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A New Class Of s-TYPE $X(u, v, l_p(E))$ Operators

Pınar Zengin Alp[*1], Merve İlkhan[2]

Abstract

In this paper, we define a new class of s-type $X(u, v; l_p(E))$ operators, $L_{u,v,E}$. Also we show that this class is a quasi-Banach operator ideal and we study on the properties of the classes which are produced via different types of s-numbers.

Keywords: operator ideals, s-numbers, block sequence spaces.

1. INTRODUCTION

Operator ideal theory is an important subject of functional analysis. There are many different ways of constructing operator ideals, one of them is using s-numbers. Some equivalents of s-numbers are Kolmogorov numbers, Weyl numbers and approximation numbers. Pietsch defined in [1] the concept of s-number sequence to combine all s-numbers in one definition. After some revisions on this definition s-number sequence is presented in [2], [3].

In this paper, by $\mathbb{N}$ and $\mathbb{R}^+$ we denote the set of all natural numbers and nonnegative real numbers, respectively.

A finite rank operator is a bounded linear operator whose dimension of the range space is finite [4].

Let X and Y be real or complex Banach spaces. The space of all bounded linear operators from X to Y and the space of all bounded linear operators between any two arbitrary Banach spaces are denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}$, respectively.

An s-number sequence is a map $s = (s_n): \mathcal{L} \to \mathbb{R}^+$ which assigns every operator $T \in \mathcal{L}$ to a non-negative scalar sequence $(s_n(T)_{n\in\mathbb{N}})$ if the following conditions hold for all Banach spaces X, Y, X_0 and Y_0:

$(S1)$ $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for every $T \in \mathcal{L}(X, Y)$,

[*] Corresponding Author: pinarzenginalp@gmail.com
[1] Düzce University, Department of Mathematics, Düzce, Turkey. ORCID: 0000-0001-9699-7199
[2] Düzce University, Department of Mathematics, Düzce, Turkey. ORCID: 0000-0002-0831-1474

$(S2)$ $s_{m+n-1}(S+T) \leq s_m(T) + s_n(T)$ for every $S,T \in \mathcal{L}(X,Y)$ and $m,n \in \mathbb{N}$,

$(S3)$ $s_n(RST) \leq \|R\| s_n(S) \|T\|$ for some $R \in \mathcal{L}(Y,Y_0)$, $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X_0,X)$, where X_0, Y_0 are arbitrary Banach spaces,

$(S4)$ If $rank(T) \leq n$, then $s_n(T) = 0$,

$(S5)$ $s_n(I: l_2^n \to l_2^n) = 1$, where I denotes the identity operator on the n-dimensional Hilbert space l_2^n, where $s_n(T)$ denotes the n-th s-number of the operator T [5].

As an example of s-numbers $a_n(T)$, the n-th approximation number, is defined as

$$a_n(T) = inf\{\|T - A\| : A \in \mathcal{L}(X,Y), \text{rank(A)} < \text{n}\},$$

where $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$ [6].

Let $T \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$. The other examples of s-number sequences are given in the following, namely Gelfand number $(c_n(T))$, Kolmogorov number $(d_n(T))$, Weyl number $(x_n(T))$, Chang number $(y_n(T))$, Hilbert number $(h_n(T))$, etc. For the definitions of these sequences we refer to [4], [7]. In the sequel there are some properties of s-number sequences.

When any metric injection $J \in \mathcal{L}(Y,Y_0)$ is given and an s-number sequence $s = (s_n)$ satisfies $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(X,Y)$ the s-number sequence is called injective [3].

Proposition 1. The number sequences $(c_n(T))$ and $(x_n(T))$ are injective [3].

When any metric surjection $S \in \mathcal{L}(X_0,X)$ is given and an s-number sequence $s = (s_n)$ satisfies $s_n(T) = s_n(TS)$ for all $T \in \mathcal{L}(X,Y)$ the s-number sequence is called surjective [3].

Proposition 2. The number sequences $(d_n(T))$ and $(y_n(T))$ are surjective [3].

Proposition 3. Let $T \in \mathcal{L}(X,Y)$. Then $h_n(T) \leq x_n(T) \leq c_n(T) \leq a_n(T)$ and $h_n(T) \leq y_n(T) \leq d_n(T) \leq a_n(T)$ [3].

Lemma 1. Let $S,T \in \mathcal{L}(X,Y)$, then $|s_n(T) - s_n(S)| \leq \|T - S\|$ for $n = 1,2,\cdots$ [1].

