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On Some New Fractional Ostrowski- and Trapezoid-Type Inequalities for Functions of Bounded Variations with Two Variables

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Abstract: In this paper, we first prove three identities for functions of bounded variations. Then, by using these equalities, we obtain several trapezoid- and Ostrowski-type inequalities via generalized fractional integrals for functions of bounded variations with two variables. Moreover, we present some results for Riemann–Liouville fractional integrals by special choice of the main results. Finally, we investigate the connections between our results and those in earlier works. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Keywords: trapezoid-type inequality; fractional integrals; functions of bounded variations



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1. Introduction

One of the most important inequalities for bounded functions is the Ostrowski inequality which gives an estimate for the deviation of the values of a smooth function from its mean value. The Ostrowski inequality is stated as follows: if $F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ is a differentiable function with a bounded derivative, then the following integral inequality

$$\left| F(\varkappa) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau) d\tau \right| \leq \left[\frac{1}{4} + \frac{(\varkappa - \frac{\kappa_1 + \kappa_2}{2})^2}{(\kappa_2 - \kappa_1)^2} \right] (\kappa_2 - \kappa_1) \|F'\|_{\infty} \quad (1)$$

is valid for every $\varkappa \in [\kappa_1, \kappa_2]$, which was proved by Ostrowski in 1938 [1]. Here, the constant $1/4$ is the best possible.

In the recent years, many versions of Ostrowski-type inequalities have been proved for some kinds of function classes, such as convex functions, bounded functions, functions of bounded variation, and so on. For example, Alomari et al. established some Ostrowski-type inequalities for s -convex functions in [2]. Moreover, some papers were devoted to study on Ostrowski-type inequalities for other kinds of convexities [3–6]. On the other hand, Set first proved the fractional version of Ostrowski inequality for s -convex functions via Riemann–Liouville fractional integrals [7]. Furthermore, many studies were focused on the proof of Ostrowski-type inequalities for certain fractional integral operators, such as k -Riemann–Liouville fractional integrals [8], local fractional integrals [9], Raina fractional integrals [10], etc. (see [11–23]). Moreover, by utilizing co-ordinated convex mapping, several Ostrowski

inequalities were presented for the Riemann integral and Riemann–Liouville fractional integrals in [24,25], respectively.

On the other hand, Dragomir extended the Ostrowski inequality for functions of bounded variation [26]. Dragomir also proved trapezoid type inequalities and midpoint type inequalities for the functions of bounded variation in [27,28], respectively. In [29], the author presented several Simpson’s type inequalities for the mappings of bounded variations. In the literature, many studies were devoted to new versions of Ostrowski-type inequalities for functions of bounded variation. For some of them, please refer to [30–42]. In [43], some important inequalities for the functions of two variables with bounded variation were given and applications in the cubature formula was provided. However, there were some minor errors in the main results of the paper [43] since the Lemma 1 in the published version of [43] is inexact. Moricz has already provided the correct version of the lemma in [44]. In [43], Budak and Sarikaya presented the corrections of these results by using the lemma proved by Moricz. For other papers on inequalities for functions of two variables with bounded variation, see [45–47].

This paper aims to establish some trapezoid and Ostrowski-type inequalities for functions of bounded variations with two variables via generalized fractional integrals. The general structure of the paper consists of six sections including an introduction. The remaining part of the paper proceeds as follows: In Section 2, we first present definitions of the functions of bounded variations and total variations. We also give the definitions of generalized fractional integrals and relations between generalized fractional integrals and other type fractional integrals. In Section 3, we prove three identities for functions of bounded variations with two variables by using the Riemann–Stieltjes integral. Trapezoid- and Ostrowski-type inequalities for functions of bounded variations with two variables are established in Sections 4 and 5, respectively. At the end of the paper, some conclusions and further directions of research are discussed in Section 6.

2. Preliminaries

In this section, we first present the definition of the functions of bounded variation (single and two variables). Then, we summarize the generalized fractional integrals and give the relations between generalized fractional integrals and the other types of fractional integrals.

2.1. Functions of Bounded Variation with One Variable

Definition 1 ([48]). For any partition $P : \kappa_1 = \varkappa_0 < \varkappa_1 < \dots < \varkappa_n = \kappa_2$ of $[\kappa_1, \kappa_2]$ and $\Delta F(\varkappa_i) = F(\varkappa_{i+1}) - F(\varkappa_i)$. If the sum

$$\sum_{i=1}^m |\Delta F(\varkappa_i)|$$

is bounded for all partitions, then the mapping $F(\varkappa)$ is called of bounded variation. We assume that F has a bounded variation on $[\kappa_1, \kappa_2]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^n |\Delta F(\varkappa_i)|$ corresponding to the partition P of $[\kappa_1, \kappa_2]$. The number

$$\bigvee_{\kappa_1}^{\kappa_2}(F) := \sup \{ \sum(P) : P \in P([\kappa_1, \kappa_2]) \}$$

is called the total variation of F on $[\kappa_1, \kappa_2]$. Here, the family of partitions of $[\kappa_1, \kappa_2]$ is denoted by $P([\kappa_1, \kappa_2])$.

In [26], for the functions of bounded variation, Dragomir proved the following Ostrowski inequality.

Theorem 1. For the mapping of bounded variation $F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ on $[\kappa_1, \kappa_2]$. The inequality

$$\left| \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau) d\tau - F(\varkappa) \right| \leq \left[\frac{1}{2} + \left| \frac{\varkappa - \frac{\kappa_1 + \kappa_2}{2}}{\kappa_2 - \kappa_1} \right| \right] V_{\kappa_1}^{\kappa_2}(F) \quad (2)$$

holds for all $\varkappa \in [\kappa_1, \kappa_2]$. The constant $\frac{1}{2}$ is the best possible.

2.2. Functions of Bounded Variation with Two Variables

Definition 2. For any set of points (\varkappa_i, γ_i) ($i = 0, 1, 2, \dots, m$) satisfying the conditions

$$\begin{aligned} \kappa_1 &= \varkappa_0 < \varkappa_1 < \dots < \varkappa_m = \kappa_2; \\ \kappa_3 &= \gamma_0 < \gamma_1 < \dots < \gamma_m = \kappa_4, \end{aligned}$$

if the sum

$$\sum_{i=1}^m |\Delta F(\varkappa_i, \gamma_i)|$$

is bounded for all such sets of points, then the mapping $F(\varkappa, \gamma)$ is called a bounded variation (see [49,50]).

As a result, the definition of total variation of a function of two variables can be described as follows:

Let F be of bounded variation on $Q = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]$ and $\Sigma(P)$ denotes the sum $\sum_{i=1}^n \sum_{j=1}^m |\Delta_{11} F(\varkappa_i, \gamma_j)|$ corresponding to the partition P of Q . The number

$$V_Q(F) := \bigvee_{\kappa_3}^{\kappa_4} \bigvee_{\kappa_1}^{\kappa_2} (F) := \sup \{ \Sigma(P) : P \in P(Q) \}$$

is called the total variation of F on Q . Here, the family of partitions of Q is denoted by $P(Q)$.

