
COPRIME ARRAYS WITH ENHANCED DEGREES OF FREEDOM

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ABSTRACT

Coprime array geometries provide robust performance for direction-of-arrival estimation problem with more sources than sensor elements. In previous works it is shown that K source directions can be resolved using only $2M + N - 1$ sensor elements where K is less than or equal to MN for M and N are integer numbers. In this paper we introduce a new approach to enhance the degrees of freedom (DOF) from MN to $2MN$ by using the same number of sensor elements. The proposed method is based on computing the covariance matrix of the observation data multiple times. Hence more DOF can be obtained. The resulting cross terms corresponding to the coherent sources are modeled as interference in a sparse recovery algorithm which is solved effectively by an alternating minimization procedure. The theoretical analysis of the proposed method is provided and its superior performance is evaluated through numerical simulations.

Keywords: Coprime arrays, Sparse recovery, Interference mitigation, Direction of arrival estimation

1. INTRODUCTION

In array signal processing direction-of-arrival (DOA) estimation is an important issue for several applications such as radar, sonar and wireless communications [1, 2]. In such a scenario, an antenna array is used for the estimation of the target/source DOA angles where the outputs of the antennas in the array are utilized. Basically, the time/phase differences among the antenna outputs are employed to determine the target DOA angles.

Several methods are proposed for the estimation of unknown source directions and one of the most popular methods in this context is the MUSIC (MUltiple Signal Classification) algorithm [3]. For an M -element sensor array, the MUSIC algorithm can identify $K \leq M-1$ source directions which poses the performance limit for M -element uniform linear arrays (ULAs).

In the last decades, nonuniform array structures gain much attention due to their ability to increase the degrees of freedom (DOF) for parameter estimation [4-7]. In [4], the performance of the minimum redundancy arrays (MRA) are discussed. While MRA provides higher DOF than usual ULAs, there is no closed form expression for the sensor positions of an MRA for a certain number of sensors M [7]. In [5], the augmentation of covariance matrices for enhancing DOF is proposed where the resulting covariance matrix is not positive semidefinite for a finite number of snapshots. In [7], nested array structures are proposed for estimating $O(N^2)$ sources with $O(N)$ sensors. Since nested arrays have more closely spaced sensors that eventually cause relatively higher mutual coupling, coprime array structures are introduced in [8] where the array is composed of less number of element pairs that are closely spaced hence less mutual coupling. In [8], it is shown that using a coprime array $K \leq MN$ sources can be identified with only $2M+N-1$ sensor elements.

In this paper, coprime arrays are considered and a new approach is proposed for further enhancing the DOF. In conventional techniques [8], covariance of the observation data is obtained where a longer virtual ULA data is exploited. Since this virtual ULA data is a single snapshot, spatial smoothing [9] is applied and the obtained covariance matrix can be used to estimate sources up to $O(MN)$. In this

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study, the covariance matrix of this single snapshot data is taken for further increasing the size of the virtual array. Thus the advantage of this approach is to increase the DOF from $O(MN)$ to $O(2MN)$. However a major drawback of taking the covariance of single snapshot data is that it introduces unwanted cross terms corresponding to the coherent sources which corrupt the array data. In order to circumvent this issue, a sparse recovery approach is proposed where the corrupting data is modeled as an interference vector. Then an alternating minimization procedure is followed and the DOA angles are accurately estimated.

2. ARRAY SIGNAL MODEL

Consider a coprime array composed of two subarrays with $2M$ (with Nd inter-element spacing) and N (with Md inter-element spacing) elements respectively in x -axis where $M < N$ and $M, N \in \mathbb{N}^+$ are coprime numbers [8]. Since the first elements of the subarrays have the same locations the total number of physical sensors is $2M + N - 1$. An example for the antenna placement for a coprime array (CPA) for $M = 2$, and $N = 3$ is shown in Figure 1. The output of the array is given by

$$x(t_i) = \sum_{k=1}^K a(\theta_k) s_k(t_i) + n(t_i), \quad i = 1, \dots, T \quad (1)$$

