

Article

A Quantum Calculus View of Hermite–Hadamard–Jensen–Mercer Inequalities with Applications

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Abstract: In this paper, we derive some new quantum estimates of generalized Hermite–Hadamard–Jensen–Mercer type of inequalities, essentially using q -differentiable convex functions. With the help of numerical examples, we check the validity of the results. We also discuss some special cases which show that our results are quite unifying. To show the efficiency of our main results, we offer some interesting applications to special means.

Keywords: convex function; quantum calculus; Jensen–Mercer inequalities; differentiable function; means

MSC: 05A30; 26A51; 26D10; 26D15



Citation: Bin-Mohsin, B.; Saba, M.; Javed, M.Z.; Awan, M.U.; Budak, H.; Nonlaopon, K. A Quantum Calculus View of Hermite–Hadamard–Jensen–Mercer Inequalities with Applications. *Symmetry* **2022**, *14*, 1246. <https://doi.org/10.3390/sym14061246>

Academic Editor: Ioan Raşa

Received: 29 May 2022

Accepted: 13 June 2022

Published: 16 June 2022

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1. Introduction and Preliminaries

A set $\mathcal{C} \subseteq \mathbb{R}$ is said to be convex if

$$(1 - \tau)\omega_1 + \tau\omega_2 \in \mathcal{C}$$

for all $\omega_1, \omega_2 \in \mathcal{C}$ and $\tau \in [0, 1]$.

A function $\Psi : \mathcal{C} \rightarrow \mathbb{R}$ is said to be convex if

$$\Psi((1 - \tau)\omega_1 + \tau\omega_2) \leq (1 - \tau)\Psi(\omega_1) + \tau\Psi(\omega_2)$$

for all $\omega_1, \omega_2 \in \mathcal{C}$ and $\tau \in [0, 1]$.

The theory of convexity has played a vital role in developing inequality theory. Many inequalities are direct consequences of applying the convexity property of the functions. One of the most studied results in this regard is Hermite–Hadamard’s inequality. This inequality provides us with necessary and sufficient conditions for a function to be convex. For details, see [1]. It reads as:

If $\Psi : I = [\omega_1, \omega_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, then

$$\Psi\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \Psi(\Theta) d\Theta \leq \frac{\Psi(\omega_1) + \Psi(\omega_2)}{2}.$$

Another significant result of convexity property of the functions is Jensen’s inequality, which reads as:

Let $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be non-negative weights such that $\sum_{i=0}^n \mu_i = 1$. If $\Psi : I = [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, then

$$\Psi\left(\sum_{i=1}^n \mu_i \Theta_i\right) \leq \sum_{i=1}^n \mu_i \Psi(\Theta_i),$$

where $\Theta_i \in [\omega_1, \omega_2]$ and $\mu_i \in [0, 1], (i = \overline{1, n})$. For more details, see [2].

The following inequality is known as Jensen-Mercer’s inequality.

Let $\Psi : I = [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function; then

$$\Psi\left(\omega_1 + \omega_2 - \sum_{i=1}^n \mu_i \Theta_i\right) \leq \Psi(\omega_1) + \Psi(\omega_2) - \sum_{i=1}^n \mu_i \Psi(\Theta_i)$$

for each $\Theta_i \in [\omega_1, \omega_2]$ and $\mu_i \in [0, 1], (i = \overline{1, n})$ with $\sum_{i=1}^n \mu_i = 1$. For more details, see [3].

Pavić [4] obtained a generalized version of Jensen–Mercer’s inequality in the following way:

Assume that $\Psi : [\omega_1, \omega_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, where $\Theta_i \in [\omega_1, \omega_2]$ are n -points. Let $\alpha, \beta, \mu_i \in [0, 1], \gamma \in [-1, 1]$ be coefficients such that $\alpha + \beta + \gamma = \sum_{i=1}^n \mu_i = 1$. Then

$$\Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma \sum_{i=1}^n \mu_i \Theta_i\right) \leq \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma \sum_{i=1}^n \mu_i \Psi(\Theta_i).$$

The classical concepts of convexity also have a close relation with the concept of symmetry. Several significant properties of symmetric convex sets can be found in the literature. A beneficial point of view of the relation between convexity and symmetry is that we work on one and apply it to the other. For some more useful information, see [5,6].

Quantum calculus (often known as calculus without limits) is the branch of mathematics in which we obtain q -analogues of mathematical objects, which can be recaptured by taking $q \rightarrow 1^-$. In recent years, the classical concepts of quantum calculus have been refined and generalized in different directions using novel and innovative ideas. For instance, Tariboon and Ntouyas [7] defined the q -derivative as:

Definition 1 ([7]). Assume $\Psi : J = [\omega_1, \omega_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and suppose $\Theta \in J$; then

$${}_{\omega_1}D_q\Psi(\Theta) = \frac{\Psi(\Theta) - \Psi(q\Theta + (1 - q)\omega_1)}{(1 - q)(\Theta - \omega_1)} \tag{1}$$

for $\Theta \neq \omega_1$ and $0 < q < 1$.