Lemma 1 ([44]). If $F(\tau, \xi)$ continuous on the $Q = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]$ and $\alpha(\tau, \xi)$ is of bounded variation, then $\alpha(\tau, \xi)$ is integrable with respect to $F(\tau, \xi)$ on Q in the Riemann–Stieltjes integrable sense and

$$\begin{aligned} & \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\tau, \xi) d_\tau d_\xi \alpha(\tau, \xi) \\ &= \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} \alpha(\tau, \xi) d_\tau d_\xi F(\tau, \xi) - \int_{\kappa_1}^{\kappa_2} \alpha(\tau, \kappa_4) d_\tau F(\tau, \kappa_4) + \int_{\kappa_1}^{\kappa_2} \alpha(\tau, \kappa_3) d_\tau F(\tau, \kappa_3) \\ & \quad - \int_{\kappa_3}^{\kappa_4} \alpha(\kappa_2, \xi) d_\xi F(\kappa_2, \xi) + \int_{\kappa_3}^{\kappa_4} \alpha(\kappa_1, \xi) d_\xi F(\kappa_1, \xi) \\ & \quad + F(\kappa_2, \kappa_4) \alpha(\kappa_2, \kappa_4) - F(\kappa_2, \kappa_3) \alpha(\kappa_2, \kappa_3) \\ & \quad - F(\kappa_1, \kappa_4) \alpha(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3) \alpha(\kappa_1, \kappa_3). \end{aligned} \quad (3)$$

Lemma 2 ([43]). Assume that ρ is integrable with respect to α on Q and α is of bounded variation on Q , then

$$\left| \int_{\kappa_3}^{\kappa_4} \int_{\kappa_1}^{\kappa_2} \rho(\varkappa, \gamma) d_\varkappa d_\gamma \alpha(\varkappa, \gamma) \right| \leq \sup_{(\varkappa, \gamma) \in Q} |\rho(\varkappa, \gamma)| V_Q(\alpha). \quad (4)$$

2.3. Generalized Fractional Integrals

In this section, we summarize some fractional integrals which will be used in our main results.

Definition 3 ([51]). Let $F \in L_1[\kappa_1, \kappa_2]$. The Riemann–Liouville fractional integrals $J_{\kappa_1+}^\alpha F$ and $J_{\kappa_2-}^\alpha F$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J_{\kappa_1+}^\alpha F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} (\varkappa - \tau)^{\alpha-1} F(\tau) d\tau, \quad \varkappa > \kappa_1$$

and

$$J_{\kappa_2+}^\alpha F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} (\tau - \varkappa)^{\alpha-1} F(\tau) d\tau, \quad \varkappa < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{\kappa_1+}^0 F(\varkappa) = J_{\kappa_2-}^0 F(\varkappa) = F(\varkappa)$.

Definition 4 ([52]). Let $\rho : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a positive increasing function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2) . The left-sides ($\mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa)$) and right-sides ($\mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa)$) fractional integral of F with respect to the function ρ on $[\kappa_1, \kappa_2]$ of order $\alpha > 0$ are defined by

$$\mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\varkappa} \frac{\rho'(\tau)F(\tau)}{[\rho(\varkappa) - \rho(\tau)]^{1-\alpha}} d\tau, \quad \varkappa > \kappa_1$$

and

$$\mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa) = \frac{1}{\Gamma(\alpha)} \int_{\varkappa}^{\kappa_2} \frac{\rho'(\tau)F(\tau)}{[\rho(\tau) - \rho(\varkappa)]^{1-\alpha}} d\tau, \quad \varkappa < \kappa_2,$$

respectively.

Riemann–Liouville fractional integrals of a function with two variables can be given as follows:

Definition 5. Let $F \in L_1(\Delta = [\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4])$. The Riemann–Liouville fractional integrals $J_{\kappa_1+,\kappa_3+}^{\alpha,\beta} F$, $J_{\kappa_1+,\kappa_4-}^{\alpha,\beta} F$, $J_{\kappa_2-,\kappa_3+}^{\alpha,\beta} F$ and $J_{\kappa_2-,\kappa_4-}^{\alpha,\beta} F$ of order $\alpha, \beta > 0$ with $\kappa_1, \kappa_3 \geq 0$ are defined by

$$J_{\kappa_1+,\kappa_3+}^{\alpha,\beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\varkappa - \tau)^{\alpha-1} (\gamma - \xi)^{\beta-1} F(\tau, \xi) d\xi d\tau, \quad \varkappa > \kappa_1, \gamma > \kappa_3,$$

$$J_{\kappa_1+,\kappa_4-}^{\alpha,\beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\varkappa - \tau)^{\alpha-1} (\xi - \gamma)^{\beta-1} F(\tau, \xi) d\xi d\tau, \quad \varkappa > \kappa_1, \gamma < \kappa_4,$$

$$J_{\kappa_2-,\kappa_3+}^{\alpha,\beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\tau - \varkappa)^{\alpha-1} (\gamma - \xi)^{\beta-1} F(\tau, \xi) d\xi d\tau, \quad \varkappa < \kappa_2, \gamma > \kappa_3$$

and

$$J_{\kappa_2-,\kappa_4-}^{\alpha,\beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\tau - \varkappa)^{\alpha-1} (\gamma - \xi)^{\beta-1} F(\tau, \xi) d\xi d\tau, \quad \varkappa < \kappa_2, \gamma < \kappa_4,$$

Definition 6 ([53]). Let $\rho : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a positive increasing function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2) and let $\omega : [\kappa_3, \kappa_4] \rightarrow \mathbb{R}$ be a positive increasing function

on $(\kappa_3, \kappa_4]$, having a continuous derivative $\omega'(\gamma)$ on (κ_3, κ_4) . If $F \in L_1(\Delta)$, then for $\alpha, \beta > 0$, the generalized fractional integral operators for functions of two variables are defined by

$$\mathcal{J}_{\kappa_1+, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} \frac{\rho'(\tau)}{[\rho(\varkappa) - \rho(\tau)]^{1-\alpha}} \frac{\omega'(\xi)}{[\omega(\gamma) - \omega(\xi)]^{1-\beta}} F(\tau, \xi) d\xi d\tau, \tag{5}$$

$\varkappa > \kappa_1, \gamma > \kappa_3,$

$$\mathcal{J}_{\kappa_1+, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\varkappa) - \rho(\tau)]^{1-\alpha}} \frac{\omega'(\xi)}{[\omega(\xi) - \omega(\gamma)]^{1-\beta}} F(\tau, \xi) d\xi d\tau, \tag{6}$$

$\varkappa > \kappa_1, \gamma < \kappa_4,$

$$\mathcal{J}_{\kappa_2-, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} \frac{\rho'(\tau)}{[\rho(\tau) - \rho(\varkappa)]^{1-\alpha}} \frac{\omega'(\xi)}{[\omega(\gamma) - \omega(\xi)]^{1-\beta}} F(\tau, \xi) d\xi d\tau, \tag{7}$$

$\varkappa < \kappa_2, \gamma > \kappa_3$

and

$$\mathcal{J}_{\kappa_2-, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\tau) - \rho(\varkappa)]^{1-\alpha}} \frac{\omega'(\xi)}{[\omega(\xi) - \omega(\gamma)]^{1-\beta}} F(\tau, \xi) d\xi d\tau, \tag{8}$$

$\varkappa < \kappa_2, \gamma < \kappa_4,$

respectively.

By using Definition 6, well-known fractional integrals can be obtained by some special choices. For example:

1. If we choose $\alpha = \beta = 1$, the operators (5)–(8) reduce to the double Riemann integral;
2. Considering $\rho(\tau) = \tau$ and $\omega(\xi) = \xi$, then the operators (5)–(8) reduce to the Riemann–Liouville fractional integrals $J_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma)$, $J_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma)$, $J_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma)$ and $J_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma)$, respectively;
3. For $\rho(\tau) = \ln \tau$ and $\omega(\xi) = \ln \xi$, the operators (5)–(8) reduce to the Hadamard fractional integrals $I_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma)$, $I_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma)$, $I_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma)$ and $I_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma)$, respectively.