where T is the number of snapshots and $n(t_i) \in \mathbb{C}^{2M+N-1}$ is temporarily and spatially white noise vector. $\{s_k(t_i)\}_{k=1, i=1}^{K, T}$ is the set of uncorrelated source signals and $a(\theta_k)$ denotes the steering vector corresponding to the k th source with direction θ_k . Since the sensor array is deployed in one dimension, i.e. x -axis, only one dimensional DOA estimation can be performed. Hence the i th element of the steering vector $a(\theta_k)$ is given as

$$[a(\theta_k)]_i = a_i(\theta_k) = \exp\{j \frac{2\pi}{\lambda} x_i \sin(\theta_k)\} \quad (2)$$

where λ is the wavelength and $x_i \in \mathbb{S}$ denotes the i th antenna position. \mathbb{S} is the set of sensor positions in the array and it is given as

$$\mathbb{S} = \{Mnd: 0 \leq n \leq N - 1\} \cup \{Nmd: 0 \leq m \leq 2M - 1\}$$

where d is the fundamental element spacing in the array and $d = \lambda/2$ to avoid spatial aliasing.

The aim in this work is to estimate DOAs $\{\theta_k\}_{1 \leq k \leq K}$ when the antenna positions $\{x_i\}_{1 \leq i \leq 2M+N-1}$ are known. Moreover it is also assumed that the number of sources K is known a priori.

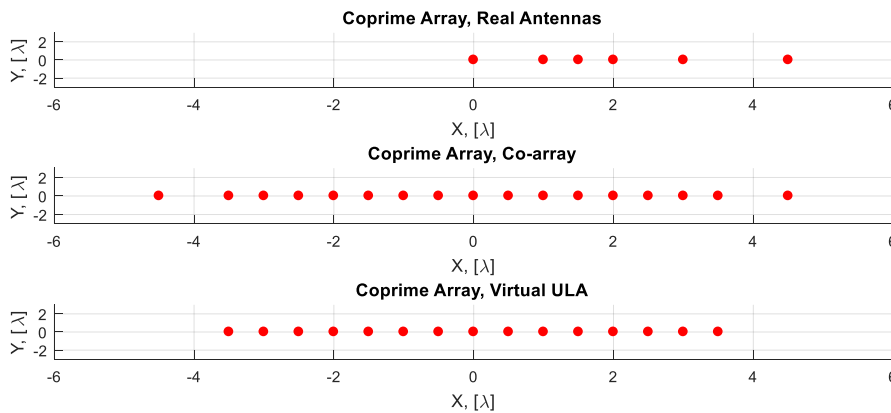


Figure 1. Antenna placement for a coprime array (CPA) for $M = 2$, and $N = 3$. Top: The positions of the physical antennas. Middle: The resulting co-array structure. Bottom: The virtual ULA structure that can be constructed by using coprime property [8].

3. DOA ESTIMATION WITH COPRIME ARRAYS WITH CONVENTIONAL TECHNIQUES

In this part, we review the method in [8] where coprime array structures are used for DOA estimation with DOF of $O(MN)$. Using the array model in (1), the covariance matrix is defined as

$$R_X = E\{x x^H\} = AR_s A^H + \sigma_n^2 I \quad (3)$$

where A is the array steering matrix and its k th column is $[A]_{:,k} = a(\theta_k)$. $R_s = \text{diag}\{\sigma_1^2, \dots, \sigma_K^2\}$ with $\{\sigma_k^2\}_{1 \leq k \leq K}$ being the variances of the source signals, σ_n^2 is the noise variance and I is the identity matrix of size $2M + N - 1$. In order to exploit the co-array structure of R_X , vectorization is applied to R_X and we get

$$y = \text{vec}\{R_X\} = Ap + v \quad (4)$$

where $A = A''A$ and $''$ denotes the Khatri-Rao product [10]. In particular, the k th column of A is given as $[A]_{:,k} = a(\theta_k) = a^*(\theta_k) \otimes a(\theta_k)$ where \otimes denotes the Kronecker product. $p = [\sigma_1^2, \dots, \sigma_K^2]^T$ represents the signal powers and $v = \text{vec}\{\sigma_n^2 I\}$. Now observe that y can be viewed as the output of a virtual array with sensor positions $\{(Mn - Nm)d: 0 \leq n \leq N - 1, 0 \leq m \leq 2M - 1\}$ which includes $2MN + 1$ contiguous terms from $-MN$ to MN . Hence a longer virtual ULA can be constructed from the row elements of y , say $y_U = y_{\mathbb{S}_U}$, where $\mathbb{S}_U = \{nd: -MN \leq n \leq MN\}$ is the set of sensor positions of virtual ULA. In order to extract the rows corresponding to \mathbb{S}_U , we also have the following definitions: $A_U = A_{\mathbb{S}_U}$ and $v_U = v_{\mathbb{S}_U}$. Then the output of this virtual array is given by

