

ALMOST α -COSYMPLECTIC f -MANIFOLDS

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Abstract. The purpose of this paper is to study a new class of framed manifolds. Such manifolds are called almost α -cosymplectic f -manifolds. For some special cases of α and s , one obtains (almost) α -cosymplectic, (almost) C -manifolds, and (almost) Kenmotsu f -manifolds. Moreover, several tensor conditions are studied. We conclude our results with a general example on α -cosymplectic f -manifolds.

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1. Introduction and preliminaries

Let M a real $(2n + s)$ -dimensional smooth manifold. M admits an f -structure ([1], [7]) if there exists a non null smooth $(1,1)$ tensor field ϕ , of the tangent bundle TM , satisfying $\phi^3 + \phi = 0$, $\text{rank } \phi = 2n$. An f -structure is a generalization of almost complex ($s = 0$) and almost contact ($s = 1$) structure ([5], [7]). In the latter case, M is orientable ([6]). Corresponding to two complementary projection operators P and Q applied to TM , defined by $P = -\phi^2$ and $Q = \phi^2 + I$, where I is the identity operator, there exist two complementary distributions D and D^\perp such that $\dim(D) = 2n$ and $\dim(D^\perp) = s$. The following relations hold $\phi P = P\phi = \phi$, $\phi Q = Q\phi = 0$, $\phi^2 P = -P$, $\phi^2 Q = 0$. Thus, we have an almost complex distribution $(D, J = \phi|_D, J^2 = -I)$ and ϕ acts on D^\perp as a null operator. It follows that $TM = D \oplus D^\perp$, $D \cap D^\perp = \{0\}$. Assume that D_p^\perp is spanned by s globally defined orthonormal vectors $\{\xi_i\}$ at each point $p \in M$, $(1 \leq i, j, \dots \leq s)$,

with its dual set $\{\eta^i\}$. Then one obtains $\phi^2 = -I + \sum_{i=1}^s \eta^i \otimes \xi_i$. In the above case, M is called a globally framed manifold (or simply an f -manifold) ([1], [4] and [5]) and we denote its framed structure by $M(\phi, \xi_i)$. From the above conditions one has $\phi \xi_i = 0, \eta^i \circ \phi = 0, \eta^i(\xi_j) = \delta_i^j$. Now, we consider a Riemannian metric g on M that is compatible with an f -structure such that

$$\begin{aligned} g(\phi X, Y) + g(X, \phi Y) &= 0, \\ g(\phi X, \phi Y) &= g(X, Y) - \sum_{i=1}^s \eta^i(X) \eta^i(Y), \quad g(X, \xi_i) = \eta^i(X). \end{aligned}$$

In the above case, we say that M is a metric f -manifold and its associated structure will be denoted by $M(\phi, \xi_i, \eta^i, g)$.

A framed structure $M(\phi, \xi_i)$ is said to be normal ([4]) if the torsion tensor N_ϕ of ϕ is zero i.e., if $N_\phi = N + 2 \sum_{i=1}^s d\eta^i \otimes \xi_i = 0$, where N denotes the Nijenhuis tensor field of ϕ .

Define a 2-form Ω on M by $\Omega(X, Y) = g(X, \phi Y)$, for any $X, Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f -manifold satisfies the following formula ([1]):

$$\begin{aligned} 2g((\nabla_X \phi)Y, Z) &= 3d\Omega(X, \phi Y, \phi Z) - 3d\Omega(X, Y, Z) \\ &\quad + g(N(Y, Z), \phi X) + N_j^2(Y, Z) \eta^j(X) \\ &\quad + 2d\eta^j(\phi Y, X) \eta^j(Z) - 2d\eta^j(\phi Z, X) \eta^j(Y), \end{aligned}$$

where the tensor field N_j^2 is defined by $N_j^2(X, Y) = (L_{\phi X} \eta^j)Y - (L_{\phi Y} \eta^j)X = 2d\eta^j(\phi X, Y) - 2d\eta^j(\phi Y, X)$, for each $j \in \{1, \dots, s\}$.

Following the terminology introduced by BLAIR ([1]), we say that a normal metric f -manifold is a K -manifold if its 2-form Ω closed (i.e., $d\Omega = 0$). Since $\eta^1 \wedge \dots \wedge \eta^s \wedge \Omega^n \neq 0$, a K -manifold is orientable. Furthermore, we say that a K -manifold is a C -manifold if each η^i is closed, an S -manifold if $d\eta^1 = d\eta^2 = \dots = d\eta^s = \Omega$.

Note that, if $s = 1$, namely if M is an almost contact metric manifold, the condition $d\Omega = 0$ means that M is quasi-Sasakian. M is said a K -contact manifold if $d\eta = \Omega$ and ξ is Killing.

FALCITELLI and PASTORE [3] introduced and studied a class of manifolds which is called almost Kenmotsu f -manifold. Such manifolds admit an f -structure with s -dimensional parallelizable kernel. A metric $f.pk$ -manifold of dimension $(2n + s)$, $s \geq 1$, with $f.pk$ -structure (ϕ, ξ_i, η^i, g) , is said to be a almost Kenmotsu $f.pk$ -manifold if the 1-forms η^i 's are closed and $d\Omega =$

$2\eta^1 \wedge \Omega$. Several foliations canonically associated with an almost Kenmotsu $f.pk$ -manifold are studied and locally conformal almost Kenmotsu $f.pk$ -manifolds are characterized by Falcitelli and Pastore.