3. Some Equalities for Functions of Bounded Variations with Two Variables

Firstly, we define the following functions which will be used frequently:

$$M_{\rho}^{\alpha}(\kappa_1, \varkappa) = \frac{[\rho(\varkappa) - \rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha + 1)},$$

$$N_{\omega}^{\beta}(\kappa_3, \gamma) = \frac{[\omega(\gamma) - \omega(\kappa_3)]^{\beta}}{\Gamma(\beta + 1)},$$

$$M_{\rho}^{\alpha}(\kappa_2, \varkappa) = \frac{[\rho(\kappa_2) - \rho(\varkappa)]^{\alpha}}{\Gamma(\alpha + 1)}$$

and

$$N_{\omega}^{\beta}(\kappa_4, \gamma) = \frac{[\omega(\kappa_4) - \omega(\gamma)]^{\beta}}{\Gamma(\beta + 1)}$$

$$M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa) = [\rho(\varkappa) - \rho(\kappa_1)]^\alpha + [\rho(\kappa_2) - \rho(\varkappa)]^\alpha$$

$$N_\omega^\beta(\kappa_3, \kappa_4; \gamma) = [\omega(\gamma) - \omega(\kappa_3)]^\beta + [\omega(\kappa_4) - \omega(\gamma)]^\beta$$

for $(\varkappa, \gamma) \in \Delta$. We also denote

$$\begin{aligned} \kappa_1 &= M_\rho^\alpha(\kappa_1, \varkappa)N_\omega^\beta(\kappa_3, \gamma)F(\kappa_1, \kappa_3) + M_\rho^\alpha(\kappa_1, \varkappa)N_\omega^\beta(\kappa_4, \gamma)F(\kappa_1, \kappa_4) \\ &+ M_\rho^\alpha(\kappa_2, \varkappa)N_\omega^\beta(\kappa_3, \gamma)F(\kappa_2, \kappa_3) + M_\rho^\alpha(\kappa_2, \varkappa)N_\omega^\beta(\kappa_4, \gamma)F(\kappa_2, \kappa_4) \end{aligned}$$

for $(\varkappa, \gamma) \in \Delta$.

Throughout this paper, we denote the second partial derivative $\frac{\partial^2 F}{\partial \tau \partial \xi}$ by $F_{\tau \xi}$. Moreover, let $\rho : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a positive increasing function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2) and let $\omega : [\kappa_3, \kappa_4] \rightarrow \mathbb{R}$ be a positive increasing function on $(\kappa_3, \kappa_4]$, having a continuous derivative $\omega'(\gamma)$ on (κ_3, κ_4) .

Now, we are in position to prove the following identity:

Lemma 3. *If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following equality:*

$$\begin{aligned} &\kappa_1 - N_\omega^\beta(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^\alpha F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-; \rho}^\alpha F(\varkappa, \kappa_3) \right] \\ &- N_\omega^\beta(\kappa_4, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^\alpha F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-; \rho}^\alpha F(\varkappa, \kappa_4) \right] \\ &- M_\rho^\alpha(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^\beta F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^\beta F(\kappa_1, \gamma) \right] \\ &- M_\rho^\alpha(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^\beta F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^\beta F(\kappa_2, \gamma) \right] + \mathcal{J}_{\kappa_1+; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \\ &+ \mathcal{J}_{\kappa_1+; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \\ &= \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} [I_1 - I_2 - I_3 + I_4], \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi), \\ I_2 &= \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi), \\ I_3 &= \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi), \\ I_4 &= \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi). \end{aligned}$$

Proof. By using Lemma 1, we have

$$\begin{aligned} I_1 &= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \tag{9} \\ &= (\omega(\gamma) - \omega(\kappa_3))^\beta (\rho(\varkappa) - \rho(\kappa_1))^\alpha F(\kappa_1, \kappa_3) - \Gamma(\alpha + 1)(\omega(\gamma) \end{aligned}$$

$$\begin{aligned}
& -\omega(\kappa_3))^\beta \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa, \kappa_3) - \Gamma(\beta + 1)(\rho(\varkappa) - \rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_1, \gamma) \\
& + \Gamma(\alpha + 1)\Gamma(\beta + 1)\mathcal{J}_{\kappa_1+\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
I_2 = & -(\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta F(\kappa_1, \kappa_4) + \Gamma(\alpha + 1)(\omega(\kappa_4) \\
& - \omega(\gamma))^\beta \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa, \kappa_4) + \Gamma(\beta + 1)(\rho(\varkappa) - \rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_1, \gamma) \\
& - \Gamma(\alpha + 1)\Gamma(\beta + 1)\mathcal{J}_{\kappa_1+\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma),
\end{aligned} \tag{10}$$

$$\begin{aligned}
I_3 = & -(\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta F(\kappa_2, \kappa_3) + \Gamma(\alpha + 1)(\omega(\gamma) \\
& - \omega(\kappa_3))^\beta \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa, \kappa_3) + \Gamma(\beta + 1)(\rho(\kappa_2) \\
& - \rho(\varkappa))^\alpha \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_2, \gamma) - \Gamma(\alpha + 1)\Gamma(\beta + 1)\mathcal{J}_{\kappa_2-\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma)
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
I_4 = & (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta F(\kappa_2, \kappa_4) - \Gamma(\alpha + 1)(\omega(\kappa_4) \\
& - \omega(\gamma))^\beta \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa, \kappa_4) - \Gamma(\beta + 1)(\rho(\kappa_2) \\
& - \rho(\varkappa))^\alpha \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_2, \gamma) + \Gamma(\alpha + 1)\Gamma(\beta + 1)\mathcal{J}_{\kappa_2-\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma).
\end{aligned} \tag{12}$$

By equalities (9)–(12), we establish

$$\begin{aligned}
& \kappa_1 - N_\omega^\beta(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa, \kappa_3) \right] - N_\omega^\beta(\kappa_4, \gamma) \\
& \times \left[\mathcal{J}_{\kappa_1+;\rho}^\alpha F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\varkappa, \kappa_4) \right] - M_\rho^\alpha(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_1, \gamma) \right] \\
& - M_\rho^\alpha(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_2, \gamma) \right] + \mathcal{J}_{\kappa_1+\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma) \\
& + \mathcal{J}_{\kappa_1+\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\varkappa, \gamma) \\
= & \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} [I_1 - I_2 - I_3 + I_4]
\end{aligned}$$

which completes the proof. \square

Lemma 4. If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following equality

$$F(\varkappa, \gamma) - \frac{\Gamma(\alpha + 1)}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa)} \left[\mathcal{J}_{\varkappa-;\rho}^\alpha F(\kappa_1, \gamma) + \mathcal{J}_{\varkappa+;\rho}^\alpha F(\kappa_2, \gamma) \right]$$

$$\begin{aligned}
 & -\frac{\Gamma(\beta + 1)}{N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\gamma-; \omega}^{\beta} F(\varkappa, \kappa_3) + \mathcal{J}_{\gamma+; \omega}^{\beta} F(\varkappa, \kappa_4) \right] \\
 & + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{M_{\rho}^{\alpha}(\kappa_1, \kappa_2; \varkappa)N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\varkappa-, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\varkappa-, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\
 & \left. + \mathcal{J}_{\varkappa+, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\varkappa+, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \\
 = & \frac{1}{M_{\rho}^{\alpha}(\kappa_1, \kappa_2; \varkappa)N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} [I_5 - I_6 - I_7 + I_8]
 \end{aligned}$$

where

$$\begin{aligned}
 I_5 &= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\kappa_1))^{\alpha} (\omega(\xi) - \omega(\kappa_3))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi), \\
 I_6 &= \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\xi))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi), \\
 I_7 &= \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\kappa_2) - \rho(\tau))^{\alpha} (\omega(\xi) - \omega(\kappa_3))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi)
 \end{aligned}$$

and

$$I_8 = \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\kappa_2) - \rho(\tau))^{\alpha} (\omega(\kappa_4) - \omega(\xi))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi).$$