We say that Ψ is q -differentiable on J provided ${}_{\omega_1}D_q\Psi(\Theta)$ exists for all $\Theta \in J$. Note that if $\omega_1 = 0$ in (1), then ${}_0D_q\Psi = \mathfrak{D}_q\Psi$, where \mathfrak{D}_q is the well-known classical q -derivative of the function $\Psi(\Theta)$ defined by

$$\mathfrak{D}_q\Psi(\Theta) = \frac{\Psi(\Theta) - \Psi(q\Theta)}{(1 - q)\Theta}.$$

Additionally, here and further we use the following notation for the q -number:

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}, \quad q \in (0, 1).$$

Jackson gave the q -Jackson integral from 0 to ω_2 for $0 < q < 1$ as follows:

$$\int_0^{\omega_2} \Psi(\tau) {}_0d_q\tau = (1 - q)\omega_2 \sum_{n=0}^{\infty} q^n \Psi(\omega_2 q^n) \tag{2}$$

provided the sum converges absolutely.

Jackson also gave the q -Jackson integral in a generic interval $[\omega_1, \omega_2]$ as:

$$\int_{\omega_1}^{\omega_2} \Psi(\tau) d_q \tau = \int_0^{\omega_2} \Psi(\tau) d_q \tau - \int_0^{\omega_1} \Psi(\tau) d_q \tau.$$

We now give the definition of a q_{ω_1} -definite integral.

Definition 2 ([7]). Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q_{ω_1} -definite integral on $[\omega_1, \omega_2]$ is defined as:

$$\int_{\omega_1}^{\omega_2} \Psi(\tau) {}_{\omega_1} d_q \tau = (1 - q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n \Psi(q^n \omega_2 + (1 - q^n) \omega_1) = (\omega_2 - \omega_1) \int_0^1 \Psi((1 - \tau) \omega_1 + \tau \omega_2) d_q \tau,$$

where $0 < q < 1$.

The following is the quantum analogue of Hermite–Hadamard’s inequality:

Theorem 1 ([8]). Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex function; then, for $0 < q < 1$, we have

$$\Psi\left(\frac{q\omega_1 + \omega_2}{1 + q}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \Psi(\Theta) {}_{\omega_1} d_q \Theta \leq \frac{qf(\omega_1) + \Psi(\omega_2)}{1 + q}. \tag{3}$$

For some recent studies regarding quantum analogues of integral inequalities involving convexity and its generalizations, see [9–11].

We now give the definition of a q^{ω_2} -definite integral.

Definition 3 ([12]). Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q^{ω_2} -definite integral on $[\omega_1, \omega_2]$ is defined as:

$$\int_{\omega_1}^{\omega_2} \Psi(\tau) {}^{\omega_2} d_q \tau = (1 - q)(\omega_2 - \omega_1) \sum_{n=0}^{\infty} q^n \Psi(q^n \omega_1 + (1 - q^n) \omega_2) = (\omega_2 - \omega_1) \int_0^1 \Psi(t\omega_1 + (1 - t)\omega_2) d_q \tau.$$

Using Definition 3, one can have the following quantum version of Hermite–Hadamard’s inequality.

Theorem 2 ([12]). Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex function; then, for $0 < q < 1$, we have

$$\Psi\left(\frac{\omega_1 + q\omega_2}{1 + q}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \Psi(\Theta) {}^{\omega_2} d_q \Theta \leq \frac{\Psi(\omega_1) + qf(\omega_2)}{1 + q}. \tag{4}$$

In the literature, there are several papers devoted to finding the bound for the left sides and right sides of Inequalities (3) and (4). These types of inequalities are called quantum trapezoid- and midpoint-type inequalities, respectively. By using q_{ω_1} -integrals, the authors established some quantum trapezoid- and quantum midpoint-type inequalities in [8,9], respectively. On the other hand, in [13], Budak proved corresponding quantum trapezoid and quantum midpoint type inequalities for q^{ω_1} -integrals. Zhao et al. presented some quantum inequalities for (α, m) -convex functions in [14]. Du et al. derived some parameterized inequalities which generalize quantum midpoint- and quantum trapezoid-type inequalities for q_{ω_1} -integrals in [15]. Moreover, in [16], Zhao et al proved some parameterized quantum inequalities involving both the q_{ω_1} -integral and q^{ω_1} -integral. For similar quantum inequalities, one can refer to the papers [17–20].

The main objective of this paper is to derive some new quantum estimates of generalized Hermite–Hadamard–Jensen–Mercer type inequalities essentially using q -differentiable convex functions. We discuss some special cases which show that our results are quite unifying. Lastly, we offer some applications of the obtained results to means that establish

our results’ efficiency. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

This paper is summarized as follows: Section 2 provides main results of the paper and certain numerical examples showing the validity of the results. Some applications to special means of real numbers are discussed in Section 3. Section 4 concludes with some suggestions for future research.

2. Main Results

In this section, we discuss our main results.

Theorem 3. Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex function; then

$$\Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{\Phi + q\Theta}{1 + q}\right)\right) \leq \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q v \leq \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{\Psi(\Phi) + qf(\Theta)}{1 + q}\right) \tag{5}$$

for all $\Theta, \Phi \in [\omega_1, \omega_2]$ with $\Theta < \Phi$ and $0 < q < 1$.

