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On New Fractional Version of Generalized Hermite-Hadamard Inequalities

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Abstract: In this study, we establish a novel version of Hermite-Hadamard inequalities through neoteric generalized Riemann-Liouville fractional integrals (RLFIs). For functions with the convex absolute values of derivatives, we create a variety of midpoint and trapezoid form inequalities, including the generalized RLFIs. Moreover, multiple fractional inequalities can be produced as special cases of the findings of this study.

Keywords: midpoint inequalities; Hermite-Hadamard inequality; generalized fractional operators

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1. Introduction

The inequalities found by Hermite and Hadamard for convex mappings are frequently considered in mathematical literature (see [1–3] and [4] (p. 137)). These inequalities explain that if ξ is a convex mapping from the interval $\mathbb{J}$ into $\mathbb{R}$ and $\vartheta_1, \vartheta_2 \in \mathbb{J}$ with $\vartheta_1 < \vartheta_2$, then

$$\xi\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\xi(\delta)d\delta \leq \frac{\xi(\vartheta_1)+\xi(\vartheta_2)}{2}. \tag{1}$$

If ξ is concave, the above inequality is satisfied reversely.

Inequality (1) is so significant that several generalizations of these inequalities involving various forms of convexities had been studied [5–7]. The Hermit–Hadamard extension for incomplete gamma functions [8] and s-type convexity for n polynomials [9] are two examples of this.

During the past few decades, many papers have focused on generalizing inequalities of the trapezoid and midpoint types, which provide limits for the two sides of inequality (1). Trapezoid and midpoint inequalities for convex functions were first derived by the authors in [10,11], respectively. Using RLFIs, Sarikaya et al. [12] expanded the inequality (1) and demonstrated several related trapezoid type inequalities. In contrast, Iqbal et al. discovered various midpoint type inequalities for convex mappings utilizing RLFIs in [13]. In addition, Jleli and Samet [14] investigated Hermite–Hadamard type inequalities and some equivalent trapezoid type inequalities for generalized fractional integrals.

Fractional derivatives have been extensively applied in the fractional calculus field and its implications for other scientific disciplines. With great success, Caputo and Riemann–Liouville derivatives were widely employed to describe complicated dynamics in physics, biology, engineering, and plentiful other domains [15–20]. It is generally known that systems with a memory impact often occur in natural events. Therefore, for each sort of data, we constantly inquire about the appropriate nonlocal model to use. In addition, other writers had investigated novel fractional generalized operators with singular, nonlocal, and local kernels [21–24]. The generalized fractional integral is one of the fundamental concepts in fractional calculus. Many novel fractional inequalities are produced through generalized fractional integrals. For further inequalities of a similar kind, please see [8,25–33].

The definitions and mathematical underpinnings of fractional calculus principles that are used later in this study are provided below.

Definition 1 ([16])**.** *Let $\chi \in L^1[\lambda,\mu]$, $\lambda < \mu \in \mathbb{R}$. The RLFIs $J^{\nu}_{\lambda+}\chi$ and $J^{\nu}_{\mu-}\chi$ of order $\nu > 0$ are given by*

$$J^{\nu}_{\lambda+}\chi(\rho) = \frac{1}{\Gamma(\nu)}\int_{\lambda}^{\rho}(\rho-\pi)^{\nu-1}\chi(\pi)d\pi, \ \ \rho > \lambda \tag{2}$$

and

$$J^{\nu}_{\mu-}\chi(\rho) = \frac{1}{\Gamma(\nu)}\int_{\rho}^{\mu}(\pi-\rho)^{\nu-1}\chi(\pi)d\pi, \ \ \rho < \mu \tag{3}$$

respectively. Here, Γ symbolizes the Gamma function and $J^{0}_{\lambda+}\chi(\rho) = J^{0}_{\mu-}\chi(\rho) = \chi(\rho)$.

The next generalized RLFIs were presented by Jarad et al. [34]. Additionally, they offered certain features and connections with a number of other fractional operators in the literature.

Definition 2 ([34])**.** *Let $\nu \in \mathbb{C}$, $Re(\nu) > 0$, and $\omega \in (0,1]$. For $\chi \in L^1[\lambda,\mu]$, the generalized RLFIs ${}^{\nu}_{\lambda}\Upsilon^{\omega}\chi$ and ${}^{\nu}\Upsilon^{\omega}_{\mu}\chi$ are defined by*

$${}^{\nu}_{\lambda}\Upsilon^{\omega}\chi(\rho) = \frac{1}{\Gamma(\nu)}\int_{\lambda}^{\rho}\left(\frac{(\rho-\lambda)^{\omega}-(\pi-\lambda)^{\omega}}{\omega}\right)^{\nu-1}\frac{\chi(\pi)}{(\pi-\lambda)^{1-\omega}}d\pi, \quad \rho > \lambda, \tag{4}$$

and

$${}^{\nu}\Upsilon^{\omega}_{\mu}\chi(\rho) = \frac{1}{\Gamma(\nu)}\int_{\rho}^{\mu}\left(\frac{(\mu-\rho)^{\omega}-(\mu-\pi)^{\omega}}{\omega}\right)^{\nu-1}\frac{\chi(\pi)}{(\mu-\pi)^{1-\omega}}d\pi, \quad \rho < \mu. \tag{5}$$