Proof. By using Lemma 1, we have

$$\begin{aligned}
 I_5 &= \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\kappa_1))^{\alpha} (\omega(\xi) - \omega(\kappa_3))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi) \tag{13} \\
 &= (\omega(\gamma) - \omega(\kappa_3))^{\beta} (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} F(\varkappa, \gamma) - \Gamma(\alpha + 1) (\omega(\gamma) - \omega(\kappa_3))^{\beta} \mathcal{J}_{\varkappa-; \rho}^{\alpha} F(\kappa_1, \gamma) \\
 &\quad - \Gamma(\beta + 1) (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} \mathcal{J}_{\gamma-; \omega}^{\beta} F(\varkappa, \kappa_3) + \Gamma(\alpha + 1)\Gamma(\beta + 1) \mathcal{J}_{\varkappa-, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3).
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 I_6 &= \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\xi))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi) \tag{14} \\
 &= -(\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} F(\varkappa, \gamma) + \Gamma(\alpha + 1) (\omega(\kappa_4) - \omega(\gamma))^{\beta} \mathcal{J}_{\varkappa-; \rho}^{\alpha} F(\kappa_1, \gamma) \\
 &\quad + \Gamma(\beta + 1) (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} \mathcal{J}_{\gamma+; \omega}^{\beta} F(\varkappa, \kappa_4) - \Gamma(\alpha + 1)\Gamma(\beta + 1) \mathcal{J}_{\varkappa-, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4),
 \end{aligned}$$

$$\begin{aligned}
 I_7 &= \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\kappa_2) - \rho(\tau))^{\alpha} (\omega(\xi) - \omega(\kappa_3))^{\beta} d_{\tau} d_{\xi} F(\tau, \xi) \tag{15} \\
 &= -(\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta} F(\varkappa, \gamma) + \Gamma(\alpha + 1) (\omega(\gamma) - \omega(\kappa_3))^{\beta} \mathcal{J}_{\varkappa+; \rho}^{\alpha} F(\kappa_2, \gamma) \\
 &\quad + \Gamma(\beta + 1) (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} \mathcal{J}_{\gamma-; \omega}^{\beta} F(\varkappa, \kappa_3) - \Gamma(\alpha + 1)\Gamma(\beta + 1) \mathcal{J}_{\varkappa+, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3)
 \end{aligned}$$

and

$$\begin{aligned}
 I_8 &= \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d\tau d\xi F(\tau, \xi) \\
 &= (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta F(\varkappa, \gamma) - \Gamma(\alpha + 1) (\omega(\kappa_4) - \omega(\gamma))^\beta \mathcal{J}_{\varkappa+, \rho}^\alpha F(\kappa_2, \gamma) \\
 &\quad - \Gamma(\beta + 1) (\rho(\kappa_2) - \rho(\varkappa))^\alpha \mathcal{J}_{\gamma+, \omega}^\beta F(\varkappa, \kappa_4) + \Gamma(\alpha + 1) \Gamma(\beta + 1) \mathcal{J}_{\varkappa+, \gamma+, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4).
 \end{aligned} \tag{16}$$

Using equalities (13)–(16), we have the desired result. \square

Lemma 5. If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following equality

$$\begin{aligned}
 &\frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \\
 &\quad - \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)} \left[\mathcal{J}_{\kappa_2-, \rho}^\alpha F(\kappa_1, \kappa_4) + \mathcal{J}_{\kappa_2-, \rho}^\alpha F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_1+, \rho}^\alpha F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_1+, \rho}^\alpha F(\kappa_2, \kappa_3) \right] \\
 &\quad - \frac{1}{4N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_4-, \omega}^\beta F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_3+, \omega}^\beta F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_4-, \omega}^\beta F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_3+, \omega}^\beta F(\kappa_1, \kappa_4) \right] \\
 &\quad + \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2) N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_2-, \kappa_4-, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_2-, \kappa_3+, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\
 &\quad \left. + \mathcal{J}_{\kappa_1+, \kappa_4-, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_1+, \kappa_3+, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \\
 = &\frac{1}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2) N_\omega^\beta(\kappa_3, \kappa_4)} \\
 &\quad \times \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} [(\rho(\kappa_2) - \rho(\tau))^\alpha - \rho(\tau) - \rho(\kappa_1))^\alpha] \\
 &\quad \times [(\omega(\kappa_4) - \omega(\xi))^\beta - (\omega(\xi) - \omega(\kappa_3))^\beta] d\tau d\xi F(\tau, \xi).
 \end{aligned} \tag{17}$$

Proof. For $\varkappa = \kappa_2$ and $\gamma = \kappa_4$ in Equation (13), we have

$$\begin{aligned}
 I_9 &= \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\xi) - \omega(\kappa_3))^\beta d\tau d\xi F(\tau, \xi) \\
 &= (\omega(\kappa_4) - \omega(\kappa_3))^\beta (\rho(\kappa_2) - \rho(\kappa_1))^\alpha F(\kappa_2, \kappa_4) - \Gamma(\alpha + 1) (\omega(\kappa_4) \\
 &\quad - \omega(\kappa_3))^\beta \mathcal{J}_{\kappa_2-, \rho}^\alpha F(\kappa_1, \kappa_4) - \Gamma(\beta + 1) (\rho(\kappa_2) - \rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_4-, \omega}^\beta F(\kappa_2, \kappa_3) \\
 &\quad + \Gamma(\alpha + 1) \Gamma(\beta + 1) \mathcal{J}_{\kappa_2-, \kappa_4-, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3).
 \end{aligned} \tag{18}$$

For $\varkappa = \kappa_2$ and $\gamma = \kappa_3$ in Equation (14), we have

$$\begin{aligned}
 I_{10} &= \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d\tau d\xi F(\tau, \xi) \\
 &\quad - (\rho(\kappa_2) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\kappa_3))^\beta F(\kappa_2, \kappa_3)
 \end{aligned} \tag{19}$$

$$\begin{aligned}
& +\Gamma(\alpha+1)(\omega(\kappa_4)-\omega(\kappa_3))^\beta \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\kappa_1, \kappa_3) \\
& +\Gamma(\beta+1)(\rho(\kappa_2)-\rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_2, \kappa_4) \\
& -\Gamma(\alpha+1)\Gamma(\beta+1)\mathcal{J}_{\kappa_2-,\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\kappa_1, \kappa_4),
\end{aligned}$$

for $\varkappa = \kappa_1$ and $\gamma = \kappa_4$ in Equation (15), we have

$$\begin{aligned}
I_{11} &= \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} (\rho(\kappa_2)-\rho(\tau))^\alpha (\omega(\xi)-\omega(\kappa_3))^\beta d_\tau d_\xi F(\tau, \xi) \quad (20) \\
&= -(\rho(\kappa_2)-\rho(\kappa_1))^\alpha (\omega(\kappa_4)-\omega(\kappa_3))^\beta F(\kappa_1, \kappa_4) \\
&+ \Gamma(\alpha+1)(\omega(\kappa_4)-\omega(\kappa_3))^\beta \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\kappa_2, \kappa_4) \\
&+ \Gamma(\beta+1)(\rho(\kappa_2)-\rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_1, \kappa_3) \\
&- \Gamma(\alpha+1)\Gamma(\beta+1)\mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\kappa_2, \kappa_3)
\end{aligned}$$

and finally, for $\varkappa = \kappa_1$ and $\gamma = \kappa_3$ in Equation (16), we find

$$\begin{aligned}
I_{12} &= \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} (\rho(\kappa_2)-\rho(\tau))^\alpha (\omega(\kappa_4)-\omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \quad (21) \\
&= (\rho(\kappa_2)-\rho(\kappa_1))^\alpha (\omega(\kappa_4)-\omega(\kappa_3))^\beta F(\kappa_1, \kappa_3) \\
&- \Gamma(\alpha+1)(\omega(\kappa_4)-\omega(\kappa_3))^\beta \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\kappa_2, \kappa_3) \\
&- \Gamma(\beta+1)(\rho(\kappa_2)-\rho(\kappa_1))^\alpha \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_1, \kappa_4) \\
&+ \Gamma(\alpha+1)\Gamma(\beta+1)\mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\kappa_2, \kappa_4).
\end{aligned}$$

Using equalities (18)–(21), we establish

$$\begin{aligned}
& \frac{1}{4\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} [I_9 - I_{10} - I_{11} + I_{12}] \\
= & \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \\
& - \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)} \left[\mathcal{J}_{\kappa_2-;\rho}^\alpha F(\kappa_1, \kappa_4) + \mathcal{J}_{\kappa_2-;\rho}^\alpha F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_1+;\rho}^\alpha F(\kappa_2, \kappa_3) \right] \\
& - \frac{1}{4N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_4-;\omega}^\beta F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_3+;\omega}^\beta F(\kappa_1, \kappa_4) \right] \\
& + \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_2-,\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_2-,\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\kappa_1, \kappa_4) \right. \\
& \left. + \mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\omega}^{\alpha,\beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\omega}^{\alpha,\beta} F(\kappa_2, \kappa_4) \right]
\end{aligned}$$

which completes the proof. \square

4. Trapezoid-Type Inequalities for Functions of Bounded Variations with Two Variables

In this section, we present some trapezoid-type inequalities for generalized fractional integrals.