A sequence space is defines as any vector subspace of ω, where ω is the space of real valued sequences.

The Cesaro sequence space ces_p is defined as

$$ces_p = \left\{x = (x_k) \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\}$$

where $1 < p < \infty$ [8], [9], [10].

If an operator $T \in \mathcal{L}(X,Y)$ satisfies $\sum_{n=1}^{\infty}(a_n(T))^p < \infty$ for $0 < p < \infty$, it is defined as an l_p type operator in [6] by Pietsch. Then Constantin defined a new class named ces-p type operators, via Cesaro sequence spaces. If an operator $T \in \mathcal{L}(X,Y)$ satisfies $\sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_n(T)\right)^p < \infty$, $1 < p < \infty$, it is called ces-p type operator. The class of ces-p type operators includes the class of l_p type operators [11]. Later on Tita [12] proved that the class of l_p type operators and ces-p type operators are coincides. Some other generalizations of l_p type operators were examined in [4], [13],[14], [15].

Continuous linear functionals on X are compose the dual of X which is denoted by X'. Let $x' \in X'$ and $y \in Y$, then the map $x' \otimes y : X \to Y$ is defined by

$$(x' \otimes y)(x) = x'(x)y, x \in X.$$

A subcollection $\mathfrak{I}$ of $\mathcal{L}$ is called an operator ideal if every component $\mathfrak{I}(X,Y) = \mathfrak{I} \cap \mathcal{L}(X,Y)$ satisfies the following conditions:

i) if $x' \in X'$, $y \in Y$, then $x' \otimes y \in \mathfrak{I}(X,Y)$,

ii) if $S,T \in \mathfrak{I}(X,Y)$, then $S + T \in \mathfrak{I}(X,Y)$,

iii) if $S \in \mathfrak{I}(X,Y), T \in \mathcal{L}(X_0,X)$ and $R \in \mathcal{L}(Y,Y_0)$, then $RST \in \mathfrak{I}(X_0,Y_0)$ [2].

Let $\mathfrak{I}$ be an operator ideal and $\alpha : \mathfrak{I} \to \mathbb{R}^+$ be a function on $\mathfrak{I}$. Then, if the following conditions satisfied:

i) If $x' \in X'$, $y \in Y$, then $\alpha(x' \otimes y) = \|x'\|\|y\|$,

ii) there exists a constant $c \geq 1$ such that $\propto (S+T) \leq c[\propto (S) + \propto (T)]$,

iii) if $S \in \Im(X,Y), T \in \mathcal{L}(X_0, X)$ and $R \in \mathcal{L}(Y, Y_0)$, then $\propto (RST) \leq \|R\| \propto (S) \|T\|$

α is called a quasi-norm on the operator ideal $\Im$ [2].

For special case $c = 1$, $\propto$ is a norm on the operator ideal $\Im$.

If $\propto$ is a quasi-norm on an operator ideal $\Im$, it is denoted by $[\Im, \alpha]$. Also if every component $\Im(X,Y)$ is complete with respect to the quasinorm α, $[\Im, \alpha]$ is called a quasi-Banach operator ideal.

Let $[\Im, \alpha]$ be a quasi-normed operator ideal and $J \in \mathcal{L}(Y, Y_0)$ be a metric injection. If for every operator $T \in \mathcal{L}(X,Y)$ and $JT \in \Im(X, Y_0)$ we have $T \in \Im(X,Y)$ and $\alpha(JT) = \alpha(T)$, $[\Im, \alpha]$ is called an injective quasi-normed operator ideal. Furthermore, let $[\Im, \alpha]$ be a quasi-normed operator ideal and $Q \in \mathcal{L}(X_0, X)$ be a metric surjection. If for every operator $T \in \mathcal{L}(X,Y)$ and $TQ \in \Im(X, Y_0)$ we have $T \in \Im(X,Y)$ and $\alpha(TQ) = \alpha(T)$, $[\Im, \alpha]$ is called an surjective quasi-normed operator ideal [2].

Let T' be the dual of T. An s-number sequence is called symmetric if $s_n(T) \geq s_n(T')$ for all $T \in \mathcal{L}$. If $s_n(T) = s_n(T')$ the s-number sequence is said to be completely symmetric [2].

For every operator ideal $\Im$, the dual operator ideal denoted by $\Im'$ is defined as

$$\Im'(X,Y) = \{T \in \mathcal{L}(X,Y) : T' \in \Im(Y', X')\},$$

where T' is the dual of T and X' and Y' are the duals of X and Y, respectively.