When $\lambda = 0$ and $\omega = 1$, Equation (4) coincides with RLFIs (2). Additionally, it corresponds with the generalized fractional integral with $\lambda = 0$ in [34] as well as the Hadamard fractional integral with $\lambda = 0$ and $\omega \to 0$ in [16]. Furthermore, Equation (5) and RLFIs (3) are the same when $\rho = 0$ and $\omega = 1$. Additionally, Equation (5) coincides with the generalized fractional integral with $\rho = 0$ in [34] and with the Hadamard fractional integral with $\rho = 0$ and $\omega \to 0$ in [16].

The current study aims to create novel versions of Hermite–Hadamard fractional inequalities for convex mappings. The proposed fractional operators are the generalized RLFIs (4) and (5). Furthermore, some extended midpoint and trapezoid inequalities are investigated under the generalized RLFIs. It is also important to note that the inequalities concerning Riemann–Liouville and Hadamard fractional integrals are produced as special cases of this study's findings.

In light of the aforementioned tendency and motivated by the continuing efforts, the remaining portions of this work are organized as follows. In Section 2, we present new versions of the Hermite–Hadamard inequality that works with the generalized RLFIs (4) and (5). In Section 3, we offer a large number of midpoint type inequalities for differentiable convex functions. In Section 4, by using functions whose absolute value derivatives are convex mappings, we construct various trapezoid inequalities. Finally, we give the paper's conclusion in Section 5.

2. New Version of Hermite–Hadamard Inequality

In this part, we discuss a new version of Hermite–Hadamard inequality that is applicable to the generalized RLFIs (4) and (5).

Theorem 1. *Assume χ is a convex function that goes from $[\lambda, \mu]$ into $\mathbb{R}$. Then, for $Re(\nu) > 0$ and $\omega \in (0, 1]$, the inequalities below are valid for the generalized RLFIs.*

$$\chi\left(\frac{\lambda+\mu}{2}\right) \leq \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) + {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right] \leq \frac{\chi(\lambda)+\chi(\mu)}{2}. \quad (6)$$

Proof. Since χ is convex on $[\lambda, \mu]$, for $\pi \in [0, 1]$, we can write

$$\begin{aligned} \chi\left(\frac{\lambda+\mu}{2}\right) &= \chi\left(\frac{1}{2}\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right) + \frac{1}{2}\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\right) \\ &\leq \frac{1}{2}\left(\chi\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right) + \chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\right) \\ &\leq \frac{\chi(\lambda)+\chi(\mu)}{2}, \end{aligned}$$

i.e.,

$$\begin{aligned} \chi\left(\frac{\lambda+\mu}{2}\right) &\leq \frac{1}{2}\left(\chi\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right) + \chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\right) \\ &\leq \frac{\chi(\lambda)+\chi(\mu)}{2}. \end{aligned} \quad (7)$$

If we multiply the inequality (7) by $\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}$ and integrate the resulting inequality on $[0, 1]$, we have

$$\begin{aligned} &\chi\left(\frac{\lambda+\mu}{2}\right)\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}d\pi \quad (8) \\ \leq\ & \frac{1}{2}\left[\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)d\pi\right. \\ &\left. + \int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)d\pi\right] \\ \leq\ & \frac{\chi(\lambda)+\chi(\mu)}{2}\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}d\pi. \end{aligned}$$

By changing variables, we achieve

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)d\pi \qquad (9)\\
=\;& \frac{2}{\mu-\lambda}\int_{\lambda}^{\frac{\lambda+\mu}{2}}\left(\frac{1-\left(\frac{2}{\mu-\lambda}(\rho-\lambda)\right)^{\omega}}{\omega}\right)^{\nu-1}\left(\frac{2}{\mu-\lambda}(\rho-\lambda)\right)^{\omega-1}\chi(\rho)d\rho\\
=\;& \left(\frac{2}{\mu-\lambda}\right)^{\omega\nu}\int_{\lambda}^{\frac{\lambda+\mu}{2}}\left(\frac{\left(\frac{\mu-\lambda}{2}\right)^{\omega}-(\rho-\lambda)^{\omega}}{\omega}\right)^{\nu-1}\frac{\chi(\rho)}{(\rho-\lambda)^{1-\omega}}d\rho\\
=\;& \frac{2^{\omega\nu}\Gamma(\nu)}{(\mu-\lambda)^{\omega\nu}}\,{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right),
\end{aligned}$$

and similarly

$$\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)d\pi = \frac{2^{\omega\nu}\Gamma(\nu)}{(\mu-\lambda)^{\omega\nu}}\,{}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right). \qquad (10)$$

