Arab. J. Math. (2022) 11:507–519
https://doi.org/10.1007/s40065-022-00381-1
Arabian Journal of Mathematics

Hüseyin Budak · Melike Yavuz · Hasan Kara

On some new generalized fractional inequalities for twice differentiable functions

Received: 4 February 2022 / Accepted: 5 June 2022 / Published online: 30 June 2022

Abstract In this paper, we establish an identity involving Sarikaya fractional integrals for twice differentiable functions. We obtain some new generalized fractional inequalities for the functions whose second derivatives in absolute value are convex by utilizing obtained equality. Utilizing the new inequalities obtained, some new inequalities for Riemann–Liouville fractional integrals and k-Riemann–Liouville fractional integrals are obtained. In addition, some of these results generalize ones obtained in earlier works.

Mathematics Subject Classification 26D07 · 26D10 · 26D15 · 26A33

1 Introduction

Convex functions are very important in the literature. A lot of research has been done for convex functions on optimization theory and inequalities in mathematics. The most important of these inequalities is the Hermite–Hadamard inequality. Many mathematicians have studied and contributed to the literature on the Hermite–Hadamard inequality and related inequalities such as trapezoid, midpoint, Simpson's inequality, and Bullen's inequality.

Many bounds have been obtained over the years for the left and right sides of the Hermite–Hadamard inequality. Dragomir and Agarwal obtained trapezoid inequalities for convex functions in [9], whereas Kirmacıestablished midpoint inequalities for convex functions in [18]. Iqbal et al. and Sarıkaya et al. established the fractional midpoint and trapezoidal type inequalities for convex functions in [16,27], respectively. Researchers have established some generalized midpoint type inequalities for Riemann–Liouville fractional integrals in [5,6].

There has also been research focusing on the Simpson-type inequality. In particular, Alomari et al. [1] studied Simpson's inequality for s-convex functions using differentiable functions. In the studies [25,26], it is established the new variants of Simpson's type inequalities based on the differentiable convex mappings. Many researchers have studied Simpson-type inequalities in the literature (see, [2,10,14,15,17,20,21]).

Bullen [7] obtained the well-known Bullen-type inequalities. Bullen-type inequalities for generalized convex functions were obtained by Sarıkaya et al. [28]. The local fractional version of Bullen-type inequality were presented in [13]. Du et al. [11] obtained Bullen-type inequalities using fractional integrals.

H. Budak · M. Yavuz · H. Kara (✉)
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
E-mail: hasan64kara@gmail.com

H. Budak
E-mail: hsyn.budak@gmail.com

M. Yavuz
E-mail: melike.yavuz.526@gmail.com

Many mathematicians have been working on twice differentiable functions recently. For instance, the inequalities for twice differentiable convex mappings associated with Hadamard's inequality in [3,4] were obtained. Moreover, Mohammed and Sarikaya established some new generalized fractional integral inequalities of midpoint and trapezoid type for twice differentiable convex functions in [19]. Sarıkaya and Aktan [22] established some new Simpson and the Hermite–Hadamard type inequalities for functions whose absolute values of derivatives are convex. In addition Hezenci et al. obtained several fractional Simpson's inequality for twice differentiable functions. In [8], some generalizations of integral inequalities of Bullen-type for twice differentiable functions involving Riemann–Liouville fractional integrals were obtained.

The goal of our research is, by using Sarikaya fractional integrals, to obtain new generalized inequalities for functions whose second derivatives in absolute values are convex functions. Some new results with some special choices will establish generalizations and connections for the classical midpoint inequalities and for the midpoint inequalities obtained for Riemann–Liouville, k-Riemann–Liouville, and different generalized fractional integrals.

Here, we give some definitions and notations which are used frequently in main section.

The well-known gamma and beta are defined as follows: For $0 < x, y < \infty$, and $x, y \in \mathbb{R}$,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

and

$$\beta(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt = 2 \int_0^{\frac{\pi}{2}} \sin(t)^{2x-1} \cos(t)^{2y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The generalized fractional integrals were introduced by Sarikaya and Ertu ğral as follows:

Definition 1.1 [24] Let us note that a function $\varphi : [0, \infty) \to [0, \infty)$ satisfy the following condition:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We consider the following left-sided and right-sided Sarikaya fractional integral operators

$$_{a+}I_\varphi f(x) = \int_a^x \frac{\varphi(x-t)}{x-t} f(t) dt, \quad x > a \tag{1.1}$$

and

$$_{b-}I_\varphi f(x) = \int_x^b \frac{\varphi(t-x)}{t-x} f(t) dt, \quad x < b, \tag{1.2}$$

respectively.