Theorem 2. *If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following inequality*

$$\begin{aligned}
 & \left| \kappa_1 - N_{\omega}^{\beta}(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^{\alpha} F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-; \rho}^{\alpha} F(\varkappa, \kappa_3) \right] \right. \\
 & \quad - N_{\omega}^{\beta}(\kappa_4, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^{\alpha} F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-; \rho}^{\alpha} F(\varkappa, \kappa_4) \right] \\
 & \quad - M_{\rho}^{\alpha}(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_1, \gamma) \right] \\
 & \quad - M_{\rho}^{\alpha}(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_2, \gamma) \right] + \mathcal{J}_{\kappa_1+; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \\
 & \quad \left. + \mathcal{J}_{\kappa_1+; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \right| \\
 & \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[(\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta} \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) \right. \\
 & \quad + (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) \\
 & \quad + (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta} \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) \\
 & \quad \left. + (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F) \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[\frac{1}{2} (\rho(\kappa_2) - \rho(\kappa_1)) + \left| \rho(\varkappa) - \frac{\rho(\kappa_1) + \rho(\kappa_2)}{2} \right| \right]^{\alpha} \\
 & \quad \times \left[\frac{1}{2} (\omega(\kappa_4) - \omega(\kappa_3)) + \left| \omega(\gamma) - \frac{\omega(\kappa_3) + \omega(\kappa_4)}{2} \right| \right]^{\beta} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_2}^{\kappa_4} (F).
 \end{aligned} \tag{22}$$

Proof. By taking the modulus in Lemma 3, we obtain

$$\begin{aligned}
 & \left| \kappa_1 - N_{\omega}^{\beta}(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^{\alpha} F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-; \rho}^{\alpha} F(\varkappa, \kappa_3) \right] \right. \\
 & \quad - N_{\omega}^{\beta}(\kappa_4, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^{\alpha} F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-; \rho}^{\alpha} F(\varkappa, \kappa_4) \right] \\
 & \quad - M_{\rho}^{\alpha}(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_1, \gamma) \right] \\
 & \quad - M_{\rho}^{\alpha}(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_2, \gamma) \right] + \mathcal{J}_{\kappa_1+; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \\
 & \quad \left. + \mathcal{J}_{\kappa_1+; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-; \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \right|
 \end{aligned} \tag{23}$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \left[\left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \right. \\ &\quad + \left| \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\ &\quad + \left| \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\ &\quad \left. + \left| \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi) \right| \right]. \end{aligned}$$

By Lemma, it follows that

$$\begin{aligned} &\left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \tag{24} \\ &\leq \sup_{(\tau, \xi) \in [\kappa_1, \varkappa] \times [\kappa_3, \gamma]} \left| (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\gamma) - \omega(\xi))^\beta \right| \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) \\ &= (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\left| \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\varkappa) - \rho(\tau))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\ &\leq (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) \tag{25} \end{aligned}$$

$$\begin{aligned} &\left| \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\ &\leq (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) \tag{26} \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\varkappa))^\alpha (\omega(\xi) - \omega(\gamma))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\ &\leq (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F). \tag{27} \end{aligned}$$

By substituting the inequalities (24)–(27) in Equation (23), we obtain

$$\begin{aligned} &\left| \kappa_1 - N_\omega^\beta(\kappa_3, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^\alpha F(\varkappa, \kappa_3) + \mathcal{J}_{\kappa_2-; \rho}^\alpha F(\varkappa, \kappa_3) \right] \right. \\ &\quad \left. - N_\omega^\beta(\kappa_4, \gamma) \left[\mathcal{J}_{\kappa_1+; \rho}^\alpha F(\varkappa, \kappa_4) + \mathcal{J}_{\kappa_2-; \rho}^\alpha F(\varkappa, \kappa_4) \right] \right| \end{aligned}$$

$$\begin{aligned}
& -M_{\rho}^{\alpha}(\kappa_1, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_1, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_1, \gamma) \right] \\
& -M_{\rho}^{\alpha}(\kappa_2, \varkappa) \left[\mathcal{J}_{\kappa_3+; \omega}^{\beta} F(\kappa_2, \gamma) + \mathcal{J}_{\kappa_4-; \omega}^{\beta} F(\kappa_2, \gamma) \right] + \mathcal{J}_{\kappa_1+, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \\
& + \mathcal{J}_{\kappa_1+, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) + \mathcal{J}_{\kappa_2-, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\varkappa, \gamma) \Big| \\
\leq & \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[(\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta} \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) \right. \\
& + (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) \\
& + (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta} \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) \\
& \left. + (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F) \right] \\
= & : Y(\varkappa, \gamma)
\end{aligned}$$

which gives the first inequality in Equation (22).

By the properties of maximum, we obtain

$$\begin{aligned}
& Y(\varkappa, \gamma) \\
\leq & \max \left\{ (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta}, (\rho(\varkappa) - \rho(\kappa_1))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta}, \right. \\
& \left. (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\gamma) - \omega(\kappa_3))^{\beta}, (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} (\omega(\kappa_4) - \omega(\gamma))^{\beta} \right\} \\
& \times \left[\bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) + \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) + \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) + \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F) \right] \\
= & \max \{ (\rho(\varkappa) - \rho(\kappa_1))^{\alpha}, (\rho(\kappa_2) - \rho(\varkappa))^{\alpha} \} \max \left\{ (\omega(\gamma) - \omega(\kappa_3))^{\beta}, (\omega(\kappa_4) - \omega(\gamma))^{\beta} \right\} \\
& \times \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_2}^{\kappa_4} (F) \\
= & \left[\frac{1}{2} (\rho(\kappa_2) - \rho(\kappa_1)) + \left| \rho(\varkappa) - \frac{\rho(\kappa_1) + \rho(\kappa_2)}{2} \right| \right]^{\alpha} \\
& \times \left[\frac{1}{2} (\omega(\kappa_4) - \omega(\kappa_3)) + \left| \omega(\gamma) - \frac{\omega(\kappa_3) + \omega(\kappa_4)}{2} \right| \right]^{\beta} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_2}^{\kappa_4} (F).
\end{aligned}$$