$$y_U = A_U p + v_U \quad (5)$$

where $A_U \in \mathbb{C}^{2MN+1 \times K}$ is the array manifold corresponding to the sensor elements with positions \mathbb{S}_U . In order to apply the MUSIC algorithm to y_U the covariance matrix $R_Y = y_U y_U^H$ should be used. Since R_Y is rank-deficient due to a single snapshot of y_U , spatial smoothing is applied to enhance the rank. Once spatial smoothing is employed, $(MN + 1) \times (MN + 1)$ covariance matrix R_Y^{SS} is constructed for DOA estimation in the MUSIC algorithm. Due to the computation of signal and noise spaces, there must be at least one vector representing the noise subspace. Hence with this approach $K \leq MN$ source directions can be resolved. In the following section, we attempt to improve the size of the virtual array hence enhancing the DOF so that more sources than MN can be accurately estimated.

4. DEGREES OF FREEDOM ENHANCEMENT

Consider the virtual ULA output data y_U given in (5) with sensor positions $\{nd: -MN \leq n \leq MN\}$ and construct the covariance matrix $R_Y = y_U y_U^H$ which can be given explicitly as

$$R_Y = A_U R_p A_U^H + A_U p v_U^H + v_U v_U^H A_U^H + v_U v_U^H \quad (6)$$

where $R_p = p p^H$ is a rank 1 covariance matrix. Note that R_p can be written as $R_p = \Lambda_p + \Sigma_p$ where $\Lambda_p = \text{diag}\{\sigma_1^4, \dots, \sigma_K^4\}$ composed of the terms in the main diagonal and $\Sigma_p = R_p - \Lambda_p$. Using this property, (6) can be rewritten as

$$R_Y = A_U \Lambda_p A_U^H + R_E + v_U v_U^H \quad (7)$$

where the first term in the right corresponds to the uncorrelated source signals and R_E represents the other terms as interference and $R_E = A_U \Sigma_p A_U^H + A_U p v_U^H + v_U p^H A_U^H$. After vectorizing R_Y we get

$$z = \text{vec}\{R_Y\} = \tilde{\Psi} u + w \quad (8)$$

where $\tilde{\Psi} = A + B$ and the k th column of $A \in \mathbb{C}^{(2MN+1)^2 \times K}$ is given as $[A]_{:,k} = a(\theta_k)^* a(\theta_k)$ where $a(\theta_k)$ is the k th column of A . $u = [\sigma_1^4, \dots, \sigma_K^4]^T$ and $w = \text{vec}\{v_U v_U^H\}$ denotes the deterministic noise term. In order to utilize the interference term, we define $B \in \mathbb{C}^{(2MN+1)^2 \times K}$ which represents the vectorization of the interference together with u , namely $Bu = \text{vec}\{R_E\}$. Note that in (8), $z \in \mathbb{C}^{(2MN+1)^2}$ denotes the response of a virtual array with positions $\mathbb{K}_C = \{(n_1 - n_2)d: -MN \leq n_1, n_2 \leq MN\}$ which includes a contiguous part from $-2MN$ to $2MN$ which is defined by $\mathbb{K}_U = \{nd: -2MN \leq n \leq 2MN\}$ where $|\mathbb{K}_U| = 4MN + 1$. By collecting the rows of z in accordance with the set \mathbb{K}_U we obtain $z = z_{\mathbb{K}_U}$ given by

$$z = \Psi u + w \tag{9}$$

where $\Psi = \mathcal{A} + B$ and $\mathcal{A} \in \mathbb{C}^{4MN+1 \times K}$ steering matrix constructed from the rows of A corresponding to \mathbb{K}_U and similarly we define $B = B_{\mathbb{K}_U}$ and $w = w_{\mathbb{K}_U}$.