The most significant feature of generalized fractional integrals is that they generalize some important types of fractional integrals such as Riemann–Liouville fractional integrals, k-Riemann–Liouville fractional integrals, Hadamard fractional integrals, Katugampola fractional integrals, conformable fractional integrals, etc.

These important special cases of the integral operators (1.1) and (1.2) are mentioned as follows:

(1) Let us consider $\varphi(t) = t$. Then, the operators (1.1) and (1.2) reduce to the Riemann integral.
(2) If we choose $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1.1) and (1.2) reduce to the Riemann–Liouville fractional integrals $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$, respectively. Here, Γ is Gamma function.
(3) For $\varphi(t) = \frac{1}{k\Gamma_k(\alpha)} t^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (1.1) and (1.2) reduce to the k-Riemann–Liouville fractional integrals $J_{a+,k}^\alpha f(x)$ and $J_{b-,k}^\alpha f(x)$, respectively. Here, Γ_k is k-Gamma function defined by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} \mathrm{d}t, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0.$$

There are many studies on fractional integrals in the literature (refer to, [12,30,31]).

2 A new identity for twice differentiable functions

In this section, we obtain the equality with one real parameter for generalized fractional integrals and twice differentiable functions.

Lemma 2.1 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping (a,b) and $x \in [a,b]$. Then, the following equality holds:*

$$(x-a)^2 \int_0^1 B_1(t) f''(tx+(1-t)a)\,\mathrm{d}t + (b-x)^2 \int_0^1 B_2(t) f''(tx+(1-t)b)\,\mathrm{d}t$$
$$= [(x-a)B_1(1) - (b-x)B_2(1)] f'(x) - [A_1(1) + A_2(1)] f(x) + {}_{x-}I_\varphi f(a) + {}_{x+}I_\varphi f(b),$$

where

$$A_1(s) = \int_0^s \frac{\varphi((x-a)u)}{u}\mathrm{d}u,$$

$$A_2(s) = \int_0^s \frac{\varphi((b-x)u)}{u}\mathrm{d}u,$$

$$B_1(t) = \int_0^t A_1(s)\,\mathrm{d}s,$$

and

$$B_2(t) = \int_0^t A_2(s)\,\mathrm{d}s.$$

Proof By using the integration by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^1 B_1(t) f''(tx+(1-t)a)\,\mathrm{d}t \\ &= B_1(t)\frac{f'(tx+(1-t)a)}{x-a}\Bigg|_0^1 - \frac{1}{x-a}\int_0^t A_1(t) f'(tx+(1-t)a)\,\mathrm{d}t \\ &= B_1(t)\frac{f'(x)}{x-a} - \frac{1}{x-a}\left[\frac{A_1(t) f(tx+(1-t)a)}{x-a}\Bigg|_0^1\right. \\ &\quad \left. -\frac{1}{x-a}\int_0^1 \frac{\varphi((x-a)t)}{t} f(tx+(1-t)a)\,\mathrm{d}t\right]. \end{aligned} \tag{2.1}$$

With help of the equality (2.1) and using the change of the variable $u = tx + (1-t)a$ for $t \in [0,1]$ it can be rewritten as follows

$$I_1 = B_1(t)\frac{f'(x)}{x-a} - \frac{A_1(t) f(x)}{(x-a)^2} - \frac{1}{(x-a)^2}\int_a^x \frac{\varphi(u-a)}{u-a} f(u)\,\mathrm{d}u$$
$$= B_1(t)\frac{f'(x)}{x-a} - \frac{A_1(t) f(x)}{(x-a)^2} - \frac{1}{(x-a)^2}\, {}_{x-}I_\varphi f(a).$$