On the other side, we have

$$\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}d\pi = \frac{1}{\nu\omega^{\nu}}. \qquad (11)$$

If we substitute the equalities (9)–(11) in (8), then we get

$$\begin{aligned}
&\chi\left(\frac{\lambda+\mu}{2}\right)\frac{1}{\nu\omega^{\nu}} \qquad (12)\\
\leq\;& \frac{2^{\omega\nu-1}\Gamma(\nu)}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)\,{}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\\
\leq\;& \frac{\chi(\lambda)+\chi(\mu)}{2}\frac{1}{\nu\omega^{\nu}},
\end{aligned}$$

which concludes the proof. □

Remark 1. *In Theorem 1, If we choose $\omega=1$, the following inequalities are achieved.*

$$\chi\left(\frac{\lambda+\mu}{2}\right)\leq\frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^{\nu}}\left[J^{\nu}_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right)\,J^{\nu}_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\leq\frac{\chi(\lambda)+\chi(\mu)}{2},$$

which are related to the integral operators in (2) and (3).

Remark 2. *In Theorem 1, if we put $\omega=\nu=1$, then inequalities (6) reduce to inequalities (1).*

3. Midpoint Type Inequalities

This section provides numerous inequalities of the midpoint type for differentiable convex functions. These findings present numerous bounds for the variation between the left and central parts of the inequality (6).

Lemma 1. *Let $\chi : [\lambda, \mu] \to \mathbb{R}$ be a differentiable mapping on (λ, μ) and $\chi' \in L^1[\lambda, \mu]$. Then, for $Re(\nu) > 0$ and $\omega \in (0, 1]$, the identity below is valid.*

$$\begin{aligned}
&\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right] - \chi\left(\frac{\lambda+\mu}{2}\right) \\
= &\frac{\omega^{\nu}(\mu-\lambda)}{4}\left[\int_0^1\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)d\pi\right. \\
&\left. - \int_0^1\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi'\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)d\pi\right].
\end{aligned}$$

Proof. Employing integration by parts gives

$$\begin{aligned}
I_1 = &\int_0^1\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)d\pi \qquad (13) \\
= &\frac{2}{\mu-\lambda}\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\Bigg|_0^1 \\
&+\frac{2\nu}{\mu-\lambda}\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)d\pi \\
= &-\frac{2}{\omega^{\nu}(\mu-\lambda)}\chi\left(\frac{\lambda+\mu}{2}\right) + \frac{2\nu}{\mu-\lambda}\left(\frac{2}{\mu-\lambda}\right)^{\omega\nu}\int_{\frac{\lambda+\mu}{2}}^{\mu}\left(\frac{\left(\frac{\mu-\lambda}{2}\right)^{\omega} - (\mu-\rho)^{\omega}}{\omega}\right)^{\nu-1}\frac{\chi(\rho)}{(\mu-\rho)^{1-\omega}}d\rho \\
= &-\frac{2}{\omega^{\nu}(\mu-\lambda)}\chi\left(\frac{\lambda+\mu}{2}\right) + \frac{2^{\omega\nu+1}\Gamma(\nu+1)}{(\mu-\lambda)^{\omega\nu+1}}\,{}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_2 = &\int_0^1\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi'\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)d\pi \qquad (14) \\
= &-\frac{2}{\mu-\lambda}\left[\frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]\chi\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)\Bigg|_0^1 \\
&-\frac{2\nu}{\mu-\lambda}\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1}(1-\pi)^{\omega-1}\chi\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)d\pi \\
= &\frac{2}{\omega^{\nu}(\mu-\lambda)}\chi\left(\frac{\lambda+\mu}{2}\right) - \frac{2\nu}{\mu-\lambda}\left(\frac{2}{\mu-\lambda}\right)^{\omega\nu}\int_{\lambda}^{\frac{\lambda+\mu}{2}}\left(\frac{\left(\frac{\mu-\lambda}{2}\right)^{\omega} - (\rho-\lambda)^{\omega}}{\omega}\right)^{\nu-1}\frac{\chi(\rho)}{(\rho-\lambda)^{1-\omega}}d\rho \\
= &\frac{2}{\omega^{\nu}(\mu-\lambda)}\chi\left(\frac{\lambda+\mu}{2}\right) - \frac{2^{\omega\nu+1}\Gamma(\nu+1)}{(\mu-\lambda)^{\omega\nu+1}}\,{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right).
\end{aligned}$$