Proof. From Definition 2 and Theorem 1, we have

$$\begin{aligned} & \int_0^1 \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Theta + (1 - \tau)\Phi)) d_q \tau \\ &= \int_0^1 \Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)) d_q \tau \\ &= \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q v \\ &\geq \Psi\left(\frac{q(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \alpha\omega_1 + \beta\omega_2 + \gamma\Phi}{1 + q}\right) \\ &= \Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{\Phi + q\Theta}{1 + q}\right)\right). \end{aligned}$$

Additionally, by the Jensen–Mercer inequality, we obtain

$$\begin{aligned} \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q v &= \int_0^1 \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta)) d_q \tau \\ &\leq \int_0^1 \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma[\tau\Psi(\Phi) + (1 - \tau)\Psi(\Theta)] d_q \tau \\ &= \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{\Psi(\Phi) + qf(\Theta)}{1 + q}\right). \end{aligned}$$

This completes the proof. \square

Remark 1. If we take $\alpha = 1, \beta = 1, \gamma = -1$ and take limit $q \rightarrow 1^-$ in Theorem 3, then (5) reduces to the following inequality:

$$\Psi\left(\omega_1 + \omega_2 - \left(\frac{\Theta + \Phi}{2}\right)\right) \leq \frac{1}{\Theta - \Phi} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(v) dv \leq \Psi(\omega_1) + \Psi(\omega_2) - \frac{\Psi(\Theta) + \Psi(\Phi)}{2}. \tag{6}$$

Theorem 4. Let $\Psi : [\omega_1, \omega_2] \rightarrow \mathbb{R}$ be a convex function; then,

$$\begin{aligned} \Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{\Theta + q\Phi}{1 + q}\right)\right) &\leq \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} d_q v \\ &\leq \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{\Psi(\Theta) + qf(\Phi)}{1 + q}\right) \end{aligned} \tag{7}$$

for all $\Theta, \Phi \in [\omega_1, \omega_2]$ with $\Theta < \Phi$ and $0 < q < 1$.

Proof. From Definition 3 and Theorem 2, we have

$$\begin{aligned} \int_0^1 \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta)) d_q \tau &= \int_0^1 \Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) d_q \tau \\ &= \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} d_q v \\ &\geq \Psi\left(\frac{q(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + \alpha\omega_1 + \beta\omega_2 + \gamma\Theta}{1 + q}\right) \\ &= \Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{\Theta + q\Phi}{1 + q}\right)\right), \end{aligned}$$

which gives the proof of first inequality in (7). Additionally, by the Jensen-Mercer inequality, we obtain

$$\begin{aligned} \int_0^1 \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Theta + (1 - \tau)\Phi)) d_q \tau &\leq \int_0^1 \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma[\tau\Psi(\Theta) + (1 - \tau)\Psi(\Phi)] d_q \tau \\ &= \alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{\Psi(\Theta) + qf(\Phi)}{1 + q}\right). \end{aligned}$$

This completes the proof. \square

Remark 2. If we take $\alpha = 1, \beta = 1, \gamma = -1$ and take limit $q \rightarrow 1^-$ in Theorem 4, then Inequality (7) reduces to the following inequality:

$$\Psi\left(\omega_1 + \omega_2 - \left(\frac{\Theta + \Phi}{2}\right)\right) \leq \frac{1}{\Theta - \Phi} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(v) dv \leq \Psi(\omega_1) + \Psi(\omega_2) - \frac{\Psi(\Theta) + \Psi(\Phi)}{2}. \tag{8}$$

We provide some examples of our main theorems in this section.

Example 1. Consider the convex function $\Psi : [-2, 3] \rightarrow \mathbb{R}$ defined by $\Psi(v) = 2v^2$ with $\Theta = 1, \Phi = 2, \alpha = 0.2, \beta = 0.3, \gamma = 0.4$ and $q = \frac{1}{7}$. Then, we have

$$\Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{q\Theta + \Phi}{1 + q}\right)\right) = \frac{25}{8}, \tag{9}$$

$$\frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v)^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} d_q v = \frac{1}{0.4} \int_{0.9}^{1.3} \Psi(v)^{0.9} d_{\frac{1}{7}} v = \frac{79}{25} \tag{10}$$

and

$$\alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{qf(\Theta) + \Psi(\Phi)}{1 + q}\right) = \frac{99}{10}. \tag{11}$$

From (9)–(11), it is evident that the Inequality (5) is valid and

$$\frac{25}{8} < \frac{79}{25} < \frac{99}{10}.$$

Example 2. Consider the convex function $\Psi : [-2, 3] \rightarrow \mathbb{R}$ defined by $\Psi(v) = 2v^2$ with $\Theta = 1, \Phi = 2, \alpha = 0.2, \beta = 0.3, \gamma = 0.4$ and $q = \frac{1}{7}$. Then, we have

$$\Psi\left(\alpha\omega_1 + \beta\omega_2 + \gamma\left(\frac{\Theta + q\Phi}{1 + q}\right)\right) = \frac{361}{200}, \tag{12}$$

$$\frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(v) {}_{(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}d_q v = \frac{1}{0.4} \int_{0.9}^{1.3} \Psi(v) {}^{1.3}d_{\frac{1}{7}} v = \frac{46}{25} \tag{13}$$

and

$$\alpha\Psi(\omega_1) + \beta\Psi(\omega_2) + \gamma\left(\frac{\Psi(\Theta) + qf(\Phi)}{1 + q}\right) = \frac{81}{10}. \tag{14}$$

From (12)–(14), it is evident that the Inequality (7) is valid and

$$\frac{361}{200} < \frac{46}{25} < \frac{81}{10}.$$

We now derive a new q -integral identity. This result will serve as an auxiliary result for our coming results.