Similarly, we get

$$I_2 = \int_0^1 B_2(t) f''(tx+(1-t)b)\,\mathrm{d}t \tag{2.2}$$
$$= B_2(t)\frac{f'(tx+(1-t)b)}{x-b}\bigg|_0^1 - \frac{1}{x-b}\int_0^t A_2(t) f'(tx+(1-t)b)\,\mathrm{d}t$$
$$= B_2(1)\frac{f'(x)}{x-b} - \frac{1}{x-b}\left[\frac{A_2(t) f(tx+(1-t)b)}{x-b}\bigg|_0^1\right.$$
$$\left. - \frac{1}{x-b}\int_0^1 \frac{\varphi((b-x)t)}{t} f(tx+(1-t)b)\,\mathrm{d}t\right]$$
$$= B_2(1)\frac{f'(x)}{x-b} - \frac{1}{x-b}\left[\frac{A_2(1) f(x)}{x-b} - \frac{1}{x-b}\int_0^1 \frac{\varphi((b-x)t)}{t} f(tx+(1-t)b)\,\mathrm{d}t\right]$$
$$= B_2(1)\frac{f'(x)}{x-b} - \frac{A_2(1) f(x)}{(x-b)^2} + \frac{1}{(x-b)^2}\int_b^x \frac{\varphi(b-u)}{\left(\frac{u-b}{x-b}\right)} \frac{f(u)\,.\mathrm{d}u}{x-b}$$
$$= B_2(1)\frac{f'(x)}{x-b} - \frac{A_2(1) f(x)}{(x-b)^2} + \frac{1}{(x-b)^2}\int_x^b \frac{\varphi(b-u)}{b-u} f(u)\,.\mathrm{d}u$$
$$= B_2(1)\frac{f'(x)}{x-b} - \frac{A_2(1) f(x)}{(x-b)^2} + \frac{1}{(x-b)^2}\, {}_{x+}I_\varphi f(b).$$

From the Eqs. (2.1) and (2.2), we have

$$(x-a)^2 I_1 + (b-x)^2 I_2$$
$$= [(x-a) B_1(1) - (b-x) B_2(1)] f'(x) - [A_1(1) + A_2(1)] f(x) + {}_{x-}I_\varphi f(a) + {}_{x+}I_\varphi f(b).$$

This ends the proof of Lemma 2.1. □

3 Some generalized inequalities for Sarikaya fractional integrals

In this section, using Sarikaya fractional integrals, we will establish some generalized inequalities for functions whose absolute value of second derivatives are convex functions with various powers. We will also obtain some new results by special choices of main results.

Theorem 3.1 *Let us consider that the assumptions of Lemma 2.1 are valid. Let us also consider that the mapping* $|f''|$ *is convex on* $[a,b]$*. Then, we get the following inequality for Sarikaya fractional integrals*

$$\left|\left[(x-a)\,B_1(1)-(b-x)\,B_2(1)\right]f^{'}(x)-\left[A_1(1)+A_2(1)\right]f(x)+{}_{x-}I_{\varphi}f(a)+{}_{x+}I_{\varphi}f(b)\right|$$
$$\leq (x-a)^2\left[Q_1^{\varphi}\left|f''(x)\right|+Q_2^{\varphi}\left|f''(a)\right|\right]+(b-x)^2\left[Q_3^{\varphi}\left|f''(x)\right|+Q_4^{\varphi}\left|f''(b)\right|\right], \tag{3.1}$$

where A_1, A_2, B_1 and B_2 are defined as in Lemma 2.1 and Q_i^{φ}, $i=1,2,3,4$, are defined by

$$Q_1^{\varphi}=\int_0^1 |B_1(t)|\,t\mathrm{d}t,$$

$$Q_2^{\varphi}=\int_0^1 |B_1(t)|\,(1-t)\,\mathrm{d}t,$$

$$Q_3^{\varphi}=\int_0^1 |B_2(t)|\,t\mathrm{d}t,$$

and

$$Q_4^{\varphi}=\int_0^1 |B_2(t)|\,(1-t)\,\mathrm{d}t.$$

Proof By taking modulus in Lemma 2.1, we have

$$\left|\left[(x-a)\,B_1(1)-(b-x)\,B_2(1)\right]f^{'}(x)-\left[A_1(1)+A_2(1)\right]f(x)+{}_{x-}I_{\varphi}f(a)+{}_{x+}I_{\varphi}f(b)\right|$$
$$\leq (x-a)^2\left|\int_0^1 B_1(t)\,f^{''}(tx+(1-t)\,a)\,\mathrm{d}t\right|+(b-x)^2\left|\int_0^1 B_2(t)\,f^{''}(tx+(1-t)\,b)\,\mathrm{d}t\right|$$
$$\leq (x-a)^2\int_0^1 |B_1(t)|\left|f^{''}(tx+(1-t)\,a)\right|\mathrm{d}t+(b-x)^2\int_0^1 |B_2(t)|\left|f^{''}(tx+(1-t)\,b)\right|\mathrm{d}t. \tag{3.2}$$