In this paper, we consider a wide subclass of f -manifolds called almost α -cosymplectic f -manifolds. Firstly, we give the concept of almost α -cosymplectic f -manifold and state general curvature properties. We derive several important formulas on almost α -cosymplectic f -manifolds. These formulas enable us to find the geometrical properties of almost α -cosymplectic f -manifolds with η -parallel tensors h_i and φh_i . We also examine the tensor fields τ_i 's which are defined by $g(\tau_i X, Y) = (\mathcal{L}_{\xi_i} g)(X, Y)$, for arbitrary vector fields X, Y on M . Then we give some results on η -parallelity, cyclic parallelity, Codazzi condition. Finally, we give an explicit example of almost α -cosymplectic f -manifold.

Throughout this paper we denote by $\bar{\eta} = \eta^1 + \eta^2 + \dots + \eta^s$, $\bar{\xi} = \xi^1 + \xi^2 + \dots + \xi^s$ and $\bar{\delta}_i^j = \delta_i^1 + \delta_i^2 + \dots + \delta_i^s$.

2. Almost α -cosymplectic f -manifolds

We introduce a notion of an almost α -cosymplectic f -manifold for any real number α which is defined as metric f -manifold with f -structure $(\varphi, \xi_i, \eta^i, g)$ satisfying the conditions $d\eta^i = 0$, $d\Omega = 2\alpha\bar{\eta} \wedge \Omega$. The manifold is called generalized almost Kenmotsu f -manifold for $\alpha = 1$.

Let M be an almost α -cosymplectic f -manifold. Since the distribution D is integrable, we have $L_{\xi_i} \eta^j = 0$, $[\xi_i, \xi_j] \in D$ and $[X, \xi_j] \in D$ for any $X \in \Gamma(D)$. Then the Levi-Civita connection is given by:

$$(2.1) \quad \begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 2\alpha g\left(\sum_{j=1}^s (g(\varphi X, Y)\xi_j - \eta^j(Y)\varphi X), Z\right) \\ &\quad + g(N(Y, Z), \varphi X), \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. Putting $X = \xi_i$ we obtain $\nabla_{\xi_i} \varphi = 0$ which implies $\nabla_{\xi_i} \xi_j \in D^\perp$ and then $\nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i$, since $[\xi_i, \xi_j] = 0$.

We put $A_i X = -\nabla_X \xi_i$ and $h_i = \frac{1}{2}(L_{\xi_i} \varphi)$, where L denotes the Lie derivative operator.

Proposition 1. *For any $i \in \{1, \dots, s\}$ the tensor field A_i is a symmetric operator such that*

- 1) $A_i(\xi_j) = 0$, for any $j \in \{1, \dots, s\}$
- 2) $A_i \circ \varphi + \varphi \circ A_i = -2\alpha \varphi$.
- 3) $tr(A_i) = -2\alpha n$

Proof. $d\eta^i = 0$ implies that A_i is symmetric.

1) For any $i, j, k \in \{1, \dots, s\}$ deriving $g(\xi_i, \xi_j) = \delta_i^j$ with respect to ξ_k , using $\nabla_{\xi_i} \xi_j = \nabla_{\xi_j} \xi_i$, we get $2g(\xi_k, A_i(\xi_j)) = 0$. Since $\nabla_{\xi_i} \xi_j \in D^\perp$, we conclude $A_i(\xi_j) = 0$.

2) For any $Z \in \Gamma(TM)$, we have $\varphi(N(\xi_i, Z)) = (L_{\xi_i} \varphi)Z$ and, on the other hand, since $\nabla_{\xi_i} \varphi = 0$,

$$(2.2) \quad L_{\xi_i} \varphi = A_i \circ \varphi - \varphi \circ A_i$$

One can easily obtain from (2.2)

$$(2.3) \quad -A_i X = -\alpha \varphi^2 X - \varphi h_i X$$

Applying (2.1) with $Y = \xi_i$, we have $2g(\varphi A_i X, Z) = -2\alpha g(\varphi X, Z) - g(\varphi N(\xi_i, Z), X)$, which implies the desired result.

3) Considering local adapted orthonormal frame $\{X_1, \dots, X_n, \varphi X_1, \dots, \varphi X_n, \xi_1, \dots, \xi_s\}$, by 1) and 2), one has

$$\text{tr} A_i = \sum_{j=1}^n g(A_i X_j, X_j) + g(A_i \varphi X_j, \varphi X_j) = -2\alpha \sum_{j=1}^n g(\varphi X_j, \varphi X_j) = -2\alpha n.$$

□

Proposition 2 ([1]). *For any $i \in \{1, \dots, s\}$ the tensor field h_i is a symmetric operator and satisfies*

- 1) $h_i \xi_j = 0$, for any $j \in \{1, \dots, s\}$
- 2) $h_i \circ \varphi + \varphi \circ h_i = 0$
- 3) $\text{tr} h_i = 0$
- 4) $\text{tr} \varphi h_i = 0$

Proposition 3. $\nabla \varphi$ satisfies the following relation:

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = \sum_{i=1}^s [-\alpha(\eta^i(Y)\varphi X + 2g(X, \varphi Y)\xi_i) - \eta^i(Y)h_i X].$$