An operator ideal $\Im$ is called symmetric if $\Im \subset \Im'$ and is called completely symmetric if $\Im = \Im'$ [2].

Let $E = (E_n)$ be a partition of finite subsets of positive integers such that

$$maxE_n < maxE_{n+1}$$

for $n = 1,2,\cdots$. Foroutannia, in [16] defined the sequence space $l_p(E)$ as

$$l_p(E) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\},$$

where $(1 \leq p < \infty)$ with the seminorm $|||x|||_{p,E}$ which defined in the following way:

$$|||x|||_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{\frac{1}{p}}$$

For example if $E_n = \{2n-1, 2n\}$ for all n, then $x = (x_n) \in l_p(E)$ if and only if $\sum_{n=1}^{\infty} |x_{2n-1} + x_{2n}|^p < \infty$. It is obvious that $|||\cdot|||_{p,E}$ cannot be a norm, since if $x = (1, -1, 0, 0, \cdots)$ and $E_n = \{2n-1, 2n\}$ for all n then $|||x|||_{p,E} = 0$ while $x \neq \theta$. In the special case $E_n = \{n\}$ for $n = 1,2,\cdots$, we have $l_p(E) = l_p$ and $|||x|||_{p,E} = \|x\|_p$.

For more information about block sequence spaces we refer to [17], [18].

2. MAIN RESULTS

Let $u = (u_n)$ and $v = (v_n)$ be positive real number sequences. In this section we give the definition of the sequence space $X(u, v; l_p(E))$ as follows:

$$X\left(u, v; l_p(E)\right) = \left\{ x \epsilon \omega : \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j x_j(T) \right)^p < \infty \right\}$$

An operator $T \in \mathcal{L}(X,Y)$ is in the class of $L_{u,v;E}(X,Y)$ if

$$\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(T) \right)^p < \infty, \quad (1 \leq p < \infty)$$

The class of all s-type $X(u, v; l_p(E))$ operators are denoted by $L_{u,v;E}$.

Theorem 1. The class $L_{u,v;E}$ is an operator ideal for $1 \leq p < \infty$ where $v_{2k-1} + v_{2k} \leq M v_k$, $(M > 0)$ and $\sum_{n=1}^{\infty} (u_n)^p < \infty$.

Proof.

$$\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(x' \otimes y) \right)^p = (u_1 v_1 s_1(x' \otimes y))^p$$
$$= u_1^p v_1^p \|x' \otimes y\|^p$$
$$= u_1^p v_1^p \|x'\|^p \|y\|^p < \infty$$

Since the rank of the operator $x' \otimes y$ is one, $s_n(x' \otimes y) = 0$ for $n \geq 2$. Therefore $x' \otimes y \in L_{u,v;E}$.

Let $S, T \in L_{u,v;E}$. Then

$$\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(S) \right)^p < \infty, \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(T) \right)^p < \infty$$

To show that $S + T \in L_{u,v;E}(X, Y)$, we begin with

$$\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(S + T) \right)^p$$
$$\leq \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_{2j-1} s_{2j-1}(S + T) + u_n \sum_{j \in E_n} v_{2j} s_{2j}(S + T) \right)^p$$
$$\leq \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} (v_{2j-1} + v_{2j}) s_{2j-1}(S + T) \right)^p$$
$$\leq \sum_{n=1}^{\infty} \left(M u_n \sum_{j \in E_n} v_j \big(s_j(S) + s_j(T) \big) \right)^p$$

By using Minkowski inequality;

$$\left(\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(S + T) \right)^p \right)^{\frac{1}{p}}$$
$$\leq M \left(\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(S) \right)^p \right)^{\frac{1}{p}} + M \left(\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(T) \right)^p \right)^{\frac{1}{p}} < \infty$$

Hence $S + T \in L_{u,v;E}(X, Y)$.

Let $R \in \mathcal{L}(Y, Y_0)$, $S \in L_{u,v;E}(X, Y)$ and $T \in \mathcal{L}(X_0, X)$

$$\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(RST) \right)^p \leq \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} \|R\| \|T\| v_j s_j(S) \right)^p$$
$$\leq \|R\|^p \|T\|^p \sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(S) \right)^p < \infty$$

So $RST \in L_{u,v;E}(X_0, Y_0)$.

Therefore $L_{u,v;E}$ is an operator ideal.

Theorem 2. $\|T\|_{u,v;E} = \frac{\left(\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(T) \right)^p \right)^{\frac{1}{p}}}{u_1 v_1}$ is a quasi-norm on the operator ideal $L_{u,v;E}$.