By using convexity of $\left|f''\right|$, we obtain

$$\left|\left[(x-a)\,B_1(1)-(b-x)\,B_2(1)\right]f^{'}(x)-\left[A_1(1)+A_2(1)\right]f(x)+{}_{x-}I_{\varphi}f(a)+{}_{x+}I_{\varphi}f(b)\right|$$
$$\leq (x-a)^2\int_0^1 |B_1(t)|\left[t\left|f''(x)\right|+(1-t)\left|f''(a)\right|\mathrm{d}t\right]$$
$$+(b-x)^2\int_0^1 |B_2(t)|\left[t\left|f''(x)\right|+(1-t)\left|f''(b)\right|\mathrm{d}t\right]$$
$$\leq (x-a)^2\left[Q_1^{\varphi}\left|f''(x)\right|+Q_2^{\varphi}\left|f''(a)\right|\right]+(b-x)^2\left[Q_3^{\varphi}\left|f''(x)\right|+Q_4^{\varphi}\left|f''(b)\right|\right].$$

This finishes the proof of Theorem 3.1. □

Corollary 3.2 *If we choose $\varphi(t)=t$ for all $t\in[a,b]$ in Theorem 3.1, then we have the following inequality*

$$\left|\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \frac{(x-a)^3}{6\,(b-a)}\left(\frac{3\left|f''(x)\right|+\left|f''(a)\right|}{4}\right)+\frac{(b-x)^3}{6\,(b-a)}\left(\frac{3\left|f''(x)\right|+\left|f''(b)\right|}{4}\right).$$

Corollary 3.3 *If we assign $x=\frac{a+b}{2}$ in Theorem 3.1, then we have the following midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{2\Lambda(1)}\left[{}_{\frac{a+b}{2}-}I_{\varphi} f(a) + {}_{\frac{a+b}{2}+}I_{\varphi} f(b)\right]\right| \leq \frac{(b-a)^2}{8\Lambda(1)}\left(\int_0^1 |\Delta(t)|\,\mathrm{d}t\right)\left[\left|f''(a)\right| + \left|f''(b)\right|\right],$$

where

$$\Lambda(s) = \int_0^s \frac{\varphi\left(\frac{b-a}{2}u\right)}{u}\mathrm{d}u,$$

$$\Delta(t) = \int_0^t \Lambda(s)\,\mathrm{d}s.$$

Proof If we choose $x=\frac{a+b}{2}$ in Theorem 3.1,

$$\begin{aligned}
&\left|2\Lambda(1) f(x) - \left[{}_{\frac{a+b}{2}-}I_{\varphi} f(a) + {}_{\frac{a+b}{2}+}I_{\varphi} f(b)\right]\right| \\
&\leq \frac{(b-a)^2}{4}\left[\left(\int_0^1 |\Delta(t)|\,t\,\mathrm{d}t\right)\left|f''\left(\frac{a+b}{2}\right)\right| + \left(\int_0^1 |\Delta(t)|\,(1-t)\,\mathrm{d}t\right)\left|f''(a)\right|\right] \\
&\quad + \frac{(b-a)^2}{4}\left[\left(\int_0^1 |\Delta(t)|\,t\,\mathrm{d}t\right)\left|f''\left(\frac{a+b}{2}\right)\right| + \left(\int_0^1 |\Delta(t)|\,(1-t)\,\mathrm{d}t\right)\left|f''(b)\right|\right] \\
&\leq \frac{(b-a)^2}{4}\left[\int_0^1 |\Delta(t)|\,t\,\mathrm{d}t + \int_0^1 |\Delta(t)|\,(1-t)\,\mathrm{d}t\right]\left[\left|f''(a)\right| + \left|f''(b)\right|\right] \\
&= \frac{(b-a)^2}{8\Lambda(1)}\left(\int_0^1 |\Delta(t)|\,t\,\mathrm{d}t\right)\left[\left|f''(a)\right| + \left|f''(b)\right|\right].
\end{aligned}$$

□

Remark 3.4 If we assign $\varphi(t)=t$ in Corollary 3.3, then we have the following midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \leq \frac{(b-a)^2}{48}\left[\left|f''(a)\right| + \left|f''(b)\right|\right],$$

which was given by Sarikaya et al. [22,23].

Remark 3.5 If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ in Corollary 3.3, then we obtain the following midpoint type inequality for Riemann–Liouville fractional integrals

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-} f(a) + J^{\alpha}_{\frac{a+b}{2}+} f(b)\right]\right| \\
&\leq \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)}\left[\left|f''(a)\right| + \left|f''(b)\right|\right],
\end{aligned}$$

which was proved by Tomar et al. [29].