By equalities (13) and (14), we obtain

$$\frac{\omega^{\nu}(\mu-\lambda)}{4}[I_1 - I_2] = \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right] - \chi\left(\frac{\lambda+\mu}{2}\right).$$

So, the proof is accomplished. □

Theorem 2. *Let $\chi : [\lambda, \mu] \to \mathbb{R}$ be a differentiable mapping on (λ, μ). If $|\chi'|$ is convex on $[\lambda, \mu]$, then we get the inequality below.*

$$\begin{aligned} &\left|\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \\ \leq\; &\frac{(\mu-\lambda)}{8}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)\left[|\chi'(\mu)|+|\chi'(\lambda)|\right], \end{aligned} \tag{15}$$

where $B(\cdot,\cdot)$ refers to the Beta function.

Proof. By Lemma 1, we have

$$\begin{aligned} &\left|\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \\ \leq\; &\frac{\omega^{\nu}(\mu-\lambda)}{4}\left[\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right|d\pi\right. \\ &\left.+\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left|\chi'\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)\right|d\pi\right]. \end{aligned} \tag{16}$$

Considering the convexity of $|\chi'|$, we acquire

$$\begin{aligned} &\left|\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \\ \leq\; &\frac{\omega^{\nu}(\mu-\lambda)}{8}\left[\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left[(1-\pi)|\chi'(\lambda)|+(1+\pi)|\chi'(\mu)|\right]d\pi\right. \\ &\left.+\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left[(1-\pi)|\chi'(\mu)|+(1+\pi)|\chi'(\lambda)|\right]\right] \\ =\; &\frac{\omega^{\nu}(\mu-\lambda)}{4}\left(\int_0^1\left[\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right]d\pi\right)\left[|\chi'(\mu)|+|\chi'(\lambda)|\right] \\ =\; &\frac{(\mu-\lambda)}{4}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)\left[|\chi'(\mu)|+|\chi'(\lambda)|\right], \end{aligned}$$

which finishes the proof. □

Remark 3. *In Theorem 2, If we set $\omega = 1$, we arrive at the next inequality.*

$$\begin{aligned} &\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^{\nu}}\left[J^{\nu}_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right) J^{\nu}_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \\ \leq\; &\frac{(\mu-\lambda)}{4}\left(\frac{\nu}{\nu+1}\right)\left[|\chi'(\mu)|+|\chi'(\lambda)|\right], \end{aligned} \tag{17}$$

which is linked to the integral fractional operators in (2) and (3).

Remark 4. *If we put $\omega = \nu = 1$ in Theorem 2, then Theorem 2 returns to ([11], Theorem 2.2).*

Theorem 3. *Let $\chi : [\lambda, \mu] \to \mathbb{R}$ be a differentiable mapping on (λ, μ). If $|\chi'|^q$ is convex on $[\lambda, \mu]$ for $q > 1$, then the inequality below is satisfied.*

$$\begin{aligned} &\left| \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}} \left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right) + {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right) \right] - \chi\left(\frac{\lambda+\mu}{2}\right) \right| \qquad (18) \\ \leq\ & \frac{\mu-\lambda}{4}\left(1 - \frac{1}{\omega}B\left(p\nu+1, \frac{1}{\omega}\right)\right)^{\frac{1}{p}} \\ &\times \left[\left(\frac{|\chi'(\lambda)|^{\mu} + 3|\chi'(\mu)|^{\mu}}{4} \right)^{\frac{1}{q}} + \left(\frac{3|\chi'(\lambda)|^{q} + |\chi'(\mu)|^{q}}{4} \right)^{\frac{1}{q}} \right] \\ \leq\ & \frac{\mu-\lambda}{4}\left(4 - \frac{4}{\omega}B\left(p\nu+1, \frac{1}{\omega}\right)\right)^{\frac{1}{p}} [|\chi'(\lambda)| + |\chi'(\mu)|], \end{aligned}$$

where $\frac{1}{p} = 1 - \frac{1}{q}$.