Lemma 1. Let $\Psi : J \rightarrow \mathbb{R}$ be a continuous function and $0 < q < 1$. If ${}_{(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}D_q\Psi$ is an integrable function on J° , then

$$\begin{aligned} & \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau) {}_{(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}d_q \tau - \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} \\ &= \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 (1 - (1 + q)\tau) {}_{(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}D_q\Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))_0 d_q \tau. \end{aligned} \tag{15}$$

Proof. Using Definitions 1 and 2, we have

$$\begin{aligned} & \int_0^1 (1 - (1 + q)\tau) {}_{(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}D_q\Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))_0 d_q \tau \\ &= \int_0^1 (1 - (1 + q)\tau) \frac{1}{\tau(1 - q)\gamma(\Phi - \Theta)} [(\Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\ & \quad - \Psi(q\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q\tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)))]_0 d_q \tau \\ &= \int_0^1 \frac{1}{\tau(1 - q)\gamma(\Phi - \Theta)} [(\Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\ & \quad - \Psi(q\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q\tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)))]_0 d_q \tau \\ & \quad - \frac{(1 + q)}{(1 - q)\gamma(\Phi - \Theta)} \int_0^1 [\Psi(\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - \tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\ & \quad - \Psi(q\tau(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q\tau)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))]_0 d_q \tau \\ &= \frac{1}{\gamma(\Phi - \Theta)} \left[\sum_{n=0}^{\infty} \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \right. \\ & \quad \left. - \sum_{n=0}^{\infty} \Psi(q^{n+1}(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^{n+1})(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \right] \\ & \quad - \frac{1 + q}{\gamma(\Phi - \Theta)} \left[\sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \right. \\ & \quad \left. - \sum_{n=0}^{\infty} q^n \Psi(q^{n+1}(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^{n+1})(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{\Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) - \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}{\gamma(\Phi - \Theta)} \\
 &\quad - \frac{1+q}{\gamma(\Phi - \Theta)} \sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &\quad + \frac{1+q}{q\gamma(\Phi - \Theta)} \sum_{n=1}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &= \frac{\Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) - \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}{\gamma(\Phi - \Theta)} \\
 &\quad - \frac{1+q}{\gamma(\Phi - \Theta)} \sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &\quad + \frac{1+q}{\gamma(\Phi - \Theta)} [\Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) - \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) \\
 &\quad + \sum_{n=1}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))] \\
 &= \frac{\Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) - \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)}{\gamma(\Phi - \Theta)} - \frac{1+q}{q\gamma(\Phi - \Theta)} \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) \\
 &\quad + \frac{1+q}{q\gamma(\Phi - \Theta)} \sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &\quad - \frac{1+q}{\gamma(\Phi - \Theta)} \sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &= -\frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{q\gamma(\Phi - \Theta)} \\
 &\quad + \frac{(1+q)(1-q)}{q\gamma(\Phi - \Theta)} \frac{\gamma(\Phi - \Theta)}{\gamma(\Phi - \Theta)} \sum_{n=0}^{\infty} q^n \Psi(q^n(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi) + (1 - q^n)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)) \\
 &= -\frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{q\gamma(\Phi - \Theta)} + \frac{(1+q)}{\gamma^2(\Phi - \Theta)^2} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau.
 \end{aligned}$$

The completes the proof. □

Remark 3. If we take $\alpha = 1, \beta = 1$ and $\gamma = -1$ in Lemma 1, then we have

$$\begin{aligned}
 &\frac{1}{\Phi - \Theta} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(\tau)_{(\omega_1 + \omega_2 - \Theta)} d_q \tau + \frac{qf(\omega_1 + \omega_2 - \Theta) + \Psi(\omega_1 + \omega_2 - \Phi)}{1+q} \\
 &= \frac{q(\Phi - \Theta)}{1+q} \int_0^1 (1 - (1+q)\tau)_{(\omega_1 + \omega_2 - \Theta)} D_q \Psi(\tau(\omega_1 + \omega_2 - \Phi) + (1 - \tau)(\omega_1 + \omega_2 - \Theta))_0 d_q \tau. \tag{16}
 \end{aligned}$$

Theorem 5. Let $\Psi : J \rightarrow \mathbb{R}$ be a continuous function. If $|\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi|$ is convex and integrable on J° , then

$$\begin{aligned}
 &\left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1+q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\
 &\leq \frac{q^2 \gamma(\Phi - \Theta)}{(1+q+q^2)(1+q)^4} \left([2(1+q)^2(1+q+q^2)] [|\alpha|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)| + \beta|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)] \right. \\
 &\quad \left. + \gamma [(1+3q^2+2q^3) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)| + (1+4q+q^2) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|] \right). \tag{17}
 \end{aligned}$$