Remark 3.6 By choosing $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha, k>0$, for all $t\in[a,b]$ in Corollary 3.3, then we have the following midpoint type inequality for k -Riemann–Liouville fractional integrals

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{a+b}{2}-,k}f(a)+J^{\alpha}_{\frac{a+b}{2}+,k}f(b)\right]\right|$$
$$\leq\frac{(b-a)^2k^2}{8(\alpha+k)(\alpha+2k)}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which was given by Mohammed and Sarikaya [19].

Theorem 3.7 *Let us note that the assumptions of Lemma 2.1 hold. If the mapping* $\left|f''\right|^q$, $q>1$ *is convex on* $[a,b]$*, then we have the following inequality for Sarikaya fractional integrals*

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_\varphi f(a)+{}_{x+}I_\varphi f(b)\right|$$
$$\leq(x-a)^2\left(\int_0^1|B_1(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\left|f''(x)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

where A_1, A_2, B_1 and B_2 are defined by as in Lemma 2.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof By using the Hölder inequality in inequality (3.2), we have

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_\varphi f(a)+{}_{x+}I_\varphi f(b)\right|$$
$$\leq(x-a)^2\left(\int_0^1|B_1(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

With the help of the convexity of $\left|f''\right|^q$, we get

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_\varphi f(a)+{}_{x+}I_\varphi f(b)\right|$$
$$\leq(x-a)^2\left(\int_0^1|B_1(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f''(x)\right|^q+(1-t)\left|f''(a)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$=(x-a)^2\left(\int_0^1|B_1(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\left|f''(x)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

This finishes the proof of Theorem 3.7. □

Corollary 3.8 *If we choose* $\varphi(t)=t$ *for all* $t\in[a,b]$ *in Theorem 3.7, then we have the following inequality*

$$\left|\left(x-\frac{a+b}{2}\right)f^{'}(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \frac{(x-a)^3}{2(b-a)(2p+1)^{\frac{1}{p}}}\left(\frac{\left|f''(x)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^3}{2(b-a)(2p+1)^{\frac{1}{p}}}\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.9 *If we assign $x=\frac{a+b}{2}$ in Theorem 3.7, then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\frac{a+b}{2}-}I_{\varphi}f(a)\;+{}_{\frac{a+b}{2}+}I_{\varphi}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{8\Lambda(1)}\left(\int_0^1|\Delta(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{2^{1+\frac{2}{q}}\Lambda(1)}\left(\int_0^1|\Delta(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

where Δ and Λ defined by as in 3.3.

Proof If we assign $x=\frac{a+b}{2}$ in Theorem 3.7, then we have the following inequality

$$2\Lambda(1)\,f\left(\frac{a+b}{2}\right)-\left[{}_{\frac{a+b}{2}-}I_{\varphi}f(a)\;{}_{\frac{a+b}{2}+}I_{\varphi}f(b)\right]$$
$$\leq \frac{(b-a)^2}{4}\left(\int_0^1|\Delta(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{4}\left(\int_0^1|\Delta(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

This gives the fist inequality. For the proof of second inequality, let $a_1=\left|f''(a)\right|^q$, $b_1=3\left|f''(b)\right|^q$, $a_2=3\left|f''(a)\right|^q$ and $b_2=\left|f''(b)\right|^q$. Using the facts that,

$$\sum_{k=1}^{n}(a_k+b_k)^s\leq\sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\;0\leq s<1$$

and $1+3^{\frac{1}{q}}\leq 4$, then the desired result can be obtained straightforwardly. □

Corollary 3.10 *If we assign $\varphi(t)=t$ in Corollary 3.9, then we have the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(b-a)^2}{4^{1+\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Corollary 3.11 *If we take* $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ *in Corollary 3.9, then we obtain the following midpoint type inequality for Riemann–Liouville fractional integrals*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{8(\alpha+1)(p(\alpha+1)+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2}{2^{1+\frac{2}{q}}(\alpha+1)(p(\alpha+1)+1)^{\frac{1}{p}}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Corollary 3.12 *By choosing* $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha,k>0$, *for all* $t\in[a,b]$ *in Corollary 3.9, then we have the following midpoint type inequality for* k *-Riemann–Liouville fractional integrals*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\frac{a+b}{2}-,k}f(a)+J^{\alpha}_{\frac{a+b}{2}+,k}f(b)\right]\right|$$
$$\leq \frac{(b-a)^2 k^{1+\frac{1}{p}}}{8(\alpha+k)(p(\alpha+k)+k)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^2 k^{1+\frac{1}{p}}}{2^{1+\frac{2}{q}}(\alpha+k)(p(\alpha+k)+k)^{\frac{1}{p}}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Theorem 3.13 *Let us note that the assumptions of Lemma 2.1 hold. If the mapping* $\left|f''\right|^q$, $q\geq 1$ *is convex on* $[a,b]$, *then we have the following inequality*