Proof. By direct computations, we get $\varphi N(X, Y) + N(\varphi X, Y) = 2 \sum_{i=1}^s \eta^i(X)h_i Y$, and $\eta^i(N(\varphi X, Y)) = 0$. From (2.1) and the equations above, the proof is completed. □

Proposition 4. *Let M be an almost α -cosymplectic f -manifold. The integral manifolds of D are almost Kaehler manifolds with mean curvature vector field $H = -\alpha \bar{\xi}$.*

Proof. Let \widetilde{M} be an integral manifold of D . We know that $(D, J = \varphi|_D, J^2 = -I)$ is an almost complex distribution and the induced metric \widetilde{g} on \widetilde{M} is a Hermitian metric. Therefore, for any $X, Y \in \Gamma(\widetilde{M})$, we have the induced 2-form on \widetilde{M} such that $\widetilde{\Omega}(X, Y) = \widetilde{g}(X, JY) = g(X, \varphi Y) = \Omega(X, Y)$ and $d\widetilde{\Omega} = 0$ on \widetilde{M} . In this manner, \widetilde{M} is an almost Kähler manifold. Computing the second fundamental form B , since, A_i 's are the Weingarten operators in the directions ξ_i , we get,

$$(2.4) \quad B(X, Y) = \sum_{i=1}^s g(A_i X, Y) \xi_i = \sum_{i=1}^s [-\alpha g(X, Y) \xi_i + g(\varphi h_i X, Y) \xi_i].$$

Using the Proposition 2 and (2.3). Now, we choose a local orthonormal frame $\{e_1, e_2, \dots, e_{2n}\}$ such that $e_{l+n} = \varphi e_l$, for $l = 1, 2, \dots, n$, in $T\widetilde{M}$. Taking $X = Y = e_p$ in (2.4) and summing over $p = 1, 2, \dots, 2n$, we get $H = \frac{1}{2n} \sum_{i=1}^s (\text{tr } A_i) \xi_i = -\alpha \bar{\xi}$. \square

Proposition 5. *Let M be an almost α -cosymplectic f -manifold and \widetilde{M} be an integral manifold of D . Then*

- 1) *when $\alpha = 0$, \widetilde{M} is totally geodesic if and only if all the operators h_i vanish;*
- 2) *when $\alpha \neq 0$, \widetilde{M} is totally umbilic if and only if all the operators h_i vanish.*

Proof. The proof is obvious through (2.4). \square

Proposition 6. *Under the same situation as in Proposition 5, M is α -cosymplectic f -manifold with structure f -structure $(\varphi, \xi_i, \eta^i, g)$ if and only if the integral manifolds of D are tangentially Kähler and all the operators h_i vanish.*

Proof. If the structure is normal, for any $X \in \Gamma(TM)$, one obtains that

$$(2.5) \quad \begin{aligned} 0 &= N(X, \xi_j) = N_\varphi(X, \xi_j) + 2 \sum_{i=1}^s d\eta^i(X, \xi_j) \xi_i \\ &= -\varphi[\varphi X, \xi_j] + \varphi^2[X, \xi_j] + 2 \sum_{i=1}^s d\eta^i(X, \xi_j) \xi_i = 2\varphi h_j X. \end{aligned}$$

Hence, all the operators h_i vanish. On the other hand, for each $X, Y \in \Gamma(D)$ we have

$$(2.6) \quad N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] - [X, Y] = N_{J=\varphi|_D}(X, Y).$$

It is obvious that $N_J = 0$ if and only if almost complex structure J is integrable. Therefore, the proof is completed by (2.5) and (2.6). \square

Theorem 1 ([1]). *A C -manifold M^{2n+s} is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .*

3. Curvature properties

Proposition 7. *Let M be an almost α -cosymplectic f -manifold. Then we have*

$$(3.1) \quad \begin{aligned} R(X, Y)\xi_i &= \alpha^2 \sum_{k=1}^s \left[\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y \right] \\ &\quad - \alpha \sum_{k=1}^s \left[\eta^k(X)\varphi h_k Y - \eta^k(Y)\varphi h_k X \right] + (\nabla_Y \varphi h_i) X - (\nabla_X \varphi h_i) Y. \end{aligned}$$

Proof. Using the Riemannian curvature tensor and (2.3), we obtain (3.1). \square

Using (2.3) and (3.1), by simple computations, we have the following proposition.

Proposition 8. *For an almost α -cosymplectic f -manifold with the f -structure $(\varphi, \xi_i, \eta^i, g)$, the following relations hold*

$$(3.2) \quad \begin{aligned} R(X, \xi_j)\xi_i &= \sum_{k=1}^s \delta_j^k \left[\alpha^2 \varphi^2 X + \alpha \varphi h_k X \right] \\ &\quad + \alpha \varphi h_i X - h_i h_j X + \varphi (\nabla_{\xi_j} h_i) X \end{aligned}$$

$$(3.3) \quad R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2 \left[-\alpha^2 \varphi^2 X + h_i h_j X \right],$$

$$(3.4) \quad \begin{aligned} (\nabla_{\xi_j} h_i) X &= -\varphi R(X, \xi_j)\xi_i + \sum_{k=1}^s \delta_j^k \left[-\alpha^2 \varphi X - \alpha h_k X \right] \\ &\quad - \alpha h_i X - \varphi h_i h_j X, \end{aligned}$$

$$(3.5) \quad S(X, \xi_i) = -2n\alpha^2 \sum_{k=1}^s \eta^k(X) - (\operatorname{div} \varphi h_i) X,$$

$$(3.6) \quad S(\xi_i, \xi_j) = -2n\alpha^2 - \operatorname{tr}(h_j h_i),$$

Corollary 1. *Let M be an almost α -cosymplectic f -manifold. The Ricci tensor satisfies the following conditions:*