Proof. Utilizing the convexity of $|\chi'|^q$ and Hölder's inequality [35], we get

$$\begin{aligned} &\int_0^1 \left| \frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega} \right)^{\nu} \right| \left| \chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu \right) \right| d\pi \qquad (19) \\ \leq\ & \left(\int_0^1 \left| \frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega} \right)^{\nu} \right|^p d\pi \right)^{\frac{1}{p}} \left(\int_0^1 \left| \chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu \right) \right|^q d\pi \right)^{\frac{1}{q}} \\ \leq\ & \frac{1}{\omega^{\nu}} \left(\int_0^1 \left(1 - (1-(1-\pi)^{\omega})^{p\nu}\right) d\pi \right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1-\pi}{2}|\chi'(\lambda)|^q + \frac{1+\pi}{2}|\chi'(\mu)|^q \right] d\pi \right)^{\frac{1}{q}} \\ =\ & \frac{1}{\omega^{\nu}} \left(1 - \frac{1}{\omega}B\left(p\nu+1, \frac{1}{\omega}\right)\right)^{\frac{1}{p}} \left(\frac{|\chi'(\lambda)|^q + 3|\chi'(\mu)|^q}{4} \right)^{\frac{1}{q}}. \end{aligned}$$

where we take advantage of the fact:

$$(\zeta - \eta)^j \leq \zeta^j - \eta^j, \qquad (20)$$

for any $\zeta > \eta \geq 0$ and $j \geq 1$.

Likewise, we can gain

$$\begin{aligned} &\int_0^1 \left| \frac{1}{\omega^{\nu}} - \left(\frac{1-(1-\pi)^{\omega}}{\omega} \right)^{\nu} \right| \left| \chi'\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu \right) \right| d\pi \qquad (21) \\ \leq\ & \frac{1}{\omega^{\nu}} \left(1 - \frac{1}{\omega}B\left(p\nu+1, \frac{1}{\omega}\right)\right)^{\frac{1}{p}} \left(\frac{3|\chi'(\lambda)|^q + |\chi'(\mu)|^q}{4} \right)^{\frac{1}{q}}. \end{aligned}$$

By substituting the inequalities (19) and (21) in (16), the first inequality of (18) is achieved. The second inequality can be fulfilled by setting $\lambda_1 = |\chi'(\lambda)|^q$, $\mu_1 = 3|\chi'(\mu)|^q$, $\lambda_2 = 3|\chi'(\lambda)|^q$ and $\mu_2 = |\chi'(\mu)|^q$. and utilizing the relation:

$$\sum_{j=1}^{n} (\lambda_j + \mu_j)^r \leq \sum_{j=1}^{n} \lambda_j^r + \sum_{j=1}^{n} \mu_j^r, \ 0 \leq r < 1$$

So, the desired result can be directly reached. □

Remark 5. *If we allow $\omega = 1$ in Theorem 3, then we get the next inequality.*

$$\begin{aligned}
&\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^{\nu}}\left[J^{\nu}_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right)+J^{\nu}_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{p\nu}{p\nu+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\chi'(\lambda)|^{\mu}+3|\chi'(\mu)|^{\mu}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\chi'(\lambda)|^{q}+|\chi'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{4p\nu}{p\nu+1}\right)^{\frac{1}{p}}[|\chi'(\lambda)|+|\chi'(\mu)|].
\end{aligned}$$

Remark 6. *If we allow $\omega = \nu = 1$ in Theorem 3, then Theorem 3 and ([11], Theorem 2.4) are identical.*

Theorem 4. *Assume $\chi : [\lambda, \mu] \to \mathbb{R}$ is a differentiable mapping on (λ, μ). If $|\chi'|^q$ is convex on $[\lambda, \mu]$, for some $q \geq 1$, then the inequality below is fulfilled.*

$$\begin{aligned}
&\left|\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)+{}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right| \qquad (22)\\
&\leq\frac{(\mu-\lambda)}{4}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{1}{4}-\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\lambda)|^{q}\right.\right. \\
&\left.+\left(\frac{3}{4}-\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^{q}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{3}{4}-\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)\right)|\chi'(\lambda)|^{q} \\
&\left.+\left(\frac{1}{4}-\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^{q}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Proof. Applying the convexity of $|\chi'|^q$, we have

$$\begin{aligned}
&\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right|d\pi \qquad (23)\\
\leq\ & \left(\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|d\pi\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left|\frac{1}{\omega^{\nu}}-\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\right|\left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right|^{q}d\pi\right)^{\frac{1}{q}} \\
\leq\ & \frac{1}{\omega^{\nu}}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left(1-(1-(1-\pi)^{\omega})^{\nu}\right)\left[\frac{1-\pi}{2}|\chi'(\lambda)|^{q}+\frac{1+\pi}{2}|\chi'(\mu)|^{q}\right]d\pi\right)^{\frac{1}{q}} \\
=\ & \frac{1}{\omega^{\nu}}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}-\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\lambda)|^{q} \\
&+\left(\frac{3}{4}-\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)\right)|\chi'(\mu)|^{q}\Bigg),
\end{aligned}$$

and similarly, we have

$$\int_0^1\left|\frac{1}{\omega^\nu}-\left(\frac{1-(1-\pi)^\omega}{\omega}\right)^\nu\right|\left|\chi'\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)\right|d\pi \tag{24}$$
$$\le \frac{1}{\omega^\nu}\left(1-\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\left(\left(\frac{3}{4}-\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)\right)|\chi'(\lambda)|^q\right.$$
$$\left.+\left(\frac{1}{4}-\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^q\right),$$

where we have employed the integral inequality of power mean [36]:

$$\int_\lambda^\mu |l(\pi)m(\pi)|d\pi \le \left(\int_\lambda^\mu |l(\pi)|d\pi\right)^{1-\frac{1}{q}}\left(\int_\lambda^\mu |l(\pi)||m(\pi)|^q d\pi\right)^{\frac{1}{q}}, \quad |l|,|l||m|^q\in L^1[\lambda,\mu]. \tag{25}$$