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_{\varphi}f(a)+{}_{x+}I_{\varphi}f(b)\right|$$
$$\leq (x-a)^2\left(\int_0^1|B_1(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(Q_1^{\varphi}\left|f''(x)\right|^q+Q_2^{\varphi}\left|f''(a)\right|^q\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(Q_3^{\varphi}\left|f''(x)\right|^q+Q_4^{\varphi}\left|f''(b)\right|^q\right)^{\frac{1}{q}},$$

where A_1, A_2, B_1 *and* B_2 *are defined as in Lemma 2.1,* Q_i^{φ}, $i=1,2,3,4$ *are defined by as in Theorem 3.1.*

Proof By applying power-mean inequality in (3.2), we obtain

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_{\varphi}f(a)+{}_{x+}I_{\varphi}f(b)\right|$$
$$\leq (x-a)^2\left(\int_0^1|B_1(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|B_1(t)|\left|f''(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$+(b-x)^2\left(\int_0^1|B_2(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|B_1(t)|\left|f''(tx+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

Since $\left|f''\right|^q$ is convex, we have

$$\left|[(x-a)B_1(1)-(b-x)B_2(1)]f'(x)-[A_1(1)+A_2(1)]f(x)+{}_{x-}I_\varphi f(a)+{}_{x+}I_\varphi f(b)\right|$$

$$\leq (x-a)^2\left(\int_0^1 |B_1(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |B_1(t)|\left[t\left|f''(x)\right|^q+(1-t)\left|f''(a)\right|^q\right]\right)^{\frac{1}{q}}\mathrm{d}t$$

$$+(b-x)^2\left(\int_0^1 |B_2(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |B_2(t)|\left[t\left|f''(x)\right|^q+(1-t)\left|f''(b)\right|^q\right]\right)^{\frac{1}{q}}\mathrm{d}t$$

$$=(x-a)^2\left(\int_0^1 |B_1(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(Q_1^\varphi\left|f''(x)\right|^q+Q_2^\varphi\left|f''(a)\right|^q\right)^{\frac{1}{q}}$$

$$+(b-x)^2\left(\int_0^1 |B_2(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(Q_3^\varphi\left|f''(x)\right|^q+Q_4^\varphi\left|f''(b)\right|^q\right)^{\frac{1}{q}}.$$

Then, we obtain the desired result of Theorem 3.13. □

Corollary 3.14 *If we choose $\varphi(t)=t$ for all $t\in[a,b]$ in Theorem 3.13, then we have the following inequality*

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$

$$\leq\frac{(x-a)^3}{6(b-a)}\left(\frac{3\left|f''(x)\right|^q+\left|f''(a)\right|^q}{4}\right)^{\frac{1}{q}}+\frac{(b-x)^3}{6(b-a)}\left(\frac{3\left|f''(x)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}.$$

Corollary 3.15 *If we take $x=\frac{a+b}{2}$ in Theorem 3.13, then we have the following midpoint type inequality for generalized fractional integrals*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{2\Lambda(1)}\left[{}_{\frac{a+b}{2}-}I_\varphi f(a)+{}_{\frac{a+b}{2}+}I_\varphi f(b)\right]\right|$$

$$\leq\frac{(b-a)^2}{8\Lambda(1)}\left(\int_0^1|\Delta(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\left(Q_1^*\left|f''\left(\frac{a+b}{2}\right)\right|^q+Q_2^*\left|f''(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(Q_1^*\left|f''\left(\frac{a+b}{2}\right)\right|^q+Q_1^*\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(b-a)^2}{8\Lambda(1)}\left(\int_0^1|\Delta(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left[\left(\frac{\left(Q_1^*+2Q_2^*\right)\left|f''(a)\right|^q+Q_1^*\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{Q_1^*\left|f''(a)\right|^q+\left(Q_1^*+2Q_2^*\right)\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right],$$

where

$$Q_1^*=\int_0^1|\Delta(t)|\,t\,\mathrm{d}t$$

and

$$Q_2^* = \int_0^1 |\Delta(t)| (1-t) \, \mathrm{d}t.$$

Remark 3.16 If we choose $\varphi(t) = t$ for all $t \in [a, b]$ in Corollary 3.15, then we have the following midpoint inequality for Riemann integrals