- 1) $S(\xi_i, \xi_i)$ always takes negative value when $\alpha \neq 0$,
- 2) If all the values $S(\xi_i, \xi_i)$ vanish then any leaf of D is totally geodesic.
- 3) If all the values $S(\xi_i, \xi_i)$ vanish and M is normal then M is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .

Proof. The proof is clear through (3.6). \square

The tensor τ was introduced by CHERN and HAMILTON [2] and is defined by $g(\tau X, Y) = (L_\xi g)(X, Y)$ for arbitrary vector fields X, Y on a contact metric manifold. Now, we define and examine this tensor field for an almost α -cosymplectic f -manifold

Proposition 9. *An almost α -cosymplectic f -manifold with f -structure $(\varphi, \xi_i, \eta^i, g)$ has tensor fields τ_i such that $\tau_i X = 2\nabla_X \xi_i$, where τ_i 's are defined by $g(\tau_i X, Y) = (L_{\xi_i} g)(X, Y)$ for arbitrary vector fields X, Y on M .*

Proof. Using the definition of the tensor fields τ_i , we get

$$\begin{aligned} (L_{\xi_i} g)(X, Y) &= g(\nabla_X \xi_i, Y) + g(X, \nabla_Y \xi_i) \\ &= 2g(-\alpha\varphi^2 X - \varphi h_i X, Y) \end{aligned}$$

for arbitrary vector fields X, Y on M . Applying the formula (2.3), the proof is completed. \square

Proposition 10. *Let M be a locally symmetric almost α -cosymplectic f -manifold. Then, $\nabla_{\xi_\gamma} h_i = 0$, for any $\gamma \in \{1, \dots, s\}$.*

Proof. Notice that (3.3) can be written as $\frac{1}{2}(R(\xi_j, \cdot)\xi_i - \varphi R(\xi_j, \varphi \cdot)\xi_i) = -\alpha^2 \varphi^2 + h_i h_j$ and since the operator $R(\xi_j, \cdot)\xi_i$ is parallel with respect to ξ_k , we get $\nabla_{\xi_k} h_i h_j = 0$. Applying ∇_{ξ_γ} to (3.4), we obtain $\nabla_{\xi_\gamma}(\nabla_{\xi_j} h_i) = -\alpha \nabla_{\xi_\gamma} h_i - \alpha \nabla_{\xi_\gamma} h_j$. Moreover, $\nabla_{\xi_k} h_i h_j = 0$ implies that $(\nabla_{\xi_k} h_i) h_j + h_i (\nabla_{\xi_k} h_j) = 0$, and applying ∇_{ξ_γ} to this equation, we get $\nabla_{\xi_\gamma} h_i = 0$. \square

Theorem 2. *Let M be a locally symmetric generalized almost Kenmotsu f -manifold. Then the following conditions are equivalent:*

- 1) M is a generalized α -Kenmotsu f -manifold;
- 2) all the operators h_i vanish.

Moreover, if any of the conditions above holds, then M cannot have constant sectional curvature.

Proof. Assuming that M is a generalized α -Kenmotsu f -manifold, we have $\nabla \xi_i = -\alpha \varphi^2$ and, by (2.3), all the operators h_i vanish. Now, supposing all the operators h_i vanish, it follows that $\nabla \xi_i = -\alpha \varphi^2$ and $\nabla \eta^i = \alpha (g - \sum_{k=1}^s \eta^k \otimes \eta^k)$ and by (3.1), $R(X, Y)\xi_i = \alpha^2 \sum_{k=1}^s [\eta^k(Y)\varphi^2 X - \eta^k(X)\varphi^2 Y]$. So, M is a generalized α -Kenmotsu f -manifold. Moreover, The sectional curvature of any 2-plane spanned by $\{Y, \xi_i\}$ is $K(Y, \xi_i) = -\alpha^2 \|\varphi Y\|^2$, for all vector fields Y on M . So, the sectional curvature of any 2-plane spanned by $\{\xi_i, \xi_j\}$, for any $i, j \in \{1, 2, \dots, s\}$, vanishes and one gets that the sectional curvature of any plane spanned by $Y \in D$ and ξ_i is equal to $-\alpha^2$. \square

4. Some tensor conditions

For any vector field X on M , we can take $X = X^T + \sum_{i=1}^s \eta^i(X)\xi_i$ where X^T is the tangential part of X and $\sum_{i=1}^s \eta^i(X)\xi_i$ is the normal part of X . We can rewrite η -parallel condition for a given almost α -cosymplectic f -manifold. We say that any $(1, 1)$ -type tensor field B is η -parallel if and only if $g((\nabla_{X^T} B)Y^T, Z^T) = 0$, for $X^T, Y^T, Z^T \in \mathcal{D}$.

