\frac{2}{v-u} \Phi(\zeta) g\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) \Big|_0^1 - \frac{2}{v-u} \int_0^1 \frac{\vartheta\left(\frac{v-u}{2}t\right)}{\zeta} g\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \right] \\ &= -\frac{2}{v-u} \Omega(1) g'\left(\frac{u+v}{2}\right) + \frac{4\Phi(0)}{(v-u)^2} g(v) - \frac{4}{(v-u)^2} \int_0^1 \frac{\vartheta\left(\frac{v-u}{2}t\right)}{\zeta} g\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta. \end{aligned}$$

By changing the variable x by $x = \frac{\zeta}{2}u + \frac{2-\zeta}{2}v$, we get

$$\begin{aligned} \mathcal{J}_1 &= -\frac{2}{v-u} \Omega(1) g'\left(\frac{u+v}{2}\right) + \frac{4\Phi(0)}{(v-u)^2} g(v) - \frac{4}{(v-u)^2} \int_{\frac{u+v}{2}}^v \frac{\vartheta(v-x)}{v-x} g(x) dx \\ &= -\frac{2}{v-u} \Omega(1) g'\left(\frac{u+v}{2}\right) + \frac{4\Phi(0)}{(v-u)^2} g(v) - \frac{4}{(v-u)^2} \left(\frac{u+v}{2}\right)^+ \mathbf{I}_\vartheta g(v). \end{aligned}$$

Analogously, we have

$$\begin{aligned} \mathcal{J}_2 &= \int_0^1 \Omega(\zeta) g''\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v\right) d\zeta \\ &= \frac{2}{v-u} \Omega(1) g'\left(\frac{u+v}{2}\right) + \frac{4\Phi(0)}{(v-u)^2} g(v) - \frac{4}{(v-u)^2} \left(\frac{u+v}{2}\right)^- \mathbf{I}_\vartheta g(u). \end{aligned}$$

From $\mathcal{J}_1$ and $\mathcal{J}_2$, we get

$$\frac{(v-u)^2}{8\Phi(0)} (\mathcal{J}_1 + \mathcal{J}_2) = \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2}\right)^+ \mathbf{I}_\vartheta g(v) + \left(\frac{u+v}{2}\right)^- \mathbf{I}_\vartheta g(u) \right],$$

which completes the proof of (14). $\square$

Corollary 3.1. With the similar assumptions of Lemma 3.1 if $\vartheta(\zeta) = \zeta$, we have the following important identity:

$$\begin{aligned} &\frac{g(u) + g(v)}{2} - \frac{1}{v-u} \int_u^v \left(\zeta - \frac{1}{2}\zeta^2\right) g(\zeta) d\zeta \\ &= \frac{(v-u)^2}{8} \left[\int_0^1 \left(\zeta - \frac{1}{2}\zeta^2\right) g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta + \int_0^1 \left(\zeta - \frac{1}{2}\zeta^2\right) g''\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v\right) d\zeta \right]. \end{aligned}$$

Corollary 3.2. With the similar assumptions of Lemma 3.1 if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then for $x = \frac{u+v}{2}$, we get

$$\begin{aligned} &\frac{g(u) + g(v)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(v-u)^\alpha} \left[I_{\left(\frac{u+v}{2}\right)^+}^\alpha g(v) + I_{\left(\frac{u+v}{2}\right)^-}^\alpha g(u) \right] \\ &= \frac{(v-u)^2}{8} \left[\int_0^1 \left(\zeta - \frac{1}{\alpha+1} \zeta^{\alpha+1}\right) g''\left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v\right) d\zeta \right. \\ &\quad \left. + \int_0^1 \left(\zeta - \frac{1}{\alpha+1} \zeta^{\alpha+1}\right) g''\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v\right) d\zeta \right]. \end{aligned}$$