In (9), we present the virtual array data of a ULA with DOF of $4MN + 1$. In the following we introduce a sparse recovery approach where $K \leq 2MN$ source directions can be uniquely identified.

5. SPARSE RECOVERY FOR DOA ESTIMATION AND INTERFERENCE MITIGATION

Since the signal model given in (9) includes the interference term Bu , spatial smoothing cannot provide accurate results. In order to solve this issue and mitigate the effect of interference, a sparse recovery approach with perturbed dictionary is proposed. In this case, the virtual array output $z \in \mathbb{C}^{4MN+1}$ is used for the estimation of DOA angles. Then (9) can be written in the following context

$$z = \Psi u + w = \Phi s + w \tag{10}$$

where $\Phi = A_\theta + B_\theta$. $A_\theta \in \mathbb{C}^{4MN+1 \times N_\theta}$ is the dictionary matrix with $K \ll N_\theta$ and its n th column is given by $[A_\theta]_{:,n} = a(\theta_n)$ which is the steering vector corresponding to angle θ_n with sensor positions \mathbb{K}_U . $s \in \mathbb{C}^{N_\theta}$ is a K -sparse vector, i.e. all entries of s but K are zero. In other words, $\|s\|_0 = K$ where the l_0 -norm is defined as the cardinality of s , that is to say, $\|s\|_0 = |\{1 \leq i \leq N_\theta: s_i \neq 0\}|$. Then the following problem setting can be written, i.e.

$$\begin{aligned} & \min_{s \in \mathbb{C}^{N_\theta}, B_\theta \in \mathbb{C}^{4MN+1 \times N_\theta}} \|s\|_0 \\ & \text{subject to: } \|z - [A_\theta | B_\theta] \begin{bmatrix} s \\ s \end{bmatrix}\|_2 \leq \varepsilon \end{aligned} \tag{11}$$

where the residual noise term is bounded by ε and The above optimization problem in (11) is NP-Hard due to the non-convexity of l_0 -norm. Moreover it is non-linear due to multiplicative unknown terms s and B_θ . In order to circumvent these issues, first l_0 -norm is relaxed to l_1 -norm and an alternating approach is followed. In this case, s and B_θ are found iteratively from

$$\begin{aligned} [s^{(j+1)}] &= \arg \min_{s \in \mathbb{C}^{N_\theta}} \|s\|_1 \text{ subject to:} \\ & \|z - [A_\theta | B_\theta^{(j)}] \begin{bmatrix} s \\ s \end{bmatrix}\|_2 \leq \varepsilon \end{aligned} \tag{12}$$

$$[B_\theta^{(j+1)}] = \arg \min_{B_\theta \in \mathbb{C}^{4MN+1 \times N_\theta}} \|z - [A_\theta | B_\theta] \begin{bmatrix} s^{(j+1)} \\ s^{(j+1)} \end{bmatrix}\|_2 \tag{13}$$

where the initial is given by $B_\theta^{(1)} = 0^{4MN+1 \times N_\theta}$ as a zero matrix. The l_1 -norm in (12) is defined as $\|s\|_1 = \sum_{i=1}^{N_\theta} |s_i|$. The optimization problem in (12) and (13) is similar to the one constructed in [11] where off-grid DOA estimation problem is considered. Once the alternating minimization program

presented in (12) and (13) converges (the convergence of alternating technique is provided in [12, Lemma 1]), the DOA angles estimates can be found from the columns of A_θ corresponding to the non-zero values of s . In the following sections the uniqueness of the alternating minimization program are discussed.

6. UNIQUENESS OF SPARSE RECOVERY

In this section, the uniqueness of the proposed method is discussed and we show that the proposed method can resolve $K \leq 2MN$ sources with $2M + N - 1$ physical sensor elements.

Theorem 1: If $z = \Phi s$ has a solution satisfying $\|s\|_0 = K < \text{spark}(\Phi)/2$, then s is the unique solution where $\text{spark}(\Phi)$ is defined as the minimum number of linearly dependent columns of Φ [12].