The starting point of the investigation of almost α -cosymplectic f -manifolds with η -parallel tensors h_i and φh_i is the following propositions:

Proposition 11. *Let M be an almost α -cosymplectic f -manifold and h_i 's are $(1, 1)$ -type tensor fields. If the tensor fields h_i 's are η -parallel, then*

$$(4.1) \quad \begin{aligned} (\nabla_X h_i)Y &= - \sum_{k=1}^s \eta^k(X) \left[\varphi l_{ki} Y + \sum_{\gamma=1}^s \delta_k^\gamma [\alpha^2 \varphi Y + \alpha h_\gamma Y] + \varphi h_i h_k Y + \alpha h_i Y \right] \\ &\quad - \sum_{k=1}^s \eta^k(Y) [\alpha h_i X + \varphi h_i h_k X] - \sum_{k=1}^s g(\alpha h_i X + \varphi h_i h_k X, Y) \xi_k, \end{aligned}$$

for all vector fields X, Y on M , where the tensor $l_{ki} = R(\cdot, \xi_k)\xi_i$ is the Jacobi operator with respect to the characteristic vector fields and h_i 's are $(1, 1)$ -type tensor fields.

Proof. Suppose that each h_i is η -parallel. Denoting the component of X orthogonal to ξ by X^T , we obtain

$$\begin{aligned}
0 &= g((\nabla_{X^T} h_i) Y^T, Z^T) \\
&= g\left(\left(\nabla_{X - \sum_{k=1}^s \eta^k(X) \xi_k} h_i\right) (Y - \sum_{k=1}^s \eta^k(Y) \xi_k), Z - \sum_{k=1}^s \eta^k(Z) \xi_k\right) \\
&= g((\nabla_X h_i) Y, Z) - \sum_{k=1}^s \eta^k(X) g((\nabla_{\xi_k} h_i) Y, Z) - \sum_{k=1}^s \eta^k(Y) g((\nabla_X h_i) \xi_k, Z) \\
&\quad - \sum_{k=1}^s \eta^k(Z) g((\nabla_X h_i) Y, \xi_k) = g((\nabla_X h_i) Y, -\varphi^2 Z) \\
&\quad - \sum_{k=1}^s \eta^k(X) g((\nabla_{\xi_k} h_i) Y, Z) - \sum_{k=1}^s \eta^k(Y) g((\nabla_X h_i) \xi_k, Z),
\end{aligned}$$

for all vector fields X, Y, Z on M . Using (2.3) and (3.4), the proof is completed. \square

Proposition 12. *Let M be an almost α -cosymplectic f -manifold. If the tensor fields φh_i 's are η -parallel, then*

$$\begin{aligned}
(\nabla_X \varphi h_i) Y &= \sum_{k=1}^s \eta^k(X) \left[l_{ki} Y - \sum_{\gamma=1}^s \delta_k^\gamma [\alpha^2 \varphi^2 Y + \alpha \varphi h_\gamma Y] + h_i h_k Y - \alpha \varphi h_i Y \right] \\
(4.2) \quad &- \sum_{k=1}^s \eta^k(Y) [\alpha \varphi h_i X - h_i h_k X] - \sum_{k=1}^s g(\alpha \varphi h_i X - h_i h_k X, Y) \xi_k.
\end{aligned}$$

Proof. We consider that φh_i is η -parallel. Thus,

$$\begin{aligned}
0 &= g((\nabla_{X^T} \varphi h_i) Y^T, Z^T) \\
&= g\left(\left(\nabla_{X - \sum_{k=1}^s \eta^k(X) \xi_k} \varphi h_i\right) (Y - \sum_{k=1}^s \eta^k(Y) \xi_k), Z - \sum_{k=1}^s \eta^k(Z) \xi_k\right) \\
&= g((\nabla_X \varphi h_i) Y, Z) - \sum_{k=1}^s \eta^k(X) g((\nabla_{\xi_k} \varphi h_i) Y, Z) \\
&\quad - \sum_{k=1}^s \eta^k(Y) g((\nabla_X \varphi h_i) \xi_k, Z) - \sum_{k=1}^s \eta^k(Z) g((\nabla_X \varphi h_i) Y, \xi_k)
\end{aligned}$$

for all vector fields X, Y on M . If we simplify the equation above, then

$$\begin{aligned} g((\nabla_X \varphi h_i) Y, Z) &= \sum_{k=1}^s \eta^k(X) g((\nabla_{\xi_k} \varphi h_i) Y, Z) + \sum_{k=1}^s \eta^k(Y) g((\nabla_X \varphi h_i) \xi_k, Z) \\ &\quad + \sum_{k=1}^s \eta^k(Z) g((\nabla_X \varphi h_i) Y, \xi_k). \end{aligned}$$

Using (2.3) and $(\nabla_{\xi_k} \varphi h_i) Y = \varphi(\nabla_{\xi_k} h_i) Y$, the proof is completed. \square

Theorem 3. *An almost α -cosymplectic f -manifold with the η -parallel tensor fields φh_i 's satisfy the following relation:*

$$(4.3) \quad R(X, Y) \xi_i = \sum_{k=1}^s \eta^k(Y) l_{ki} X - \eta^k(X) l_{ki} Y,$$

where $l_{ki} = R(., \xi_k) \xi_i$ is the Jacobi operator with respect to the characteristic vector fields ξ_k and ξ_i .