Proof. Using Lemma 1, the Jensen–Mercer inequality and the convexity of $|\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi|$ on J° , we have

$$\begin{aligned}
 & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{q\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\
 &= \left| \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 (1 - (1 + q)\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau \right| \\
 &\leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 |1 - (1 + q)\tau| (\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)| + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)| \\
 &\quad + \gamma [\tau |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)| + (1 - \tau) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)|])_0 d_q \tau \\
 &= \frac{q\gamma(\Phi - \Theta)}{1 + q} \left[\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)| \int_0^1 |1 - (1 + q)\tau|_0 d_q \tau \right. \\
 &\quad + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)| \int_0^1 |1 - (1 + q)\tau|_0 d_q \tau + \gamma \left(|\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)| \int_0^1 |1 - (1 + q)\tau|_0 d_q \tau \right. \\
 &\quad \left. \left. + |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)| \int_0^1 |1 - (1 + q)\tau|(1 - \tau)_0 d_q \tau \right) \right] \\
 &\leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \left[\frac{2q\alpha}{(1 + q)^2} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)| + \frac{2q\beta}{(1 + q)^2} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)| \right. \\
 &\quad \left. + \gamma \left(\frac{q(1 + 4q + q^2)}{(1 + q + q^2)(1 + q)^3} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)| + \frac{q(1 + 3q^2 + 2q^3)}{(1 + q + q^2)(1 + q)^3} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)| \right) \right] \\
 &\leq \frac{q^2\gamma(\Phi - \Theta)}{(1 + q + q^2)(1 + q)^4} ([2(1 + q)^2(1 + q + q^2)] [\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)| + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)|] \\
 &\quad + \gamma [(1 + 4q + q^2) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)| + (1 + 3q^2 + 2q^3) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)|]).
 \end{aligned}$$

The proof is complete. □

Remark 4. If we take $\alpha = 1, \beta = 1$ and $\gamma = -1$ in Theorem 5, then the inequality (17) reduces to the integral inequality:

$$\begin{aligned}
 & \left| \frac{qf(\omega_1 + \omega_2 - \Theta) + \Psi(\omega_1 + \omega_2 - \Phi)}{1 + q} + \frac{1}{\Phi - \Theta} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(\tau)_{(\omega_1 + \omega_2 - \Theta)} d_q \tau \right| \\
 &\leq \frac{-q^2(\Phi - \Theta)}{(1 + q + q^2)(1 + q)^4} ([2(1 + q)^2(1 + q + q^2)] [|\omega_1 + \omega_2 - \Theta D_q \Psi(\omega_1)| + \beta |\omega_1 + \omega_2 - \Theta D_q \Psi(\omega_2)|] \\
 &\quad - [(1 + 3q^2 + 2q^3) |\omega_1 + \omega_2 - \Theta D_q \Psi(\Theta)| + (1 + 4q + q^2) |\omega_1 + \omega_2 - \Theta D_q \Psi(\Phi)|]). \tag{18}
 \end{aligned}$$

Theorem 6. Let $\Psi : J \rightarrow \mathbb{R}$ be a continuous function. If $|\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi|^r$ is convex and integrable on J° , and $r \geq 1$, then

$$\begin{aligned}
 & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\
 &\leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \left(\frac{2q}{(1 + q)^2} \right)^r \left(\frac{q}{(1 + q + q^2)(1 + q)^3} (2(1 + q)(1 + q + q^2)) [\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)|^r \right. \\
 &\quad + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)|^r] + \gamma [(1 + 4q + q^2) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)|^r \\
 &\quad \left. + (1 + 3q^2 + 2q^3) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)|^r] \right)^{\frac{1}{r}}. \tag{19}
 \end{aligned}$$

Proof. From Lemma 1, we have

$$\begin{aligned} & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\ &= \left| \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 (1 - (1 + q)\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau \right| \\ &\leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau. \end{aligned}$$

Using the power-mean inequality,

$$\begin{aligned} & \int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau \\ &\leq \left(\int_0^1 |1 - (1 + q)\tau|_0 d_q \tau \right)^{1 - \frac{1}{r}} \\ &\times \left(\int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0^r d_q \tau \right)^{\frac{1}{r}}. \end{aligned} \tag{20}$$

Now using the Jensen–Mercer inequality and convexity of $|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi|^r$, it follows that

$$\begin{aligned} & \int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0^r d_q \tau \\ &\leq \int_0^1 |1 - (1 + q)\tau| (\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \\ &\quad + \gamma [\tau |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r + (1 - \tau) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r])_0 d_q \tau \\ &\leq \left[\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r \int_0^1 |1 - (1 + q)\tau|_0 d_q \tau \right. \\ &\quad + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \int_0^1 |1 - (1 + q)\tau|_0 d_q \tau + \gamma \left(|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r \int_0^1 |1 - (1 + q)\tau| \tau_0 d_q \tau \right. \\ &\quad \left. + |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r \int_0^1 |1 - (1 + q)\tau| (1 - \tau)_0 d_q \tau \right) \Big] \\ &= \frac{2q\alpha}{(1 + q)^2} |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r + \frac{2q\beta}{(1 + q)^2} |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \\ &\quad + \gamma \left(\frac{q(1 + 4q + q^2)}{(1 + q + q^2)(1 + q)^3} |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r + \frac{q(1 + 3q^2 + 2q^3)}{(1 + q + q^2)(1 + q)^3} |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r \right) \\ &\leq \frac{q}{(1 + q + q^2)(1 + q)^3} ([2(1 + q)^2(1 + q + q^2)] [\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r] \\ &\quad + \gamma [(1 + 4q + q^2) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r + (1 + 3q^2 + 2q^3) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r]). \end{aligned} \tag{21}$$