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) \, \mathrm{d}t \right| \\
&\le \frac{(b-a)^2}{48} \left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(a)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(b)\right|^q}{4} \right)^{\frac{1}{q}} \\
&\le \frac{(b-a)^2}{48} \left[\left(\frac{5\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{5\left|f''(b)\right|^q + 3\left|f''(a)\right|^q}{8} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 3.17 *By choosing $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, $\alpha > 0$ for all $t \in [a, b]$ in Corollary 3.15, then we have the following midpoint type inequality for Riemann–Liouville fractional integrals*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\frac{a+b}{2}-} f(a) + J^\alpha_{\frac{a+b}{2}+} f(b) \right] \right| \\
&\le \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \\
&\times \left[\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(a)\right|^q}{\alpha+3} \right)^{\frac{1}{q}} + \left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q + \left|f''(b)\right|^q}{\alpha+3} \right)^{\frac{1}{q}} \right] \\
&\le \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \\
&\times \left[\left(\frac{(\alpha+4)\left|f''(a)\right|^q + (\alpha+2)\left|f''(b)\right|^q}{2(\alpha+3)} \right)^{\frac{1}{q}} + \left(\frac{(\alpha+2)\left|f''(a)\right|^q + (\alpha+4)\left|f''(b)\right|^q}{2(\alpha+3)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 3.18 *By choosing $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, $\alpha, k > 0$, for all $t \in [a, b]$ in Corollary 3.15, then we have the following midpoint type inequality for k-Riemann–Liouville fractional integrals*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}} \left[J^\alpha_{\frac{a+b}{2}-,k} f(a) + J^\alpha_{\frac{a+b}{2}+,k} f(b) \right] \right| \\
&\le \frac{k^2(b-a)^2}{8(\alpha+k)(\alpha+2k)} \\
&\times \left[\left(\frac{(\alpha+2k)\left|f''\left(\frac{a+b}{2}\right)\right|^q + k\left|f''(a)\right|^q}{\alpha+3k} \right)^{\frac{1}{q}} + \left(\frac{(\alpha+2k)\left|f''\left(\frac{a+b}{2}\right)\right|^q + k\left|f''(b)\right|^q}{\alpha+3k} \right)^{\frac{1}{q}} \right] \\
&\le \frac{k^2(b-a)^2}{8(\alpha+k)(\alpha+2k)} \\
&\times \left[\left(\frac{(\alpha+4k)\left|f''(a)\right|^q + (\alpha+2k)\left|f''(b)\right|^q}{2(\alpha+3k)} \right)^{\frac{1}{q}} + \left(\frac{(\alpha+2k)\left|f''(a)\right|^q + (\alpha+4k)\left|f''(b)\right|^q}{2(\alpha+3k)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

4 Conclusion

In this research, some generalized inequalities for twice differentiable functions using Sarikaya fractional integrals are obtained. Moreover, we prove that our results generalize the inequalities obtained in some earlier works. What's more, we obtain new inequalities for Riemann–Liouville fractional integrals and k-Riemann–Liouville fractional integrals. In the future works, mathematicians can focus to generalize our results by utilizing some other kinds of convex function classes.

Acknowledgements The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding There is no funding.

Declarations

Conflict of interest The authors declare that they have no competing interest.

Author contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Availability of data materials Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