Proof. Using (3.1) and (4.2), we get

$$\begin{aligned} R(X, Y) \xi_i &= \alpha^2 \sum_{k=1}^s \left[\eta^k(Y) \varphi^2 X - \eta^k(X) \varphi^2 Y \right] \\ &\quad - \alpha \sum_{k=1}^s \left[\eta^k(X) \varphi h_k Y - \eta^k(Y) \varphi h_k X \right] \\ &\quad + \sum_{k=1}^s \eta^k(Y) \left[l_{ki} X - \sum_{\gamma=1}^s \delta_k^\gamma \left[\alpha^2 \varphi^2 X + \alpha \varphi h_\gamma X \right] + h_i h_k X - \alpha \varphi h_i X \right] \\ &\quad - \sum_{k=1}^s \eta^k(X) \left[\alpha \varphi h_i Y - h_i h_k Y \right] - \sum_{k=1}^s g(\alpha \varphi h_i Y - h_i h_k Y, X) \xi_k \\ &\quad - \sum_{k=1}^s \eta^k(X) \left[l_{ki} Y - \sum_{\gamma=1}^s \delta_k^\gamma \left[\alpha^2 \varphi^2 Y + \alpha \varphi h_\gamma Y \right] + h_i h_k Y - \alpha \varphi h_i Y \right] \\ &\quad + \sum_{k=1}^s \eta^k(Y) \left[\alpha \varphi h_i X - h_i h_k X \right] + \sum_{k=1}^s g(\alpha \varphi h_i X - h_i h_k X, Y) \xi_k. \end{aligned}$$

Then, we can easily write (4.3) by simplifying the equation above. \square

Theorem 4. *An almost α -cosymplectic f -manifold has negative pointwise constant ξ_i -sectional curvature.*

Proof. Let M be an almost α -cosymplectic f -manifold with a pointwise constant ξ_i -sectional curvature $K(p), p \in M$. It means that $g(R(X^T, \xi_i)\xi_i, X^T) = K_i(p)g(X^T, X^T)$ for all tangent vectors X^T orthogonal to ξ_i at the point $p \in M$, i.e., $X^T \in \mathcal{D}$. Putting $X^T = X - \sum_{k=1}^s \eta^k(X)\xi_k$ and using the symmetries of curvature tensor R , we see that the equation above is equivalent to $\varphi l_{ii}X = K_i\varphi X$, for any vector field X , where K_i is a smooth function in M . From the equation (3.4), we get

$$(\nabla_{\xi_i} h_i)X = -K_i\varphi X + \sum_{k=1}^s \delta_i^k [-\alpha^2\varphi X - \alpha h_k X] - \alpha h_i X - \varphi h_i^2 X$$

Separating the equation above to symmetric and skew-symmetric parts, we obtain

$$(\nabla_{\xi_i} h_i)X = -\alpha \left[\sum_{k=1}^s \delta_i^k h_k X + h_i X \right]$$

and

$$(4.4) \quad -K_i\varphi X - \alpha^2\varphi X - \varphi h_i^2 X = 0.$$

Let $\{E_1, E_2, \dots, E_{2n}, \xi_1, \dots, \xi_s\}$ be an orthonormal basis of the tangent space at any point. Firstly, we apply inner product with φX both two sides in (4.4). Then, the sum for $1 \leq j \leq 2n$ of the relation (4.4) with $X = E_j$ yields $K_i = -(\alpha^2 + \frac{\|h_i\|^2}{2n})$. \square

Remark 1. The conditions " h_i is a Codazzi tensor" and " φh_i is a Codazzi tensor" are equivalent.

Proposition 13. *Let M be an almost α -cosymplectic f -manifold. If the tensor field φh_i 's (or h_i 's) are Codazzi, then the following conditions hold:*

- 1) *If $\alpha = 0$ then the integral manifolds of D are totally geodesic.*
- 2) *If $\alpha = 0$ and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .*
- 3) *The integral manifolds of D are totally umbilic when $\alpha \neq 0$.*

Proof. Let the tensor field φh_i be Codazzi. Taking $X = \xi_j$, $Y \in D$, we get $(\nabla_{\xi_j} h_i)Y - (\nabla_Y h_i)\xi_j = 0$. By using (3.4), we obtain $-\varphi l_{ji}Y = \alpha^2 \varphi Y + \alpha h_j Y$. By (3.3), we have $h_i h_j Y = 0$, for any i, j , so $h_i = 0$, for any i , and the statement follows by Proposition 5. \square

Theorem 5. *Let M be an almost α -cosymplectic f -manifold. If the tensors τ_i 's are parallel and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .*

Proof. Let the tensor fields τ_i 's are the parallel tensor field. It means that $(\nabla_X \tau_i)Y = 0$, for all $i \in \{1, 2, \dots, s\}$ and $X, Y \in \Gamma(TM)$. Putting $Y = \xi_j$ for any $j \in \{1, 2, \dots, s\}$ and contracting the equation above with respect to X , we get $-2n\alpha^2 + \alpha \text{trace}(\varphi h_j) + \alpha \text{trace}(\varphi h_i) - \text{trace}(h_i h_j) = 0$. If we examine the last equation for all values of i and j and, we see that suffices $\alpha = 0$ and $h_\varsigma = 0$ for all $\varsigma \in \{1, 2, \dots, s\}$. Hence, the proof is obvious by Theorem 1. \square