Applying the fact that $\int_0^1 |1 - (1 + q)\tau|_0 d_q \tau = \frac{2q}{(1 + q)^2}$ and substituting (20) into (21), we get

$$\begin{aligned} & \int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau \\ &\leq \left(\frac{2q}{(1 + q)^2} \right)^r \times \left(\frac{q}{(1 + q + q^2)(1 + q)^3} (2(1 + q)(1 + q + q^2) [\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r \right. \\ &\quad + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r] + \gamma [(1 + 4q + q^2) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r \\ &\quad \left. + (1 + 3q^2 + 2q^3) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r] \right)^{\frac{1}{r}}. \end{aligned}$$

This completes the proof. □

Remark 5. If we take $\alpha = 1, \beta = 1$ and $\gamma = -1$ in Theorem 6, then the inequality (19) reduces to the integral inequality:

$$\begin{aligned} & \left| \frac{qf(\omega_1 + \omega_2 - \Theta) + \Psi(\omega_1 + \omega_2 - \Phi)}{1 + q} + \frac{1}{\Phi - \Theta} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(\tau)_{\omega_1 + \omega_2 - \Theta} d_q \tau \right| \\ & \leq \frac{-q(\Phi - \Theta)}{1 + q} \left(\frac{2q}{(1 + q)^2} \right)^r \left(\frac{q}{(1 + q + q^2)(1 + q)^3} (2(1 + q)(1 + q + q^2) [|_{\omega_1 + \omega_2 - \Theta} D_q \Psi(\omega_1)|^r \right. \\ & \quad \left. + |_{\omega_1 + \omega_2 - \Theta} D_q \Psi(\omega_2)|^r] - [(1 + 4q + q^2) |_{\omega_1 + \omega_2 - \Theta} D_q \Psi(\Phi)|^r \right. \\ & \quad \left. + (1 + 3q^2 + 2q^3) |_{\omega_1 + \omega_2 - \Theta} D_q \Psi(\Theta)|^r] \right)^{\frac{1}{r}}. \end{aligned} \tag{22}$$

Theorem 7. Let $\Psi : J \rightarrow \mathbb{R}$ be a continuous function. If $|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi|^r$ is convex and integrable on J° , where $p, r > 1$ and $\frac{1}{p} + \frac{1}{r} = 1$, then

$$\begin{aligned} & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1 + q} k_1^{\frac{1}{p}} \left(\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \right. \\ & \quad \left. + \frac{\gamma}{1 + q} [|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r + q |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r] \right)^{\frac{1}{r}}. \end{aligned} \tag{23}$$

where

$$k_1 = \frac{1 - q}{1 + q} \sum_{n=0}^{\infty} q^n \left(\frac{1 - (1 + q)q^n}{1 + q} \right)^p + (1 - q) \sum_{n=0}^{\infty} q^n ((1 + q)q^n - 1)^p - \frac{1 - q}{1 + q} \sum_{n=0}^{\infty} q^n \left(\frac{(1 + q)q^n - 1}{1 + q} \right)^p.$$

Proof. From Lemma 1, we have

$$\begin{aligned} & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\ & = \left| \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 (1 - (1 + q)\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0 d_q \tau \right| \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \int_0^1 |1 - (1 + q)\tau|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} |D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))_0| d_q \tau. \end{aligned}$$

Now using the Jensen–Mercer inequality, Hölder’s inequality and the convexity of $|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi|^r$, it follows that

$$\begin{aligned} & \left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \left(\int_0^1 |1 - (1 + q)\tau|^p d_q \tau \right)^{\frac{1}{p}} \left(\int_0^1 |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma(\tau\Phi + (1 - \tau)\Theta))|^r d_q \tau \right)^{\frac{1}{r}} \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1 + q} \left(\int_0^1 |1 - (1 + q)\tau|^p d_q \tau \right)^{\frac{1}{p}} \left(\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \right. \\ & \quad \left. + \gamma [\tau |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r + (1 - \tau) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r] \right)^{\frac{1}{r}} d_q \tau \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1 + q} (k_1)^{\frac{1}{p}} \left(\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)|^r \int_0^1 d_q \tau + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)|^r \int_0^1 d_q \tau \right. \\ & \quad \left. + \gamma \left[|_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)|^r \int_0^1 \tau d_q \tau + |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)|^r \int_0^1 (1 - \tau) d_q \tau \right] \right)^{\frac{1}{r}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{q\gamma(\Phi - \Theta)}{1 + q} (k_1)^{\frac{1}{p}} (\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)|^r + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)|^r \\
 &\quad + \gamma \left[\frac{1}{1 + q} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)|^r + \frac{q}{1 + q} |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)|^r \right])^{\frac{1}{r}} \\
 &\leq \frac{q\gamma(\Phi - \Theta)}{1 + q} (k_1)^{\frac{1}{p}} \times (\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)|^r + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)|^r \\
 &\quad + \frac{\gamma}{1 + q} [|\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)|^r + q |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)|^r])^{\frac{1}{r}}.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 k_1 &= \int_0^1 |1 - (1 + q)\tau|^p d_q \tau = \int_0^{\frac{1}{1+q}} (1 - (1 + q)\tau)^p d_q \tau + \int_{\frac{1}{1+q}}^1 ((1 + q)\tau - 1)^p d_q \tau \\
 &= \int_0^{\frac{1}{1+q}} (1 - (1 + q)\tau)^p d_q \tau + \int_0^1 ((1 + q)\tau - 1)^p d_q \tau - \int_0^{\frac{1}{1+q}} ((1 + q)\tau - 1)^p d_q \tau \\
 &= \frac{1 - q}{1 + q} \sum_{n=0}^{\infty} q^n \left(\frac{1 - (1 + q)q^n}{1 + q} \right)^p + (1 - q) \sum_{n=0}^{\infty} q^n ((1 + q)q^n - 1)^p \\
 &\quad - \frac{1 - q}{1 + q} \sum_{n=0}^{\infty} q^n \left(\frac{(1 + q)q^n - 1}{1 + q} \right)^p.
 \end{aligned}$$