By considering (23) and (24) in (16), we obtain the desired inequality (22). □

Remark 7. *If we set $\omega=1$ in Theorem 4, then we have the inequality below.*

$$\left|\frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^\nu}\left[J^\nu_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right)+J^\nu_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]-\chi\left(\frac{\lambda+\mu}{2}\right)\right|$$
$$\le \frac{\mu-\lambda}{4}\left(\frac{\nu}{\nu+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{1}{4}-\frac{1}{2(\nu+1)(\nu+2)}\right)|\chi'(\lambda)|^q+\left(\frac{3}{4}-\frac{2\nu+3}{2(\nu+1)(\nu+2)}\right)|\chi'(\mu)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{3}{4}-\frac{2\nu+3}{2(\nu+1)(\nu+2)}\right)|\chi'(\lambda)|^q+\left(\frac{1}{4}-\frac{1}{2(\nu+1)(\nu+2)}\right)|\chi'(\mu)|^q\right)^{\frac{1}{q}}\right].$$

Remark 8. *If we choose $\omega=\nu=1$ in Theorem 4, then Theorem 4 reduces to ([37], Remark 4.10).*

4. Trapezoid Type Inequalities

In this section, we set up some trapezoid inequalities by using functions whose derivatives in absolute value are convex mappings. These inequalities give bounds for the difference between the middle and right terms of the inequalities (6).

Lemma 2. *Let $\chi:[\lambda,\mu]\to\mathbb{R}$ be a differentiable mapping on (λ,μ) and $\chi'\in L[\lambda,\mu]$. Then, for $Re(\nu)>0$ and $\omega\in(0,1]$, we get the following identity:*

$$\frac{\chi(\lambda)+\chi(\mu)}{2}-\frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^\nu}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^\omega\chi\left(\frac{\lambda+\mu}{2}\right)\ {}^{\nu}\Upsilon^\omega_\mu\chi\left(\frac{\lambda+\mu}{2}\right)\right]$$
$$=\frac{\omega^\nu(\mu-\lambda)}{4}\left[\int_0^1\left(\frac{1-(1-\pi)^\omega}{\omega}\right)^\nu\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)d\pi\right.$$
$$\left.-\int_0^1\left(\frac{1-(1-\pi)^\omega}{\omega}\right)^\nu\chi'\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)d\pi\right].$$

Proof. Applying integration by parts, we have

$$I_3 = \int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu} \chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right) d\pi \tag{26}$$

$$= \frac{2}{\mu-\lambda}\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu} \chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\Bigg|_0^1$$

$$-\frac{2\nu}{\mu-\lambda}\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu-1} (1-\pi)^{\omega-1} \chi\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right) d\pi$$

$$= \frac{2}{\omega^{\nu}(\mu-\lambda)}\chi(\mu) - \frac{2^{\omega\nu+1}\Gamma(\nu+1)}{(\mu-\lambda)^{\omega\nu+1}}\, {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)$$

and similarly,

$$I_4 = \int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu} \chi'\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right) d\pi \tag{27}$$

$$= -\frac{2}{\omega^{\nu}(\mu-\lambda)}\chi(\lambda) + \frac{2^{\omega\nu+1}\Gamma(\nu+1)}{(\mu-\lambda)^{\omega\nu+1}}\, {}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right).$$

By equalities (26) and (27), we obtain

$$\frac{\omega^{\nu}(\mu-\lambda)}{4}[I_3 - I_4] = \frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)\, {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right].$$

This completes the proof. □

Theorem 5. *Let $\chi : [\lambda, \mu] \to \mathbb{R}$ be a differentiable mapping on (λ, μ). If $|\chi'|$ is convex on $[\lambda, \mu]$, then the inequality below holds.*

$$\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)\, {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \tag{28}$$

$$\leq \frac{(\mu-\lambda)}{8\omega}B\left(\nu+1, \frac{1}{\omega}\right)[|\chi'(\mu)| + |\chi'(\lambda)|].$$