Proposition 14. *Let M be an almost α -cosymplectic f -manifold. If the tensor fields τ_i 's are η -parallel, then*

$$(4.5) \quad (\nabla_X \varphi h_i)Y = \sum_{k=1}^s \left[\eta^k(X) (\nabla_{\xi_k} \varphi h_i)Y - \eta^k(Y) \varphi h_i \nabla_X \xi_k + g((\nabla_X \varphi h_i) \xi_k, Y) \xi_k \right].$$

Proof. Suppose that τ_i is η -parallel. It satisfies equation $g((\nabla_{X^T} \tau_i)Y^T, Z^T) = 0$ for any vector fields X^T, Y^T, Z^T on D . By simple computations, we get

$$(4.6) \quad (\nabla_X \tau_i)Y = \sum_{k=1}^s \left[g((\nabla_X \tau_i)Y, \xi_k) \xi_k + \eta^k(Y) (\nabla_X \tau_i) \xi_k + \eta^k(X) (\nabla_{\xi_k} \tau_i)Y \right].$$

On the other hand, one can easily obtain that

$$(4.7) \quad (\nabla_X \tau_i)Y = \sum_{v=1}^s [-2\alpha \eta^v(Y) \nabla_X \xi_v - 2\alpha g(\nabla_X \xi_v, Y) \xi_v] - 2(\nabla_X \varphi h_i)Y.$$

From (4.6) and (4.7) we have the desired result. \square

Theorem 6. *Let M be an almost α -cosymplectic f -manifold. If the tensor fields τ_i 's are η -parallel, then $R(X, Y)\xi_i = \sum_{k=1}^s \eta^k(Y)l_{ki}X - \eta^k(X)l_{ki}Y$.*

Proof. Using equation (4.5), we obtain the following difference:

$$(4.8) \quad \begin{aligned} (\nabla_Y \varphi h_i)X - (\nabla_X \varphi h_i)Y &= \sum_{k=1}^s \eta^k(Y) (\nabla_{\xi_k} \varphi h_i)X - \sum_{k=1}^s \eta^k(X) (\nabla_{\xi_k} \varphi h_i)Y \\ &+ \sum_{k=1}^s \eta^k(Y) \varphi h_i \nabla_X \xi_k - \sum_{k=1}^s \eta^k(X) \varphi h_i \nabla_Y \xi_k. \end{aligned}$$

Using (3.4) and (4.8), we get, $R(X, Y)\xi_i = \sum_{k=1}^s \eta^k(Y)l_{ki}X - \eta^k(X)l_{ki}Y$. Hence, the proof is completed. \square

Proposition 15. *Let M be an almost α -cosymplectic f -manifold. If the tensor field φh_i 's are cyclically parallel, then the following conditions hold:*

- 1) *If $\alpha = 0$ then the integral manifolds of D are totally geodesic.*
- 2) *If $\alpha = 0$ and M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .*
- 3) *The integral manifolds of D are totally umbilic when $\alpha \neq 0$.*

Proof. The hypothesis can be written

$$g((\nabla_X \varphi h_i)Y, \xi_j) + g((\nabla_Y \varphi h_i)\xi_j, X) + g((\nabla_{\xi_j} \varphi h_i)X, Y) = 0$$

for all vector fields X, Y on M . From this equation, we get the following equation $(\nabla_{\xi_j} h_i)X = 2\alpha h_i X + \varphi(h_i \circ h_j + h_j \circ h_i)X$. Making use of (3.2), we obtain $R(X, \xi_i)\xi_i = \sum_{k=1}^s \delta_i^k [\alpha^2 \varphi^2 X + \alpha \varphi h_k X] + 3\alpha \varphi h_i X - 3h_i^2 X$. Applying φ to the last equation, substituting φX for X and using (3.3), we get $h_i^2 = 0$. So, we obtain $\text{trace}(h_i^2) = 0$, for any i , and apply Proposition 5. \square

Theorem 7. *Let M be an almost α -cosymplectic f -manifold. If the tensors τ_i 's are cyclically parallel, then the following conditions hold:*

- 1) *The integral manifolds of D are totally geodesic*
- 2) *If M is normal then M is a locally decomposable Riemannian manifold which is locally the product of a Kaehler manifold M_1^{2n} and an Abelian Lie group M_2^s .*

Proof. As $\tau_i X = -2\alpha \varphi^2 X - 2\varphi h_i X$, the hypothesis can be written $g((\nabla_X \tau_i)Y, Z) + g((\nabla_Y \tau_i)Z, X) + g((\nabla_Z \tau_i)X, Y) = 0$, for arbitrary vector

fields X, Y, Z on M . Using (2.3) and replacing Z by ξ_j , we reduce the following relation:

$$(4.9) \quad \varphi(\nabla_{\xi_j} h_i)X = 2\alpha^2 \varphi^2 X + 2\alpha \varphi h_j X + 2\alpha \varphi h_i X - h_i h_j X - h_j h_i X.$$

Substitution of the (4.9) into (3.2), we get

$$(4.10) \quad l_{ji}X - \varphi l_{ji} \varphi X = 6\alpha^2 \varphi^2 X - 4h_i h_j X - 2h_j h_i X.$$

From equality of (3.3) and (4.10), we have $2\alpha^2 \varphi^2 X - h_j h_i X - h_i h_j X = 0$. Hence, the proof is clear. \square