Proof.

$$\frac{\left(\sum_{n=1}^{\infty} \left(u_n \sum_{j \in E_n} v_j s_j(x' \otimes y) \right)^p \right)^{\frac{1}{p}}}{u_1 v_1}$$
$$= \frac{\left(u_1^p v_1^p \|x' \otimes y\|^p \right)^{\frac{1}{p}}}{u_1 v_1} = \|x' \otimes y\|$$
$$= \|x'\| \|y\|.$$

Since rank of the operator $x' \otimes y$ is one, $s_n(x' \otimes y) = 0$ for $n \geq 2$. Therefore $\|x' \otimes y\|_{u,v;E} = \|x'\| \|y\|$.

Let $S, T \in L_{u,v;E}$. Then

$$\sum_{j \in E_n} v_j s_j(S + T) \leq \sum_{j \in E_n} v_{2j-1} s_{2j-1}(S + T) + \sum_{j \in E_n} v_{2j} s_{2j}(S + T)$$
$$\leq \sum_{j \in E_n} (v_{2j-1} + v_{2j}) s_{2j-1}(S + T)$$
$$\leq M \sum_{j \in E_n} v_j \big(s_j(S) + s_j(T) \big)$$

By using Minkowski inequality;

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(S+T)\right)^p\right)^{\frac{1}{p}} \leq M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(S)\right)^p\right)^{\frac{1}{p}} + M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(T)\right)^p\right)^{\frac{1}{p}} < \infty$$

Hence $\|S+T\|_{u,v;E} \leq M\big(\|S\|_{u,v;E} + \|T\|_{u,v;E}\big)$.

Let $R \in \mathcal{L}(Y, Y_0)$, $S \in L_{u,v;E}(X,Y)$ and $T \in \mathcal{L}(X_0, X)$

$$\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(RST)\right)^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} \|R\|\|T\| v_j s_j(S)\right)^p\right)^{\frac{1}{p}} \leq \|R\|\|T\|\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(S)\right)^p\right)^{\frac{1}{p}} < \infty$$

Hence

$$\|RST\|_{u,v;E} \leq \|R\|\|T\|\|S\|_{u,v;E}.$$

Therefore $\|T\|_{u,v;E}$ is a quasi-norm on $L_{u,v;E}$.

Theorem 3. Let $1 \leq p < \infty$. $\left[L_{u,v;E}, \|T\|_{u,v;E}\right]$ is a quasi-Banach operator ideal.

Proof: Let X, Y be any two Banach spaces and $1 \leq p < \infty$. The following inequality holds

$$\|T\|_{u,v;E} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(T)\right)^p\right)^{\frac{1}{p}}}{u_1 v_1} \geq \|T\|$$

for $T \in L_{u,v;E}$.

Let (T_m) be a Cauchy in $L_{u,v;E}(X,Y)$. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|T_m - T_l\|_{u,v;E} < \varepsilon \qquad (2.1)$$

For all $m, l \geq n_0$. It follows that

$$\|T_m - T_l\| \leq \|T_m - T_l\|_{u,v;E} < \varepsilon.$$

Then (T_m) is a Cauchy sequence in $\mathcal{L}(X,Y)$. $\mathcal{L}(X,Y)$ is a Banach space since Y is a Banach space. Therefore $\|T_m - T\| \to 0$ as $m \to \infty$ for $T \in \mathcal{L}(X,Y)$. Now we show that $\|T_m - T\|_{u,v;E} \to 0$ as $m \to \infty$ for $T \in L_{u,v;E}(X,Y)$.

The operators $T_l - T_m, T - T_m$ are in the class $\mathcal{L}(X,Y)$ for $T_m, T_l, T \in \mathcal{L}(X,Y)$.

$$|s_n(T_l - T_m) - s_n(T - T_m)| \leq \|T_l - T_m - (T - T_m)\| = \|T_l - T\|$$

Since $T_l \to T$ as $l \to \infty$ that is $\|T_l - T\| < \varepsilon$ we obtain

$$s_n(T_l - T_m) \to s_n(T - T_m) \text{ as } l \to \infty. \qquad (2.2)$$

It follows from (2.1) that the statement

$$\|T_m - T_l\|_{u,v;E} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(T_m - T_l)\right)^p\right)^{\frac{1}{p}}}{u_1 v_1} < \varepsilon$$

holds for all $m, l \geq n_0$. We obtain from (2.2) that

$$\frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n} v_j s_j(T_m - T)\right)^p\right)^{\frac{1}{p}}}{u_1 v_1} < \varepsilon \text{ as } l \to \infty.$$

Hence we have $\|T_m - T\|_{u,v;E} < \varepsilon$ for all $m \geq n_0$.