Corollary 3.3. With the similar assumptions of Lemma 3.1 if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then

$$\begin{aligned} & \frac{g(u) + g(v)}{2} - \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{(\frac{u+v}{2})^+}^{\alpha,k} g(v) + I_{(\frac{u+v}{2})^-}^{\alpha,k} g(u) \right] \\ &= \frac{(v-u)^2}{8} \left[\int_0^1 \left(\zeta - \frac{k}{\alpha+k} \zeta^{\frac{\alpha}{k}+1} \right) g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) d\zeta \right. \\ & \quad \left. + \int_0^1 \left(\zeta - \frac{k}{\alpha+k} \zeta^{\frac{\alpha}{k}+1} \right) g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) d\zeta \right]. \end{aligned}$$

Theorem 3.1. Let $g : \mathfrak{J} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathfrak{J}^\circ$ such that $g'' \in L([u, v])$, where $u, v \in \mathfrak{J}^\circ$ with $u < v$. If the function $|g''|$ is convex on $[u, v]$, then

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[I_{(\frac{u+v}{2})^+} g(v) + I_{(\frac{u+v}{2})^-} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left(\int_0^1 |\Omega(\zeta)| d\zeta \right) (|g''(u)| + |g''(v)|). \end{aligned} \quad (16)$$

Proof. Using equality (14) and the convexity of $|g''|$ on $[u, v]$, we have

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[I_{(\frac{u+v}{2})^+} g(v) + I_{(\frac{u+v}{2})^-} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left| \int_0^1 \Omega(\zeta) g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) d\zeta + \int_0^1 \Omega(\zeta) g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) d\zeta \right| \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left[\int_0^1 |\Omega(\zeta)| \left| g'' \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) \right| d\zeta + \int_0^1 |\Omega(\zeta)| \left| g'' \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) \right| d\zeta \right] \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left[\int_0^1 |\Omega(\zeta)| \left(\frac{\zeta}{2} |g''(u)| + \frac{2-\zeta}{2} |g''(v)| \right) d\zeta + \int_0^1 |\Omega(\zeta)| \left(\frac{2-\zeta}{2} |g''(u)| + \frac{\zeta}{2} |g''(v)| \right) d\zeta \right] \\ & = \frac{(v-u)^2}{8\Phi(0)} \left(\int_0^1 |\Omega(\zeta)| d\zeta \right) (|g''(u)| + |g''(v)|), \end{aligned}$$

which completes the proof of (16). $\square$

Corollary 3.4. With the similar assumptions of Theorem 3.1 if $\vartheta(\zeta) = \zeta$, we have

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{v-u} \int_u^v g(\zeta) d\zeta \right| \leq \frac{(v-u)^2}{24} (|g''(u)| + |g''(v)|).$$

Corollary 3.5. With the similar assumptions of Theorem 3.1 if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then for $x = \frac{u+v}{2}$, we have

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(v-u)^\alpha} \left[I_{(\frac{u+v}{2})^+}^\alpha g(v) + I_{(\frac{u+v}{2})^-}^\alpha g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8} \left(\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \right) (|g''(u)| + |g''(v)|). \end{aligned}$$

Corollary 3.6. With the similar assumptions of Theorem 3.1 if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, we have

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha + k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{(\frac{u+v}{2})^+}^{\alpha,k} g(v) + I_{(\frac{u+v}{2})^-}^{\alpha,k} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8} \left(\frac{\alpha(\alpha+3k)}{2(\alpha+k)(\alpha+2k)} \right) (|g''(u)| + |g''(v)|). \end{aligned}$$

Theorem 3.2. Let $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on $\mathcal{I}^0$ such that $g'' \in L([u, v])$, where $u, v \in \mathcal{I}^0$ with $u < v$. If the function $|g''|^q$ is convex on $[u, v]$, then

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{2^{\frac{3}{q}} \Phi(0)} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|), \end{aligned} \quad (17)$$

where $\frac{1}{p} + \frac{1}{q} = 1, q > 1$.