Example 1. Let, $n = 1$ and $s = 2$. We consider the 4-dimensional manifold $M = \{(x, y, z_1, z_2) \in \mathbb{R}^4\}$, where (x, y, z_1, z_2) are the standart coordinates in \mathbb{R}^4 . The vector field $e_1 = f_1(z_1, z_2) \frac{\partial}{\partial x} + f_2(z_1, z_2) \frac{\partial}{\partial y}$, $e_2 = -f_2(z_1, z_2) \frac{\partial}{\partial x} + f_1(z_1, z_2) \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z_1}$, $e_4 = \frac{\partial}{\partial z_2}$, where f_1 and f_2 are given by

$$\begin{aligned} f_1(z_1, z_2) &= c_2 e^{-\alpha(z_1+z_2)} \cos(z_1 + z_2) - c_1 e^{-\alpha(z_1+z_2)} \sin(z_1 + z_2), \\ f_2(z_1, z_2) &= c_1 e^{-\alpha(z_1+z_2)} \cos(z_1 + z_2) + c_2 e^{-\alpha(z_1+z_2)} \sin(z_1 + z_2) \end{aligned}$$

for constant $c_1, c_2, \alpha \in \mathbb{R}$. It is obvious that $\{e_1, e_2, e_3, e_4\}$ are linearly independent at each point of M . Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

for all $i, j \in \{1, 2, 3, 4\}$ and given by the tensor product $g = \frac{1}{f_1^2 + f_2^2} (dx \otimes dx + dy \otimes dy) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2$. Let η^1 and η^2 be the 1-form defined by $\eta^1(X) = g(X, e_3)$ and $\eta^2(X) = g(X, e_4)$, respectively, for any vector field X on M and φ be the $(1, 1)$ tensor field defined by $\varphi(e_1) = e_2, \varphi(e_2) = -e_1, \varphi(e_3) = \xi_1 = 0, \varphi(e_4) = \xi_2 = 0$. Also, let h_i 's be the $(1, 1)$ tensor fields defined by $h_i(e_1) = -e_1, h_i(e_2) = e_2, h_i(e_3) = 0$ and $h_i(e_4) = 0$. Then using linearity of g and φ , we have

$$\begin{aligned} \varphi^2 X &= -X + \eta^1(X) e_3 + \eta^2(X) e_4 \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta^1(X) \eta^1(Y) - \eta^2(X) \eta^2(Y) \\ \eta^1(e_3) &= 1 \text{ and } \eta^2(e_4) = 1 \end{aligned}$$

for any vector fields on M

It remains to prove that $d\Omega = 2\bar{\eta} \wedge \Omega$ and Nijenhuis torsion tensor of φ is zero. It follows that $\Omega(e_1, e_2) = -1$ and otherwise $\Omega(e_i, e_j) = 0$ for $i \leq j$. Therefore, the essential non-zero component of Ω is $\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -\frac{1}{f_1^2 + f_2^2} = -\frac{e^{2\alpha(z_1+z_2)}}{c_1^2 + c_2^2}$, and hence

$$(4.11) \quad \Omega = -\frac{2e^{2\alpha(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy.$$

Consequently, the exterior derivative $d\Omega$ is given by

$$(4.12) \quad d\Omega = -\frac{4\alpha e^{2\alpha(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy \wedge (dz_1 + dz_2).$$

Since $\eta^1 = dz_1$ and $\eta^2 = dz_2$, by (4.11) and (4.12), we find $d\Omega = 2\alpha(\eta^1 + \eta^2) \wedge \Omega$. Let ∇ be the Levi-Civita connection with respect to the metric g . Then, we obtain $[e_1, e_3] = [e_1, e_4] = \alpha e_1 - e_2$, $[e_2, e_3] = [e_2, e_4] = e_1 + \alpha e_2$, $[e_1, e_2] = 0$, $[e_3, e_4] = 0$. In conclusion, it can be noted that Nijenhuis torsion tensor of φ is zero. Thus, the manifold is an α -cosymplectic f -manifold.

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REFERENCES

1. BLAIR, D.E. – *Geometry of manifolds with structural group $U(n) \times O(s)$* , J. Differential Geometry, 4 (1970), 155–167.
2. CHERN, S.S.; HAMILTON, R.S. – *On Riemannian metrics adapted to three-dimensional contact manifolds*, Lecture Notes in Math., 1111, Workshop Bonn 1984 (Bonn, 1984), 279–308, Springer, Berlin, 1985.
3. FALCITELLI, M.; PASTORE, A.M. – *Almost Kenmotsu f -manifolds*, Balkan J. Geom. Appl., 12 (2007), 32–43.
4. GOLDBERG, S.I.; YANO, K. – *On normal globally framed f -manifolds*, Tohoku Math. J., 22 (1970), 362–370.
5. GOLDBERG, S.I.; YANO, K. – *Globally framed f -manifolds*, Illinois J. Math., 15 (1971), 456–474.

6. YANO, K.; KON, M. – *Structures on Manifolds*, Series in Pure Mathematics, 3, World Scientific Publishing Co., Singapore, 1984.
7. YANO, K. – *On a structure defined by a tensor field f of type $(1, 1)$ satisfying $f^3 + f = 0$* , Tensor (N.S.), 14 (1963), 99–109.

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