This completes the proof. \square

Remark 6. If we take $\alpha = 1, \beta = 1$ and $\gamma = -1$ in Theorem 7, then the inequality (23) reduces to the integral inequality:

$$\begin{aligned}
 &\left| \frac{qf(\omega_1 + \omega_2 - \Theta) + \Psi(\omega_1 + \omega_2 - \Phi)}{1 + q} + \frac{1}{\Phi - \Theta} \int_{\omega_1 + \omega_2 - \Theta}^{\omega_1 + \omega_2 - \Phi} \Psi(\tau)_{\omega_1 + \omega_2 - \Theta} d_q \tau \right| \\
 &\leq \frac{-q(\Phi - \Theta)}{1 + q} (k_1)^{\frac{1}{p}} \times (|\omega_1 + \omega_2 - \Theta D_q \Psi(\omega_1)|^r + |\omega_1 + \omega_2 - \Theta D_q \Psi(\omega_2)|^r \\
 &\quad - \frac{1}{1 + q} [|\omega_1 + \omega_2 - \Theta D_q \Psi(\Phi)|^r + q |\omega_1 + \omega_2 - \Theta D_q \Psi(\Theta)|^r])^{\frac{1}{r}}. \tag{24}
 \end{aligned}$$

Example 3. Consider the convex function $\Psi : [-2, 3] \rightarrow \mathbb{R}, \Psi(v) = 2v^2$ with $\Theta = 1, \Phi = 2, \alpha = 0.2, \beta = 0.3, \gamma = 0.4$ and $q = \frac{1}{7}$. Then, we have

$$\left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| = \frac{1}{50}, \tag{25}$$

and

$$\begin{aligned}
 &\frac{q^2\gamma(\Phi - \Theta)}{(1 + q + q^2)(1 + q)^4} ([2(1 + q)^2(1 + q + q^2)] [\alpha |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_1)| + \beta |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\omega_2)|] \\
 &\quad + \gamma [(1 + 3q^2 + 2q^3) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Theta)| + (1 + 4q + q^2) |\alpha\omega_1 + \beta\omega_2 + \gamma\Theta D_q \Psi(\Phi)|]) = \frac{17}{200}. \tag{26}
 \end{aligned}$$

From (25) and (26), it is evident that the Inequality (17) is valid and

$$\frac{1}{50} < \frac{17}{200}.$$

Example 4. Let us again consider the convex function $\Psi : [-2, 3] \rightarrow \mathbb{R}, \Psi(v) = 2v^2$ with $\Theta = 1, \Phi = 2, \alpha = 0.2, \beta = 0.3, \gamma = 0.4, r = 1$ and $q = \frac{1}{7}$. Then, we have

$$\left| \frac{qf(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) + \Psi(\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)}{1 + q} - \frac{1}{\gamma(\Phi - \Theta)} \int_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta}^{\alpha\omega_1 + \beta\omega_2 + \gamma\Phi} \Psi(\tau)_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} d_q \tau \right| = \frac{1}{50} \tag{27}$$

and

$$\begin{aligned} & \frac{q\gamma(\Phi - \Theta)}{1 + q} \left(\frac{2q}{(1 + q)^2} \right)^r \left(\frac{q}{(1 + q + q^2)(1 + q)^3} (2(1 + q)(1 + q + q^2) [\alpha |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_1)]^r \right. \\ & + \beta |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\omega_2)]^r + \gamma [(1 + 4q + q^2) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Phi)]^r \\ & \left. + (1 + 3q^2 + 2q^3) |_{\alpha\omega_1 + \beta\omega_2 + \gamma\Theta} D_q \Psi(\Theta)]^r \right)^{\frac{1}{r}} = \frac{259}{100}. \end{aligned} \tag{28}$$

From (27) and (28), it is evident that the inequality (19) is valid and

$$\frac{1}{50} < \frac{259}{100}.$$

3. Applications

In this section, we discuss some applications to means. For arbitrary real numbers, we consider the following means:

1. The arithmetic mean:

$$A(\omega_1, \omega_2) = \frac{\omega_1 + \omega_2}{2};$$

2. The generalized log-mean:

$$L_p(\omega_1, \omega_2) = \left[\frac{\omega_2^{p+1} - \omega_1^{p+1}}{(p + 1)(\omega_2 - \omega_1)} \right]^{\frac{1}{p}},$$

where $p \in \mathbb{R} \setminus \{-1, 0\}$, $\omega_1, \omega_2 \in \mathbb{R}$ and $\omega_1 \neq \omega_2$.