This completes the proof. \square

Corollary 1. If we take $\rho(\tau) = \tau$, $\tau \in [\kappa_1, \kappa_2]$ and $\omega(\xi) = \xi$, $\xi \in [\kappa_3, \kappa_4]$ in Theorem 2, then we have the following trapezoid-type inequalities for Riemann–Liouville fractional integrals

$$\left| \frac{(\varkappa - \kappa_1)^{\alpha} (\gamma - \kappa_3)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} F(\kappa_1, \kappa_3) + \frac{(\varkappa - \kappa_1)^{\alpha} (\kappa_4 - \gamma)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} F(\kappa_1, \kappa_4) \right|$$

$$\begin{aligned}
& + \frac{(\kappa_2 - \varkappa)^\alpha (\gamma - \kappa_3)^\beta}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} F(\kappa_2, \kappa_3) + \frac{(\kappa_2 - \varkappa)^\alpha (\kappa_4 - \gamma)^\beta}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} F(\kappa_2, \kappa_4) \\
& - \frac{(\gamma - \kappa_3)^\beta}{\Gamma(\beta + 1)} [J_{\kappa_1+}^\alpha F(\varkappa, \kappa_3) + J_{\kappa_2-}^\alpha F(\varkappa, \kappa_3)] \\
& - \frac{(\kappa_4 - \gamma)^\beta}{\Gamma(\beta + 1)} [J_{\kappa_1+}^\alpha F(\varkappa, \kappa_4) + J_{\kappa_2-}^\alpha F(\varkappa, \kappa_4)] \\
& - \frac{(\varkappa - \kappa_1)^\alpha}{\Gamma(\alpha + 1)} [J_{\kappa_3+}^\beta F(\kappa_1, \gamma) + J_{\kappa_4-}^\beta F(\kappa_1, \gamma)] \\
& - \frac{(\kappa_2 - \varkappa)^\alpha}{\Gamma(\alpha + 1)} [J_{\kappa_3+}^\beta F(\kappa_2, \gamma) + J_{\kappa_4-}^\beta F(\kappa_2, \gamma)] \\
& + J_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma) + J_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma) + J_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma) + J_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma) \\
\leq & \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \\
& \times \left[(\varkappa - \kappa_1)^\alpha (\gamma - \kappa_3)^\beta \underset{\kappa_1}{\overset{\varkappa}{\mathbb{V}}} \underset{\kappa_3}{\overset{\gamma}{\mathbb{V}}}(F) + (\varkappa - \kappa_1)^\alpha (\kappa_4 - \gamma)^\beta \underset{\kappa_1}{\overset{\varkappa}{\mathbb{V}}} \underset{\gamma}{\overset{\kappa_4}{\mathbb{V}}}(F) \right. \\
& \left. + (\kappa_2 - \varkappa)^\alpha (\gamma - \kappa_3)^\beta \underset{\varkappa}{\overset{\kappa_2}{\mathbb{V}}} \underset{\kappa_3}{\overset{\gamma}{\mathbb{V}}}(F) + (\kappa_2 - \varkappa)^\alpha (\kappa_4 - \gamma)^\beta \underset{\varkappa}{\overset{\kappa_2}{\mathbb{V}}} \underset{\gamma}{\overset{\kappa_4}{\mathbb{V}}}(F) \right] \\
\leq & \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \left[\frac{\kappa_2 - \kappa_1}{2} + \left| \varkappa - \frac{\kappa_1 + \kappa_2}{2} \right| \right]^\alpha \\
& \times \left[\frac{\kappa_4 - \kappa_3}{2} + \left| \gamma - \frac{\kappa_3 + \kappa_4}{2} \right| \right]^\beta \underset{\kappa_1}{\overset{\kappa_2}{\mathbb{V}}} \underset{\kappa_2}{\overset{\kappa_4}{\mathbb{V}}}(F).
\end{aligned}$$

Corollary 2. If we take $\varkappa = \frac{\kappa_1 + \kappa_2}{2}$ and $\gamma = \frac{\kappa_3 + \kappa_4}{2}$ in Corollary 1, then we have

$$\begin{aligned}
& \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\
& - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} [J_{\kappa_1+}^\alpha F(\varkappa, \kappa_3) + J_{\kappa_2-}^\alpha F(\varkappa, \kappa_3)] \\
& - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(\kappa_2 - \kappa_1)^\alpha} [J_{\kappa_1+}^\alpha F(\varkappa, \kappa_4) + J_{\kappa_2-}^\alpha F(\varkappa, \kappa_4)] \\
& - \frac{2^{\beta-1}\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta} [J_{\kappa_3+}^\beta F(\kappa_1, \gamma) + J_{\kappa_4-}^\beta F(\kappa_1, \gamma)] \\
& - \frac{2^{\beta-1}\Gamma(\beta + 1)}{(\kappa_4 - \kappa_3)^\beta} [J_{\kappa_3+}^\beta F(\kappa_2, \gamma) + J_{\kappa_4-}^\beta F(\kappa_2, \gamma)] \\
& \left. - \frac{2^{\alpha+\beta-2}\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(\kappa_2 - \kappa_1)^\alpha (\kappa_4 - \kappa_3)^\beta} [J_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma) + J_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma) \right. \\
& \left. + J_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\varkappa, \gamma) + J_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\varkappa, \gamma)] \right| \\
\leq & \frac{1}{4} \underset{\kappa_1}{\overset{\kappa_2}{\mathbb{V}}} \underset{\kappa_2}{\overset{\kappa_4}{\mathbb{V}}}(F).
\end{aligned}$$

Remark 1. If we assign $\alpha = \beta = 1$ in Corollary 2, then we have

$$\begin{aligned} & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\ & - \frac{1}{2(\kappa_2 - \kappa_1)} \left[\int_{\kappa_1}^{\kappa_2} F(x, \kappa_4) dx + \int_{\kappa_1}^{\kappa_2} F(x, \kappa_3) dx \right] \\ & - \frac{1}{2(\kappa_4 - \kappa_3)} \left[\int_{\kappa_3}^{\kappa_4} F(\kappa_2, \gamma) d\gamma + \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \gamma) d\gamma \right] \\ & \left. + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(x, \gamma) d\gamma dx \right| \\ & \leq \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F). \end{aligned}$$

which is given by Budak and Sarikaya in [43].

Theorem 3. If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following inequality

$$\begin{aligned} & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\ & - \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)} \left[\mathcal{J}_{\kappa_2^-; \rho}^\alpha F(\kappa_1, \kappa_4) + \mathcal{J}_{\kappa_2^-; \rho}^\alpha F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_1^+; \rho}^\alpha F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_1^+; \rho}^\alpha F(\kappa_2, \kappa_3) \right] \\ & - \frac{1}{4N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_4^-; \omega}^\beta F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_3^+; \omega}^\beta F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_4^-; \omega}^\beta F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_3^+; \omega}^\beta F(\kappa_1, \kappa_4) \right] \\ & + \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_2^-, \kappa_4^-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_2^-, \kappa_3^+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\ & \left. + \mathcal{J}_{\kappa_1^+, \kappa_4^-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_1^+, \kappa_3^+; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\ & \leq \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F). \end{aligned}$$

Proof. By using Lemma 2, we have

$$\begin{aligned} & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\ & - \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)} \left[\mathcal{J}_{\kappa_2^-; \rho}^\alpha F(\kappa_1, \kappa_4) + \mathcal{J}_{\kappa_2^-; \rho}^\alpha F(\kappa_1, \kappa_3) \right. \\ & \left. + \mathcal{J}_{\kappa_1^+; \rho}^\alpha F(\kappa_2, \kappa_4) + \mathcal{J}_{\kappa_1^+; \rho}^\alpha F(\kappa_2, \kappa_3) \right] \\ & - \frac{1}{4N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_4^-; \omega}^\beta F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_3^+; \omega}^\beta F(\kappa_2, \kappa_4) \right. \\ & \left. + \mathcal{J}_{\kappa_4^-; \omega}^\beta F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_3^+; \omega}^\beta F(\kappa_1, \kappa_4) \right] \Big| \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \left[\mathcal{J}_{\kappa_2-, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\kappa_2-, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\
 & \left. + \mathcal{J}_{\kappa_1+, \kappa_4-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\kappa_1+, \kappa_3+; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \\
 = & \frac{1}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \\
 & \times \left| \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} [(\rho(\kappa_2) - \rho(\tau))^\alpha - \rho(\tau) - \rho(\kappa_1)]^\alpha \right. \\
 & \times \left. [(\omega(\kappa_4) - \omega(\xi))^\beta - (\omega(\xi) - \omega(\kappa_3))^\beta] d_\tau d_\xi F(\tau, \xi) \right| \\
 \leq & \frac{1}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \\
 & \times \sup_{(\tau, \gamma) \in \Delta} |(\rho(\kappa_2) - \rho(\tau))^\alpha - \rho(\tau) - \rho(\kappa_1)|^\alpha \\
 & \times \left| (\omega(\kappa_4) - \omega(\xi))^\beta - (\omega(\xi) - \omega(\kappa_3))^\beta \right| \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F) \\
 = & \frac{1}{4\Gamma(\alpha + 1)\Gamma(\beta + 1)} \frac{(\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta}{M_\rho^\alpha(\kappa_1, \kappa_2)N_\omega^\beta(\kappa_3, \kappa_4)} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F) \\
 = & \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F).
 \end{aligned}$$