Proof. By Lemma 2, we acquire

$$\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)\, {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \tag{29}$$

$$\leq \frac{\omega^{\nu}(\mu-\lambda)}{4}\left[\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\left|\chi'\left(\frac{1-\pi}{2}\lambda + \frac{1+\pi}{2}\mu\right)\right| d\pi\right.$$

$$\left. + \int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\left|\chi'\left(\frac{1+\pi}{2}\lambda + \frac{1-\pi}{2}\mu\right)\right| d\pi\right].$$

Since $|\chi'|$ is convex, we can write

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)\ {}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \\
\leq\ & \frac{\omega^{\nu}(\mu-\lambda)}{8}\left[\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}[(1-\pi)|\chi'(\lambda)|+(1+\pi)|\chi'(\mu)|]d\pi\right. \\
&\left.+\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}[(1-\pi)|\chi'(\mu)|+(1+\pi)|\chi'(\lambda)|]\right] \\
=\ & \frac{\omega^{\nu}(\mu-\lambda)}{4}\left(\int_0^1\left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}d\pi\right)[|\chi'(\mu)|+|\chi'(\lambda)|] \\
=\ & \frac{(\mu-\lambda)}{4\omega}B\left(\nu+1,\frac{1}{\omega}\right)[|\chi'(\mu)|+|\chi'(\lambda)|],
\end{aligned}$$

which finishes the proof. □

Remark 9. *If we assign $\omega=1$ in Theorem 5, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^{\nu}}\left[J^{\nu}_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right)+J^{\nu}_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{1}{\nu+1}\right)[|\chi'(\mu)|+|\chi'(\lambda)|],
\end{aligned}$$

which is related to the integral fractional operators in (2) *and* (3)*.*

Remark 10. *If we choose $\omega=\nu=1$ in Theorem 5, then Theorem 5 reduces to ([10], Theorem 2.2).*

Theorem 6. *Let $\chi:[\lambda,\mu]\to\mathbb{R}$ be a differentiable mapping on (λ,μ). If $|\chi'|^{q}$ is convex on $[\lambda,\mu]$ for $q>1$, then we have the next inequality*

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^{\nu}}{(\mu-\lambda)^{\omega\nu}}\left[{}^{\nu}_{\lambda}\Upsilon^{\omega}\chi\left(\frac{\lambda+\mu}{2}\right)+{}^{\nu}\Upsilon^{\omega}_{\mu}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \qquad (30) \\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\chi'(\lambda)|^{\mu}+3|\chi'(\mu)|^{\mu}}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\chi'(\lambda)|^{q}+|\chi'(\mu)|^{q}}{4}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{4}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}}[|\chi'(\lambda)|+|\chi'(\mu)|],
\end{aligned}$$

where $\frac{1}{p}=1-\frac{1}{q}$.

Proof. According to the convexity of $|\chi'|^q$ and the well-known Hölder inequality, we get

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\pi)^\omega}{\omega}\right)^\nu \left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right| d\pi \qquad (31)\\
\leq\ & \left(\int_0^1 \left(\frac{1-(1-\pi)^\omega}{\omega}\right)^{\nu p} d\pi\right)^{\frac{1}{p}} \left(\int_0^1 \left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right| d\pi\right)^{\frac{1}{q}}\\
\leq\ & \frac{1}{\omega^\nu}\left(\int_0^1 (1-(1-\pi)^\omega)^{p\nu} d\pi\right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1-\pi}{2}|\chi'(\lambda)|^q+\frac{1+\pi}{2}|\chi'(\mu)|^q\right] d\pi\right)^{\frac{1}{q}}\\
=\ & \frac{1}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}} \left(\frac{|\chi'(\lambda)|^\mu+3|\chi'(\mu)|^b}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{\omega^\nu}-\left(\frac{1-(1-\pi)^\omega}{\omega}\right)^\nu\right| \left|\chi'\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)\right| d\pi \qquad (32)\\
\leq\ & \frac{1}{\omega^\nu}\left(\frac{1}{\omega}B\left(p\nu+1,\frac{1}{\omega}\right)\right)^{\frac{1}{p}} \left(\frac{3|\chi'(\lambda)|^q+|\chi'(\mu)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

If we substitute from (31) and (32) in (29), we obtain the first inequality of (30). The second inequality of (30) is obvious from the inequality (20). □

Remark 11. *If we take $\omega = 1$ in Theorem 6, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^\nu}\left[J^\nu_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right) + J^\nu_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right|\\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{1}{p\nu+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\chi'(\lambda)|^b+3|\chi'(\mu)|^\mu}{4}\right)^{\frac{1}{q}} + \left(\frac{3|\chi'(\lambda)|^q+|\chi'(\mu)|^q}{4}\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{\mu-\lambda}{4}\left(\frac{4}{p\nu+1}\right)^{\frac{1}{p}}[|\chi'(\lambda)|+|\chi'(\mu)|].
\end{aligned}$$