Proof. Using equality (14), the convexity of $|g''|^q, q > 1$ and Hölder's inequality, we get

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left\{ \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| g'' \left(\frac{\zeta}{2}u + \frac{2-\zeta}{2}v \right) \right|^q d\zeta \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left(\int_0^1 \left| g'' \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}v \right) \right|^q d\zeta \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\int_0^1 \left(\frac{\zeta}{2}|g''(u)|^q + \frac{2-\zeta}{2}|g''(v)|^q \right) d\zeta \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left(\frac{2-\zeta}{2}|g''(u)|^q + \frac{\zeta}{2}|g''(v)|^q \right) d\zeta \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{8\Phi(0)} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Let $c_1 = |g''(u)|^q, c_2 = 3|g''(u)|^q, d_1 = 3|g''(v)|^q, d_2 = |g''(v)|^q$, then using the fact that

$$\sum_{i=1}^n (c_i + d_i)^m \leq \sum_{i=1}^n c_i^m + \sum_{i=1}^n d_i^m, \quad 0 \leq m < 1,$$

and $1 + 3^{\frac{1}{q}} \leq 4$, we get

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} g(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{2^{\frac{3}{q}} \Phi(0)} \left(\int_0^1 |\Omega(\zeta)|^p d\zeta \right)^{\frac{1}{p}} (|g''(u)| + |g''(v)|). \end{aligned}$$

This ends the proof of (17). $\square$

Corollary 3.7. With the similar assumptions of Theorem 3.2 if $\vartheta(\zeta) = \zeta$, we get the following inequality:

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{1}{v-u} \int_u^v \left(\zeta - \frac{1}{2}\zeta^2 \right) g(\zeta) d\zeta \right| \\ & \leq \frac{(v-u)^2}{8} \left(\frac{1}{p+1} - \frac{1}{3} \left(\frac{1}{2} \right)^p \right)^{\frac{1}{p}} \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{2^{\frac{3}{q}}} \left(\frac{1}{p+1} - \frac{1}{3} \left(\frac{1}{2} \right)^p \right)^{\frac{1}{p}} (|g''(u)|^q + |g''(v)|^q). \end{aligned}$$

Corollary 3.8. Under assumption of Theorem 3.2 if $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then for $x = \frac{u+v}{2}$, we get

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(v-u)^\alpha} \left[I_{(\frac{u+v}{2})^+}^\alpha g(v) + I_{(\frac{u+v}{2})^-}^\alpha g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8} \left(\frac{1}{p+1} - \left(\frac{1}{\alpha+1} \right)^p \left(\frac{1}{(\alpha+1)p+1} \right) \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{2^{\frac{3}{q}}} \left(\frac{1}{p+1} - \left(\frac{1}{\alpha+1} \right)^p \left(\frac{1}{(\alpha+1)p+1} \right) \right)^{\frac{1}{p}} (|g''(u)|^q + |g''(v)|^q). \end{aligned}$$

Corollary 3.9. With the similar assumptions of Theorem 3.2 if $\vartheta(\zeta) = \frac{\zeta^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we get

$$\begin{aligned} & \left| \frac{g(u) + g(v)}{2} - \frac{2^{\frac{\alpha}{k}-1} \Gamma_k(\alpha+k)}{(v-u)^{\frac{\alpha}{k}}} \left[I_{(\frac{u+v}{2})^+}^{\alpha,k} g(v) + I_{(\frac{u+v}{2})^-}^{\alpha,k} g(u) \right] \right| \\ & \leq \frac{(v-u)^2}{8} \left(\frac{1}{p+1} - \left(\frac{k}{\alpha+k} \right)^p \left(\frac{k}{(\alpha+k)p+1} \right) \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\frac{|g''(u)|^q + 3|g''(v)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|g''(u)|^q + |g''(v)|^q}{4} \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(v-u)^2}{2^{\frac{3}{q}}} \left(\frac{1}{p+1} - \left(\frac{k}{\alpha+k} \right)^p \left(\frac{k}{(\alpha+k)p+1} \right) \right)^{\frac{1}{p}} (|g''(u)|^q + |g''(v)|^q). \end{aligned}$$

4. Application: Moments of the random variable

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied by many authors, for more details see [25–32].