This completes the proof. \square

Corollary 3. *If we take $\rho(\tau) = \tau$, $\tau \in [\kappa_1, \kappa_2]$ and $\omega(\xi) = \xi$, $\xi \in [\kappa_3, \kappa_4]$ in Theorem 3, then we have*

$$\begin{aligned}
 & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\
 & - \frac{\Gamma(\alpha + 1)}{4(\kappa_2 - \kappa_1)^\alpha} \left[\mathbf{J}_{\kappa_2-}^\alpha F(\kappa_1, \kappa_4) + \mathbf{J}_{\kappa_2-}^\alpha F(\kappa_1, \kappa_3) + \mathbf{J}_{\kappa_1+}^\alpha F(\kappa_2, \kappa_4) + \mathbf{J}_{\kappa_1+}^\alpha F(\kappa_2, \kappa_3) \right] \\
 & - \frac{\Gamma(\beta + 1)}{4(\kappa_4 - \kappa_3)^\beta} \left[\mathbf{J}_{\kappa_4-}^\beta F(\kappa_2, \kappa_3) + \mathbf{J}_{\kappa_3+}^\beta F(\kappa_2, \kappa_4) + \mathbf{J}_{\kappa_4-}^\beta F(\kappa_1, \kappa_3) + \mathbf{J}_{\kappa_3+}^\beta F(\kappa_1, \kappa_4) \right] \\
 & + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{4(\kappa_2 - \kappa_1)^\alpha (\kappa_4 - \kappa_3)^\beta} \left[\mathbf{J}_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathbf{J}_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\
 & \left. + \mathbf{J}_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathbf{J}_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\
 \leq & \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F).
 \end{aligned}$$

Remark 2. *If we assign $\alpha = \beta = 1$ in Corollary 3, then we have*

$$\begin{aligned}
 & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\
 & \left. - \frac{1}{2(\kappa_2 - \kappa_1)} \left[\int_{\kappa_1}^{\kappa_2} F(\varkappa, \kappa_4) d\varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa, \kappa_3) d\varkappa \right] \right|
 \end{aligned}$$

$$\begin{aligned} & -\frac{1}{2(\kappa_4 - \kappa_3)} \left[\int_{\kappa_3}^{\kappa_4} F(\kappa_2, \gamma) d\gamma + \int_{\kappa_3}^{\kappa_4} F(\kappa_1, \gamma) d\gamma \right] \\ & + \frac{1}{(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)} \int_{\kappa_1}^{\kappa_2} \int_{\kappa_3}^{\kappa_4} F(\varkappa, \gamma) d\gamma d\varkappa \Big| \\ & \leq \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F). \end{aligned}$$

which is given by Budak and Sarikaya in [43].

Corollary 4. If we take $\rho(\tau) = \ln \tau$, $\tau \in [\kappa_1, \kappa_2]$ and $\omega(\xi) = \ln \xi$, $\xi \in [\kappa_3, \kappa_4]$ in Theorem 3, then we have the following trapezoid-type inequality for Hadamard fractional integrals

$$\begin{aligned} & \left| \frac{F(\kappa_2, \kappa_4) + F(\kappa_2, \kappa_3) + F(\kappa_1, \kappa_4) + F(\kappa_1, \kappa_3)}{4} \right. \\ & - \frac{1}{4M_{\ln}^{\alpha}(\kappa_1, \kappa_2)} \left[I_{\kappa_2-}^{\alpha} F(\kappa_1, \kappa_4) + I_{\kappa_2-}^{\alpha} F(\kappa_1, \kappa_3) + I_{\kappa_1+}^{\alpha} F(\kappa_2, \kappa_4) + I_{\kappa_1+}^{\alpha} F(\kappa_2, \kappa_3) \right] \\ & - \frac{1}{4N_{\ln}^{\beta}(\kappa_3, \kappa_4)} \left[I_{\kappa_4-}^{\beta} F(\kappa_2, \kappa_3) + I_{\kappa_3+}^{\beta} F(\kappa_2, \kappa_4) + I_{\kappa_4-}^{\beta} F(\kappa_1, \kappa_3) + I_{\kappa_3+}^{\beta} F(\kappa_1, \kappa_4) \right] \\ & + \frac{1}{4M_{\ln}^{\alpha}(\kappa_1, \kappa_2)N_{\ln}^{\beta}(\kappa_3, \kappa_4)} \left[I_{\kappa_2-, \kappa_4-}^{\alpha, \beta} F(\kappa_1, \kappa_3) + I_{\kappa_2-, \kappa_3+}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\ & \left. + I_{\kappa_1+, \kappa_4-}^{\alpha, \beta} F(\kappa_2, \kappa_3) + I_{\kappa_1+, \kappa_3+}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\ & \leq \frac{1}{4} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F). \end{aligned}$$

5. Ostrowski-Type Inequalities for Functions of Bounded Variations with Two Variables

In this section, we prove some Ostrowski-type inequalities for generalized fractional integrals.

Theorem 4. If $F : \Delta \rightarrow \mathbb{R}$ be a mapping of bounded variation on Δ , then for $\alpha, \beta > 0$, we have the following inequality

$$\begin{aligned} & \left| F(\varkappa, \gamma) - \frac{\Gamma(\alpha + 1)}{M_{\rho}^{\alpha}(\kappa_1, \kappa_2; \varkappa)} \left[\mathcal{J}_{\varkappa-, \rho}^{\alpha} F(\kappa_1, \gamma) + \mathcal{J}_{\varkappa+, \rho}^{\alpha} F(\kappa_2, \gamma) \right] \right. \tag{28} \\ & - \frac{\Gamma(\beta + 1)}{N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\gamma-, \omega}^{\beta} F(\varkappa, \kappa_3) + \mathcal{J}_{\gamma+, \omega}^{\beta} F(\varkappa, \kappa_4) \right] \\ & + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{M_{\rho}^{\alpha}(\kappa_1, \kappa_2; \varkappa)N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\varkappa-, \gamma-, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\varkappa-, \gamma+, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\ & \left. + \mathcal{J}_{\varkappa+, \gamma-, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\varkappa+, \gamma+, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\ & = \frac{1}{M_{\rho}^{\alpha}(\kappa_1, \kappa_2; \varkappa)N_{\omega}^{\beta}(\kappa_3, \kappa_4; \gamma)} \end{aligned}$$

$$\begin{aligned}
& \times \left[(\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \underset{\kappa_1}{\overset{\varkappa}{\mathbb{V}}} \underset{\kappa_3}{\overset{\gamma}{\mathbb{V}}}(F) \right. \\
& + (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \underset{\kappa_1}{\overset{\varkappa}{\mathbb{V}}} \underset{\gamma}{\overset{\kappa_4}{\mathbb{V}}}(F) \\
& + (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \underset{\varkappa}{\overset{\kappa_2}{\mathbb{V}}} \underset{\kappa_3}{\overset{\gamma}{\mathbb{V}}}(F) \\
& \left. + (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \underset{\varkappa}{\overset{\kappa_2}{\mathbb{V}}} \underset{\gamma}{\overset{\kappa_4}{\mathbb{V}}}(F) \right] \\
\leq & \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa) N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \\
& \times \left[\frac{1}{2}(\rho(\kappa_2) - \rho(\kappa_1)) + \left| \rho(\varkappa) - \frac{\rho(\kappa_1) + \rho(\kappa_2)}{2} \right| \right]^\alpha \\
& \times \left[\frac{1}{2}(\omega(\kappa_4) - \omega(\kappa_3)) + \left| \omega(\gamma) - \frac{\omega(\kappa_3) + \omega(\kappa_4)}{2} \right| \right]^\beta \underset{\kappa_1}{\overset{\kappa_2}{\mathbb{V}}} \underset{\kappa_3}{\overset{\kappa_4}{\mathbb{V}}}(F).
\end{aligned}$$