Remark 12. *If we choose $\omega = \nu = 1$ in Theorem 6, then Theorem 6 reduces to ([37], Remark 5.5).*

Theorem 7. *Assume $\chi : [\lambda, \mu] \to \mathbb{R}$ is a differentiable mapping on (λ, μ). If $|\chi'|^q$ is convex on $[\lambda, \mu]$ for $q \geq 1$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2} - \frac{2^{\omega\nu-1}\Gamma(\nu+1)\omega^\nu}{(\mu-\lambda)^{\omega\nu}}\left[{}^\nu_\lambda\Upsilon^\omega\chi\left(\frac{\lambda+\mu}{2}\right) + {}^\nu\Upsilon^\omega_\mu\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right| \qquad (33)\\
\leq\ & \frac{\mu-\lambda}{4\omega}\left(B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\frac{1}{2}\mu\left(\nu+1,\frac{2}{\omega}\right)|\chi'(\lambda)|^q + \left(B\left(\nu+1,\frac{1}{\omega}\right)-\frac{1}{2}\mu\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(B\left(\nu+1,\frac{1}{\omega}\right)-\frac{1}{2}B\left(\nu+1,\frac{2}{\omega}\right)|\chi'(\lambda)|^q+\frac{1}{2}\mu\left(\nu+1,\frac{2}{\omega}\right)|\chi'(\mu)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Employing the integral inequality of power mean (25) and the convexity of $|\chi'|^q$ yields

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right|d\pi \qquad (34)\\
\le\ & \left(\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}d\pi\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\left|\chi'\left(\frac{1-\pi}{2}\lambda+\frac{1+\pi}{2}\mu\right)\right|^{q}d\pi\right)^{\frac{1}{q}}\\
\le\ & \frac{1}{\omega^{\nu}}\left(\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 (1-(1-\pi)^{\omega})^{\nu}\left[\frac{1-\pi}{2}|\chi'(\lambda)|^{q}+\frac{1+\pi}{2}|\chi'(\mu)|^{q}\right]d\pi\right)^{\frac{1}{q}}\\
=\ & \frac{1}{\omega^{\nu}}\left(\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)|\chi'(\lambda)|^{q}+\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^{q}\right),
\end{aligned}$$

and similarly, we acquire

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\pi)^{\omega}}{\omega}\right)^{\nu}\left|\chi'\left(\frac{1+\pi}{2}\lambda+\frac{1-\pi}{2}\mu\right)\right|d\pi \qquad (35)\\
\le\ & \frac{1}{\omega^{\nu}}\left(\frac{1}{\omega}B\left(\nu+1,\frac{1}{\omega}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{1}{2\omega}\left(2B\left(\nu+1,\frac{1}{\omega}\right)-B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\lambda)|^{q}+\left(\frac{1}{2\omega}B\left(\nu+1,\frac{2}{\omega}\right)\right)|\chi'(\mu)|^{q}\right).
\end{aligned}$$

By considering (34) and (35) in (29), we obtain the desired inequality (33). □

Remark 13. *If we assign $\omega = 1$ in Theorem 7, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{\chi(\lambda)+\chi(\mu)}{2}-\frac{2^{\nu-1}\Gamma(\nu+1)}{(\mu-\lambda)^{\nu}}\left[J^{\nu}_{\lambda+}\chi\left(\frac{\lambda+\mu}{2}\right)+J^{\nu}_{\mu-}\chi\left(\frac{\lambda+\mu}{2}\right)\right]\right|\\
\le\ & \frac{\mu-\lambda}{4(\nu+1)}\left[\left(\left(\frac{1}{2(\nu+2)}\right)|\chi'(\lambda)|^{q}+\left(\frac{2\nu+3}{2(\nu+2)}\right)|\chi'(\mu)|\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\left(\frac{2\nu+3}{2(\nu+2)}\right)|\chi'(\lambda)|^{q}+\left(\frac{1}{2(\nu+2)}\right)|\chi'(\mu)|\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 14. *If we choose $\omega = \nu = 1$ in Theorem 7, then Theorem 7 reduces to ([37], Remark 5.6).*

5. Conclusions

Novel versions of Hermite–Hadamard inequalities through beneficial generalized RLFIs have been established in this study. Further, numerous midpoint and trapezoid form inequalities, including the suggested fractional integrals, have been proved for functions with the convex absolute values of derivatives. When $\omega = 1$, it was evident that the findings of this study could be simplified to the results gained by the usual RLFIs in (2) and (3). In addition, when $\omega = \nu = 1$, the findings of Kirmaci [11] and Budak et al. [37] may be derived. However, by using the more generic fractional operators listed in [20], one can expand and enhance these findings.

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