Let χ be a random variable whose probability function is $h: \mathcal{J} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathcal{J}$ such that $u, v \in \mathcal{J}$ with $u < v$. Denote by $M_r(x)$ the r th moment about any arbitrary point x of the random variable χ , $r \geq 0$, defined as

$$M_r(x) = \int_u^v (z-x)^r h(z) dz, \quad r = 0, 1, 2, \dots \quad (18)$$

In view of (18), we obtain the following propositions:

Proposition 4.1. Let χ be a random variable whose probability function is $h: \mathcal{J} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathcal{J}$ such that $u, v \in \mathcal{J}$ with $u < v$. Then

$$\begin{aligned} & \left[I_{(\frac{u+v}{2})^+}^\vartheta M_r(v) + I_{(\frac{u+v}{2})^-}^\vartheta M_r(u) \right] - 2\Upsilon(1) M_r\left(\frac{u+v}{2}\right) \\ & = \frac{r(r-1)(v-u)}{4} \left[\int_0^1 \Omega(\zeta) M_{r-2}\left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v\right) d\zeta + \int_0^1 \Omega(\zeta) M_{r-2}\left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b\right) d\zeta \right] \\ & = \frac{r(r-1)(v-u)}{4} \left\{ \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{r-2} h(z) dz \right) d\zeta \right. \\ & \quad \left. + \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{r-2} h(z) dz \right) d\zeta \right\} \end{aligned} \quad (19)$$

for $r \geq 2k$, $k = 1, 2, \dots$

Proof. Proof of this proposition follows from setting $g(x) = M_r(x)$ in the equality (9). $\square$

Proposition 4.2. Let χ be a random variable whose probability function is $h : \mathfrak{I} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathfrak{I}$ such that $u, v \in \mathfrak{I}$ with $u < v$. If the function $|h|$ is bounded, then

$$\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] - 2\Upsilon(1)M_r \left(\frac{u+v}{2} \right) \right| \\ \leq \frac{r(1+(-1)^r)(v-u)^r \|h\|_{\infty}}{2^{r+1}} \int_0^1 \Omega(\zeta) (\zeta^{r-1} - (\zeta-2)^{r-1}) d\zeta$$

for $r \geq 2k, k = 1, 2, \dots$

Proof. By using Proposition 4.1 and since $|h|$ is a bounded function, we have

$$\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] - 2\Upsilon(1)M_r \left(\frac{u+v}{2} \right) \right| \\ \leq \frac{r(r-1)(v-u)}{4} \left\{ \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{r-2} |h(z)| dz \right) d\zeta \right. \\ \left. + \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{r-2} |h(z)| dz \right) d\zeta \right\} \\ \leq \frac{r(1+(-1)^r)(v-u)^r \|h\|_{\infty}}{2^{r+1}} \int_0^1 \Omega(\zeta) (\zeta^{r-1} - (\zeta-2)^{r-1}) d\zeta.$$

This completes the proof. $\square$

Corollary 4.1. Under the similar assumptions of Proposition 4.2, if

1. $\vartheta(\zeta) = \zeta$, then

$$\left| \frac{1}{v-u} \int_u^v g(\zeta) d\zeta - f \left(\frac{u+v}{2} \right) \right| \\ \leq \frac{r(1+(-1)^r)(v-u)^{r+1} \|h\|_{\infty}}{2^{r+4}} \left(\frac{1}{r+2} + (-1)^r 2^{r+2} \beta_{\frac{1}{2}}(3, r) \right).$$

2. $\vartheta(\zeta) = \frac{\zeta^{\alpha}}{\Gamma(\alpha)}$, then for $x = \frac{u+v}{2}$, we have

$$\left| \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(v-u)^{\alpha}} \left[I_{\left(\frac{u+v}{2} \right)^+}^{\alpha} g(v) + I_{\left(\frac{u+v}{2} \right)^-}^{\alpha} g(u) \right] - (\alpha+1) f \left(\frac{u+v}{2} \right) \right| \\ \leq \frac{r(1+(-1)^r)(v-u)^{\alpha+r} \|h\|_{\infty}}{2^{\alpha+r+2} \Gamma(\alpha+2)} \left(\frac{1}{\alpha+r+1} + (-1)^r 2^{\alpha+r+1} \beta_{\frac{1}{2}}(\alpha+2, r) \right).$$