Proof. By taking the modulus in Lemma 3, we obtain

$$\begin{aligned}
& \left| F(\varkappa, \gamma) - \frac{\Gamma(\alpha + 1)}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa)} \left[\mathcal{J}_{\varkappa-, \rho}^\alpha F(\kappa_1, \gamma) + \mathcal{J}_{\varkappa+, \rho}^\alpha F(\kappa_2, \gamma) \right] \right. \\
& - \frac{\Gamma(\beta + 1)}{N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\gamma-, \omega}^\beta F(\varkappa, \kappa_3) + \mathcal{J}_{\gamma+, \omega}^\beta F(\varkappa, \kappa_4) \right] \\
& + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa) N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\varkappa-, \gamma-, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\varkappa-, \gamma+, \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\
& \left. + \mathcal{J}_{\varkappa+, \gamma-, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\varkappa+, \gamma+, \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\
= & \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa) N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \\
& \times \left[\left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\xi) - \omega(\kappa_3))^\beta d_\tau d_\xi F(\tau, \xi) \right| \right. \\
& + \left| \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\
& + \left| \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\xi) - \omega(\kappa_3))^\beta d_\tau d_\xi F(\tau, \xi) \right| \\
& \left. + \left| \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \right].
\end{aligned}$$

By Lemma 2, we obtain

$$\left| \int_{\kappa_1}^{\varkappa} \int_{\kappa_3}^{\gamma} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\xi) - \omega(\kappa_3))^\beta d_\tau d_\xi F(\tau, \xi) \right|$$

$$\leq (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F),$$

$$\left| \int_{\kappa_1}^{\varkappa} \int_{\gamma}^{\kappa_4} (\rho(\tau) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \leq (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F),$$

$$\left| \int_{\varkappa}^{\kappa_2} \int_{\kappa_3}^{\gamma} (\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\xi) - \omega(\kappa_3))^\beta d_\tau d_\xi F(\tau, \xi) \right| \leq (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F)$$

and

$$\left| \int_{\varkappa}^{\kappa_2} \int_{\gamma}^{\kappa_4} (\rho(\kappa_2) - \rho(\tau))^\alpha (\omega(\kappa_4) - \omega(\xi))^\beta d_\tau d_\xi F(\tau, \xi) \right| \leq (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F).$$

It follows that

$$\begin{aligned} & \left| F(\varkappa, \gamma) - \frac{\Gamma(\alpha + 1)}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa)} \left[\mathcal{J}_{\varkappa-; \rho}^\alpha F(\kappa_1, \gamma) + \mathcal{J}_{\varkappa+; \rho}^\alpha F(\kappa_2, \gamma) \right] \right. \\ & - \frac{\Gamma(\beta + 1)}{N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\gamma-; \omega}^\beta F(\varkappa, \kappa_3) + \mathcal{J}_{\gamma+; \omega}^\beta F(\varkappa, \kappa_4) \right] \\ & + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa)N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \left[\mathcal{J}_{\varkappa-, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) + \mathcal{J}_{\varkappa-, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) \right. \\ & \left. + \mathcal{J}_{\varkappa+, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\varkappa+, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\ & = \frac{1}{M_\rho^\alpha(\kappa_1, \kappa_2; \varkappa)N_\omega^\beta(\kappa_3, \kappa_4; \gamma)} \\ & \times \left[(\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) \right. \\ & + (\rho(\varkappa) - \rho(\kappa_1))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) \\ & + (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\gamma) - \omega(\kappa_3))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) \\ & \left. + (\rho(\kappa_2) - \rho(\varkappa))^\alpha (\omega(\kappa_4) - \omega(\gamma))^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F) \right] \end{aligned}$$

which completes the proof of first inequality in Equation (28). The proof of the second inequality is obvious from the proof of Theorem 2. \square

Remark 3. If we take $\rho(\tau) = \tau$, $\tau \in [\kappa_1, \kappa_2]$ and $\omega(\xi) = \xi$, $\xi \in [\kappa_3, \kappa_4]$ in Theorem 4, then we have the following Ostrowski-type inequalities for Riemann–Liouville fractional integrals

$$\begin{aligned}
& \left| F(\varkappa, \gamma) - \frac{\Gamma(\alpha + 1)}{(\varkappa - \kappa_1)^\alpha + (\kappa_2 - \varkappa)^\alpha} \left[\mathcal{J}_{\varkappa-; \rho}^\alpha F(\kappa_1, \gamma) + \mathcal{J}_{\varkappa+; \rho}^\alpha F(\kappa_2, \gamma) \right] \right. \\
& - \frac{\Gamma(\beta + 1)}{(\gamma - \kappa_3)^\beta + (\kappa_4 - \gamma)^\beta} \left[\mathcal{J}_{\gamma-; \omega}^\beta F(\varkappa, \kappa_3) + \mathcal{J}_{\gamma+; \omega}^\beta F(\varkappa, \kappa_4) \right] \\
& + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{[(\varkappa - \kappa_1)^\alpha + (\kappa_2 - \varkappa)^\alpha][(\gamma - \kappa_3)^\beta + (\kappa_4 - \gamma)^\beta]} \left[\mathcal{J}_{\varkappa-, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_3) \right. \\
& \left. + \mathcal{J}_{\varkappa-, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_1, \kappa_4) + \mathcal{J}_{\varkappa+, \gamma-; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_3) + \mathcal{J}_{\varkappa+, \gamma+; \rho, \omega}^{\alpha, \beta} F(\kappa_2, \kappa_4) \right] \Big| \\
& = \frac{1}{[(\varkappa - \kappa_1)^\alpha + (\kappa_2 - \varkappa)^\alpha][(\gamma - \kappa_3)^\beta + (\kappa_4 - \gamma)^\beta]} \\
& \times \left[(\varkappa - \kappa_1)^\alpha (\gamma - \kappa_3)^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\kappa_3}^{\gamma} (F) + (\varkappa - \kappa_1)^\alpha (\kappa_4 - \gamma)^\beta \bigvee_{\kappa_1}^{\varkappa} \bigvee_{\gamma}^{\kappa_4} (F) \right. \\
& \left. + (\kappa_2 - \varkappa)^\alpha (\gamma - \kappa_3)^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\kappa_3}^{\gamma} (F) + (\kappa_2 - \varkappa)^\alpha (\kappa_4 - \gamma)^\beta \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4} (F) \right] \\
& \leq \frac{1}{[(\varkappa - \kappa_1)^\alpha + (\kappa_2 - \varkappa)^\alpha][(\gamma - \kappa_3)^\beta + (\kappa_4 - \gamma)^\beta]} \\
& \times \left[\frac{\kappa_2 - \kappa_1}{2} + \left| \varkappa - \frac{\kappa_1 + \kappa_2}{2} \right| \right]^\alpha \left[\frac{\kappa_4 - \kappa_3}{2} + \left| \gamma - \frac{\kappa_3 + \kappa_4}{2} \right| \right]^\beta \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_4} (F).
\end{aligned}$$

6. Conclusions

In this paper, we present several trapezoid and Ostrowski-type inequalities for functions of bounded variation with two variables via generalized fractional integrals. It is also shown that several results are given by special cases of the main results. We deduce that the findings proved in this work are naturally universal, contribute to the theory of inequalities and have applications for determining the uniqueness of solutions in fractional boundary value problems. The findings of this study can be applied to symmetry. The results for the case of symmetric functions can be obtained by applying the concepts of symmetric convex functions, which will be studied in future work. It is an interesting and new problem and forthcoming researchers can use the techniques of this study to derive similar inequalities for different kinds of fractional integrals in their future works.

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