References

1. Alomari, M.; Darus, M.; Dragomir, S.S.: New inequalities of Simpson's type for s-convex functions with applications. RGMIA Res. Rep. Coll. **12**(4) (2009)
2. Ali, M.A.; Kara, H.; Tariboon, J.; Asawasamrit, S.; Budak, H.; Hezenci, F.: Some new Simpson's formula type inequalities for twice differentiable convex functions via generalized fractional operators. Symmetry, **13**(12), Art. 2249 (2021)
3. Barani, A.; Barani, S.; Dragomir, S.S.: Refinements of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are quasi-convex. RGMIA Res. Rep. Coll. **14**, Art. 69 (2011)
4. Barani, A.; Barani, S.; Dragomir, S.S.: Refinements of Hermite-Hadamard inequalities for functions when a power of the absolute value of the second derivative is P-convex. J. Appl. Math. **2012**, Art. 615737 (2012) https://doi.org/10.1155/2012/615737
5. Budak, H.; Agarwal, P.: New generalized midpoint type inequalities for fractional integral. Miskolc Math. Notes **20**(2), 781–793 (2019)
6. Budak, H.; Kapucu, R.: New generalization of midpoint type inequalities for fractional integral. An. Ştiint. Univ Al. I. Cuza Iaşi. Mat. (N.S) **67**(1) (2021)
7. Bullen, P.S.: Error estimates for some elementary quadrature rules. Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika (602/633), 97–103 (1978)
8. Ciobotariu-Boer, V.: On Some Common Generalizations of two classes of integral inequalities for twice differentiable functions. **1**, 43–50 (2018)
9. Dragomir, S.S.; Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**(5), 91–95 (1998)
10. Du, T.; Li, Y.; Yang, Z.: A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions. Appl. Math. Comput. **293**, 358–369 (2017)
11. Du, T.; Luo, C.; Cao, Z.: On the Bullen-type inequalities via generalized fractional integrals and their applications. Fractals **29**(7), Article ID 2150188, 20 pages (2021)
12. Du, T.S.; Zhou, T.C.: On the fractional double integral inclusion relations having exponential kernels via interval-valued co-ordinated convex mappings. Chaos Solitons Fractals **156**, Article ID 111846, 19 pages (2022)
13. Erden, S.; Sarikaya, M.Z.: Generalized Bullen-type inequalities for local fractional integrals and its applications. Palest. J. Math. **9**(2), 945–956 (2020)
14. Ertuğral, F.; Sarikaya, M.Z.: Simpson type integral inequalities for generalized fractional integral. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matem áticas **113**(4), 3115–3124 (2019)
15. Hezenci, F.; Budak, H.; Kara, H.: New version of Fractional Simpson type inequalities for twice differentiable functions. Adv. Differ. Equ. **2021**(460), 1–10 (2021)

16. Iqbal, M.; Bhatti, M.I.; Nazeer, K.: Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals. Bull. Korean Math. Soc. **52**(3), 707–716 (2015)
17. İşcan, İ: Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions. J. Math. Article ID 346305, 10 pages (2014)
18. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. Appl. Math. Comput. **147**(5), 137–146 (2004). https://doi.org/10.1016/S0096-3003(02)00657-4
19. Mohammed, P.O.; Sarikaya, M.Z.: On generalized fractional integral inequalities for twice differentiable convex functions. J. Comput. Appl. Math. **372**, 112740 (2020)
20. Ozdemir, M.E.; Akdemir, A.O.; Kavurmacı, H.: On the Simpson's inequality for convex functions on the coordinates. Turk. J. Anal. Number Theory **2**(5), 165–169 (2014)
21. Park, J.: On Simpson-like type integral inequalities for differentiable preinvex functions. Appl. Math. Sci. **7**(121), 6009–6021 (2013)
22. Sarikaya, M.Z.; Aktan, N.: On the generalization of some integral inequalities and their applications. Math. Comput. Model. **54**(9–10), 2175–2182 (2011)
23. Sarikaya, M.Z.; Saglam, A.; Yıldırım, H.: New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex. Int. J. Open Probl. Comput. Sci. Math. **5**(3), 1–14 (2012)
24. Sarikaya, M.Z.; Ertugral, F.: On the generalized Hermite-Hadamard inequalities. Ann. Univ. Craiova-Math. Comput. Sci. Ser. **47**(1), 193–213 (2020)
25. Sarikaya, M.Z.; Set, E.; Özdemir, M.E.: On new inequalities of Simpson's type for convex functions. RGMIA Res. Rep. Coll. **13**(2), Article2 (2010)
26. Sarikaya, M.Z.; Set, E.; Özdemir, M.E.: On new inequalities of Simpson's type for s-convex functions. Comput. Math. Appl. **60**(8), 2191–2199 (2020) **1–38**
27. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N.: Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
28. Sarikaya, M.Z.; Budak, H.: Some integral inequalities for local fractional integrals. Int. J. Anal. Appl. **14**(1), 9–19 (2017)
29. Tomar, M.; Set, E.; Sarıkaya, M.Z.: Hermite-Hadamard type Riemann-Liouville fractional integral inequalities for convex functions. AIP Conf. Proc. **1726**, 020035 (2016). https://doi.org/10.1063/1.4945861
30. Xu, L.; Yu, S.H.; Du, T.S.: Properties and integral inequalities arising from the generalized n-polynomial convexity in the frame of fractal space. Fractals (2022). https://doi.org/10.1142/S0218348X22500840
31. Zhou, T.C.; Yuan, Z.R.; Du, T.S.: On the fractional integral inclusions having exponential kernels for interval-valued convex functions. Math. Sci. **2021**, 1–14 (2021). https://doi.org/10.1007/s40096-021-00445-x