Proposition 1. Let $n \in \mathbb{N}$ and $0 < q < 1$. If $0 < (\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) < (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)$; then

$$\begin{aligned} & \left| \frac{2A(q(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)^n, (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)^n)}{1 + q} - \frac{1 - q}{(1 - q^{n+1})} L_n^n[\alpha\omega_1 + \beta\omega_2 + \gamma\Phi, \alpha\omega_1 + \beta\omega_2 + \gamma\Theta] \right| \\ & \leq \frac{q^2\gamma(\Phi - \Theta)}{(1 + q + q^2)(1 + q)^4} \left([2(1 + q)^2(1 + q + q^2)] \left[\alpha \left(\frac{\omega_1^n - (q\omega_1 + (1 - q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1 - q)(\omega_1 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right) \right. \right. \\ & + \beta \left(\frac{\omega_2^n - (q\omega_2 + (1 - q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1 - q)(\omega_2 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right) \left. \right] \\ & + \gamma \left[(1 + 3q^2 + 2q^3) \left(\frac{\Theta^n - (q\Theta + (1 - q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1 - q)(\Theta - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right) \right. \\ & \left. + (1 + 4q + q^2) \left(\frac{\Phi^n - (q\Phi + (1 - q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1 - q)(\Phi - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right) \right] \right). \end{aligned}$$

Proof. The proof is obvious from Theorem 5 applied for $\Psi(\Theta) = \Theta^n$. □

Proposition 2. Let $n \in \mathbb{N}$ and $0 < q < 1$. If $0 < (\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) < (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)$; then

$$\begin{aligned} & \left| \frac{2A(q(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)^n, (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)^n)}{1+q} - \frac{1-q}{(1-q^{n+1})} L_n^n[\alpha\omega_1 + \beta\omega_2 + \gamma\Phi, \alpha\omega_1 + \beta\omega_2 + \gamma\Theta] \right| \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1+q} \left(\frac{2q}{(1+q)^2} \right)^r \\ & \times \left(\frac{q}{(1+q+q^2)(1+q)^3} \left(2(1+q)(1+q+q^2) \left[\alpha \left(\frac{\omega_1^n - (q\omega_1 + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\omega_1 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right. \right. \right. \\ & \quad \left. \left. + \beta \left(\frac{\omega_2^n - (q\omega_2 + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\omega_2 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right] \right. \\ & \quad \left. + \gamma \left[(1+4q+q^2) \left(\frac{\Phi^n - (q\gamma + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\Phi - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right. \right. \\ & \quad \left. \left. + (1+3q^2+2q^3) \left(\frac{\Theta^n - (q\Theta + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\Theta - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right] \right) \right)^{\frac{1}{r}}. \end{aligned}$$

Proof. The proof is obvious from Theorem 6 applied for $\Psi(\Theta) = \Theta^n$. \square

Proposition 3. Let $n \in \mathbb{N}$ and $0 < q < 1$. If $0 < (\alpha\omega_1 + \beta\omega_2 + \gamma\Theta) < (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)$; then

$$\begin{aligned} & \left| \frac{2A(q(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta)^n, (\alpha\omega_1 + \beta\omega_2 + \gamma\Phi)^n)}{1+q} - \frac{1-q}{(1-q^{n+1})} L_n^n[\alpha\omega_1 + \beta\omega_2 + \gamma\Phi, \alpha\omega_1 + \beta\omega_2 + \gamma\Theta] \right| \\ & \leq \frac{q\gamma(\Phi - \Theta)}{1+q} (k_1)^{\frac{1}{p}} \\ & \times \left(\alpha \left(\frac{\omega_1^n - (q\omega_1 + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\omega_1 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r + \beta \left(\frac{\omega_2^n - (q\omega_2 + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\omega_2 - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right. \\ & \left. + \frac{\gamma}{1+q} \left[\left(\frac{\Phi^n - (q\gamma + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\Phi - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r + q \left(\frac{\Theta^n - (q\Theta + (1-q)(\alpha\omega_1 + \beta\omega_2 + \gamma\Theta))^n}{(1-q)(\Theta - \alpha\omega_1 - \beta\omega_2 - \gamma\Theta)} \right)^r \right] \right)^{\frac{1}{r}}. \end{aligned}$$

Proof. The proof is obvious from Theorem 7 applied for $\Psi(\Theta) = \Theta^n$. \square

4. Conclusions

We have derived some new quantum estimates of the generalized Hermite–Hadamard–Jensen–Mercer type of inequalities by using the class of q -differentiable convex functions. We have discussed several special cases which can be deduced from the main results of the paper. This shows that the results obtained in this paper are unifying and represent a significant q -generalizations of the classical results. In order to elaborate on the efficiency of the main results, we have also presented some applications to means. We would like to mention here that the results of this paper can be extended by using post-quantum calculus and other convexity classes. This will be an interesting problem for future research.

Author Contributions: All authors contributed equally to this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Saud University grant number RSP-2021/158.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions. This research is supported by Project number (RSP-2021/158), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare that they have no competing interest.

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