Proposition 4.3. Let χ be a random variable whose probability function is $h : \mathfrak{I} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathfrak{I}$ such that $u, v \in \mathfrak{I}$ with $u < v$ and $h \in L_p[u, v], p > 1$. If the function $|h|$ is bounded, then

$$\left| \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] - 2\Upsilon(1)M_r \left(\frac{u+v}{2} \right) \right| \leq r(r-1) \left(1 + (-1)^{\frac{(r-2)q+1}{q}} \right) \|h\|_p \\ \times \left[\frac{(v-u)^{(r-1)q+1}}{2^{rq+1}((r-2)q+1)} \right]^{\frac{1}{q}} \int_0^1 \Omega(\zeta) (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1})^{\frac{1}{q}} d\zeta$$

for $r \geq 2k, k = 1, 2, \dots$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Since $h \in L_p[u, v]$, then by Holder's integral inequality we have

$$\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{r-2} h(z) dz \\ \leq \left(\int_u^v g^p(z) dz \right)^{\frac{1}{p}} \left(\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{(r-2)q} dz \right)^{\frac{1}{q}} \\ = \|h\|_p \left[\frac{(v-u)^{(r-2)q+1} (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1})}{2^{(r-2)q+1}((r-2)q+1)} \right]^{\frac{1}{q}}. \quad (20)$$

Analogously,

$$\begin{aligned} & \int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{r-2} h(z) dz \\ & \leq \left(\int_u^v g^p(z) dz \right)^{\frac{1}{p}} \left(\int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{(r-2)q} dz \right)^{\frac{1}{q}} \\ & = \|h\|_p \left[\frac{(-1)^{(r-2)q+2}(v-u)^{(r-2)q+1} (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1})}{2^{(r-2)q+1}((r-2)q+1)} \right]^{\frac{1}{q}}. \end{aligned} \quad (21)$$

Using (20) and (21) in (19), we get the desired result. $\square$

Proposition 4.4. Let χ be a random variable whose probability function is $h : \mathcal{I} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathcal{I}$ such that $u, v \in \mathcal{I}$ with $u < v$. Then

$$\begin{aligned} & \frac{M_r(u) + M_r(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] \\ & = \frac{r(r-1)(v-u)}{8\Phi(0)} \left[\int_0^1 \Omega(\zeta) M_{r-2} \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) d\zeta - \int_0^1 \Omega(\zeta) M_{r-2} \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) d\zeta \right] \\ & = \frac{r(r-1)(v-u)}{8\Phi(0)} \left\{ \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{r-2} h(z) dz \right) d\zeta \right. \\ & \quad \left. - \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{r-2} h(z) dz \right) d\zeta \right\} \end{aligned} \quad (22)$$

for $r \geq 2k, k = 1, 2, \dots$

Proof. Proof of this proposition follows from setting $g(x) = M_r(x)$ in the equality (14). $\square$

Proposition 4.5. Let χ be a random variable whose probability function is $h : \mathcal{I} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathcal{I}$ such that $u, v \in \mathcal{I}$ with $u < v$. If the function $|h|$ is bounded, then

$$\begin{aligned} & \left| \frac{M_r(u) + M_r(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] \right| \\ & \leq \frac{r(1+(-1)^r)(v-u)^r \|h\|_{\infty}}{2^{r+2}\Phi(0)} \int_0^1 \Omega(\zeta) (\zeta^{r-1} - (\zeta-2)^{r-1}) d\zeta \end{aligned}$$

for $r \geq 2k, k = 1, 2, \dots$

Proof. By using Proposition 4.4 and since $|h|$ is a bounded function, we have

$$\begin{aligned} & \left| \frac{M_r(u) + M_r(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ \mathbf{I}_{\vartheta} M_r(v) + \left(\frac{u+v}{2} \right)^- \mathbf{I}_{\vartheta} M_r(u) \right] \right| \\ & \leq \frac{r(r-1)(v-u)}{8\Phi(0)} \left\{ \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{\zeta}{2}a + \frac{2-\zeta}{2}v \right) \right)^{r-2} |h(z)| dz \right) d\zeta \right. \\ & \quad \left. + \int_0^1 \Omega(\zeta) \left(\int_u^v \left(z - \left(\frac{2-\zeta}{2}u + \frac{\zeta}{2}b \right) \right)^{r-2} |h(z)| dz \right) d\zeta \right\} \\ & \leq \frac{r(1+(-1)^r)(v-u)^r \|h\|_{\infty}}{2^{r+2}\Phi(0)} \int_0^1 \Omega(\zeta) (\zeta^{r-1} - (\zeta-2)^{r-1}) d\zeta. \end{aligned}$$