Proof: Proof is by contradiction. First, assume that there exists at most one s where $\|s\|_0 = K$ and $\text{spark}(\Phi) \leq 2K$. Then suppose that there exists an h with $\|h\|_0 = 2K$ and $h \in \text{Null}\{\Phi\}$ i.e. the null space of Φ . This means that there exist some set of at most $2K$ columns that are linearly dependent. Since $\|h\|_0 = 2K$, we can write $h = s - s'$ for $\|s\|_0 = \|s'\|_0 = K$ with $s \neq s'$. Using $h \in \text{Null}\{\Phi\}$, we have $\Phi(s - s')$, in other words, $\Phi s = \Phi s'$. This leads to the fact that there exist two solutions s and s' . However, this contradicts with our assumption that there exists at most one s with $\|s\|_0 = K$. Therefore we must have $\text{spark}(\Phi) > 2K$. Since $\text{spark}(\Phi) \leq \bar{M} + 1$ where $\bar{M} = 4MN + 1$, $K < \text{spark}(\Phi)/2$ leads to the final condition $2K < \bar{M}$. Then we can conclude that the uniqueness condition for sparse recovery is $K \leq 2MN$.

7. NUMERICAL SIMULATIONS

In this section, numerical simulation results are illustrated to show the superior performance of the proposed method. The proposed method is compared with the conventional technique [7] where the coprime arrays are used in the MUSIC algorithm. In the experiments, two different scenarios are conducted for the evaluation of the methods.

7.1. Scenario 1: $M = 3$, $N = 4$ And $K = 17$ For $MN \leq K \leq 2MN$

In this scenario, we select $M = 3$, $N = 4$ and $K = 17$ where at most $MN = 12$ sources can be resolved using conventional technique and the DOF of the proposed method is $2MN = 24$. In order to design a closely spaced source scenario, the source directions are located equally spaced as $\bar{\theta}_k \in [-0.4, 0.4]$ where $\bar{\theta}_k = \sin(\theta_k)$ is defined to obtained a normalized spectra. Note that the y-axis denotes the MUSIC spectra [3]. The dictionary matrix A_θ is constructed with resolution $\delta_\theta = |\bar{\theta}_n - \bar{\theta}_{n-1}| = 1/2^{10}$ for $N_\theta = 2^{10}$ and $\bar{\theta}_n = \sin(\theta_n)$. The same dictionary is also used for the computation of the MUSIC pseudo-spectra. The results are presented in Figure 2. As it is seen the conventional technique (i.e. CPA (coprime array) with MUSIC) cannot estimate the source locations accurately due to insufficient DOF whereas the proposed method provides a better spatial spectra where source locations can accurately be estimated.

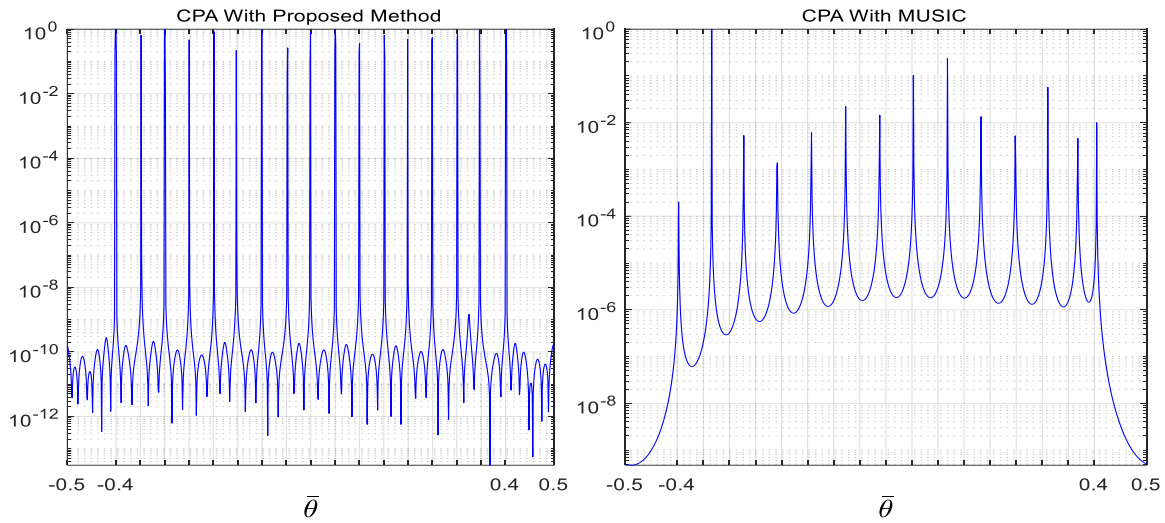


Figure 2. The spatial spectrums of the algorithms: CPA with the MUSIC algorithm (a) and CPA with the proposed method (b). $M = 3$, $N = 4$ and $K = 17$. SNR 0dB and the number of snapshots $T = 1000$.