This completes the proof. $\square$

Corollary 4.2. Under the similar assumptions of Proposition 4.5, if

1. $\vartheta(\zeta) = \zeta$, then

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{v-u} \int_u^v \left(\zeta - \frac{1}{2} \zeta^2 \right) g(\zeta) d\zeta \right| \leq \frac{r(1 + (-1)^r)(v-u)^{r+1} \|h\|_\infty}{2^{r+4}} \left(\frac{1}{r+2} + (-1)^r 2^{r+2} \beta_{\frac{1}{2}}(3, r) \right).$$

2. $\vartheta(\zeta) = \frac{\zeta^\alpha}{\Gamma(\alpha)}$, then for $x = \frac{u+v}{2}$, we have

$$\left| \frac{g(u) + g(v)}{2} - \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(v-u)^\alpha} \left[I_{\left(\frac{u+v}{2}\right)^+}^\alpha g(v) + I_{\left(\frac{u+v}{2}\right)^-}^\alpha g(u) \right] \right| \leq \frac{r(1 + (-1)^r)(v-u)^{\alpha+r} \|h\|_\infty}{2^{\alpha+r+2} \Gamma(\alpha+2)} \left(\frac{1}{\alpha+r+1} + (-1)^r 2^{\alpha+r+1} \beta_{\frac{1}{2}}(\alpha+2, r) \right).$$

Proposition 4.6. Let χ be a random variable whose probability function is $h: \mathfrak{I} \subset \mathbb{R} \rightarrow \mathbb{R}_+$, where h is a convex function on the interval of real numbers $\mathfrak{I}$ such that $u, v \in \mathfrak{I}$ with $u < v$ and $h \in L_p[u, v]$, $p > 1$. If the function $|h|$ is bounded, then

$$\left| \frac{M_r(u) + M_r(v)}{2} - \frac{1}{2\Phi(0)} \left[\left(\frac{u+v}{2} \right)^+ I_\vartheta M_r(v) + \left(\frac{u+v}{2} \right)^- I_\vartheta M_r(u) \right] \right| \leq \frac{r(r-1) \left(1 + (-1)^{\frac{(r-2)q+1}{q}} \right) \|h\|_p}{\Phi(0)} \times \left[\frac{(v-u)^{(r-1)q+1}}{2^{(r+1)q+1}((r-2)q+1)} \right]^{\frac{1}{q}} \int_0^1 \Omega(\zeta) (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1}) d\zeta$$

for $r \geq 2k$, $k = 1, 2, \dots$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Proceed as in Proposition 4.3, we get

$$\int_u^v \left(z - \left(\frac{\zeta}{2} u + \frac{2-\zeta}{2} v \right) \right)^{r-2} h(z) dz \leq \|h\|_p \left[\frac{(v-u)^{(r-2)q+1} (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1})}{2^{(r-2)q+1}((r-2)q+1)} \right]^{\frac{1}{q}}, \quad (23)$$

and

$$\int_u^v \left(z - \left(\frac{2-\zeta}{2} u + \frac{\zeta}{2} v \right) \right)^{r-2} h(z) dz \leq \|h\|_p \left[\frac{(-1)^{(r-2)q+2} (v-u)^{(r-2)q+1} (\zeta^{(r-2)q+1} - (\zeta-2)^{(r-2)q+1})}{2^{(r-2)q+1}((r-2)q+1)} \right]^{\frac{1}{q}}. \quad (24)$$

Using (23) and (24) in (22), we get the desired result. $\square$

5. Conclusion

In this paper, we established some generalized fractional integral inequalities of midpoint- and trapezoid-type for twice differentiable convex functions. In view of this, we obtained some new inequalities in a form of classical integral and Riemann–Liouville fractional integrals. As application, we have applied our new inequalities to construct new inequalities involving moments of a continuous random variable. Lastly, we conclude that our results presented in this paper would provide generalizations of those given in earlier works.

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