7.2. Scenario 2: $M = 3$, $N = 5$ and $K = 21$ for $MN \leq K \leq 2MN$

In this scenario a larger array is considered with $M = 3$, $N = 5$ and $K = 21$ sources assumed in $\bar{\theta}_k \in [-0.4, 0.4]$. In Figure 3, the performance of the algorithms are shown. As it is seen the proposed method gives peaks at true source locations whereas the conventional technique cannot provide accurate results due to insufficient rank of the covariance matrix and short aperture of virtual ULA. The superior performance of the proposed method can be attributed to the enhancement of the DOF of the virtual ULA observation and mitigation the coherent source terms in covariance computation process.

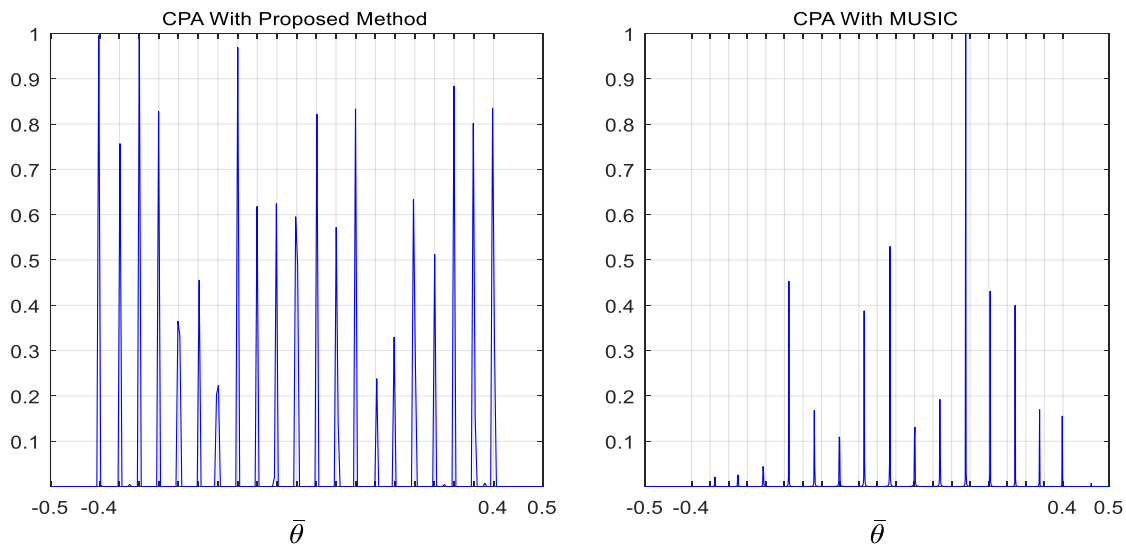


Figure 3. The spatial spectrums of the algorithms: CPA with the MUSIC algorithm (a) and CPA with the proposed method (b). $M = 3$, $N = 5$ and $K = 21$. SNR -10 dB and the number of snapshots $T = 2000$.

8. CONCLUSIONS AND REMARKS

In this paper, a new approach is proposed for DOA estimation with enhanced degrees of freedom using coprime arrays. In conventional techniques, $K \leq MN$ sources can be identified by using coprime arrays whereas the proposed method can provide the DOF of $O(2MN)$. The DOF is improved by computing the covariance matrix of the single snapshot virtual array data. In this case, cross terms are present and they corrupt the array data which constitutes one of the drawbacks of this approach. In order to circumvent this issue, these cross terms are modeled as interference and treated in a sparse recovery problem. In the future, the proposed method can be applied to different sparse array structures and more robust algorithms can be developed.

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