Finally we show that $T \in L_{u,v;E}(X,Y)$,

$$\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(T)\right)^p$$
$$\leq \sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_{2j-1}s_{2j-1}(T) + u_n\sum_{j\in E_n}v_{2j}s_{2j}(T)\right)^p$$
$$\leq \sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}(v_{2j-1} + v_{2j})s_{2j-1}(T-T_m+T_m)\right)^p$$
$$\leq M\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\big(s_j(T-T_m)+s_j(T_m)\big)\right)^p$$

By using Minkowski inequality; since $T_m \in L_{u,v;E}(X,Y)$ for all m and $\|T_m - T\|_{u,v;E} \to 0$ as $m \to \infty$, we have

$$M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\big(s_j(T-T_m)+s_j(T_m)\big)\right)^p\right)^{\frac{1}{p}}$$
$$\leq M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\left(s_j(T-T_m)\right)\right)^p\right)^{\frac{1}{p}} + M\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\big(s_j(T_m)\big)\right)^p\right)^{\frac{1}{p}} < \infty$$

which means that $\in L_{u,v;E}(X,Y)$.

Proposition 1. The inclusion $L^p_{u,v;E} \subseteq L^q_{u,v,E}$ holds for $1 < p \leq q < \infty$.

Proof: Since $l_p \subseteq l_q$ for $1 < p \leq q < \infty$ we have $L^p_{u,v;E} \subseteq L^q_{u,v,E}$.

Let $\mu = \big(\mu_n(T)\big)$ be one of the sequences $a = \big(a_n(T)\big)$, $c = \big(c_n(T)\big)$, $d = \big(d_n(T)\big)$, $x = \big(x_n(T)\big)$, $y = \big(y_n(T)\big)$ and $h = \big(h_n(T)\big)$. Then we define the space $L^{(\mu)}_{u,v;E}$ and the norm $\|T\|^{(\mu)}_{u,v;E}$ as follows:

$$L^{(\mu)}_{u,v;E}(X,Y) = \left\{T \in \mathcal{L}(X,Y): \sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\big(\mu_j(T)\big)\right)^p < \infty\right\},\ (1<p<\infty)$$

and

$$\|T\|^{(\mu)}_{u,v,E} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_j\mu_j(T)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1}.$$

Theorem 4. Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{u,v,E}, \|T\|^{(s)}_{u,v,E}\right]$ is injective, if s-number sequence is injective.

Proof. Let $1 < p < \infty$ and $T \in \mathcal{L}(X,Y)$ and $I \in \mathcal{L}(Y,Y_0)$ be any metric injection. Suppose that $IT \in L^{(s)}_{u,v,E}(X,Y_0)$. Then

$$\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(IT)\right)^p < \infty$$

Since $s = (s_n)$ is injective, we have

$$s_n(T) = s_n(IT) \text{ for all } T \in \mathcal{L}(X,Y), n = 1,2,\dots. \tag{2.3}$$

Hence we get

$$\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(T)\right)^p = \sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(IT)\right)^p < \infty$$

Thus $T \in L^{(s)}_{u,v,E}(X,Y)$ and we have from (2.3)

$$\|IT\|^{(s)}_{u,v,E} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(IT)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1}$$
$$= \frac{\left(\sum_{n=1}^{\infty}\left(u_n\sum_{j\in E_n}v_js_j(T)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}}v_1} = \|T\|^{(s)}_{u,v,E}$$

Hence the operator ideal $\left[L^{(s)}_{u,v,E}, \|T\|^{(s)}_{u,v,E}\right]$ is injective.

Corollary 1. Since the number sequences $(c_n(T))$ and $(x_n(T))$ are injective, the quasi-

Banach operator ideals $\left[L_{u,v,E}^{(c)}, \|T\|_{u,v,E}^{(c)}\right]$ and $\left[L_{u,v,E}^{(x)}, \|T\|_{u,v,E}^{(x)}\right]$ are injective [3].

Theorem 5. Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L_{u,v,E}^{(s)}, \|T\|_{u,v,E}^{(s)}\right]$ is surjective, if s-number sequence is surjective.

Proof. Let $1 < p < \infty$ and $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(X_0, X)$ be any metric injection. Suppose that $TS \in L_{u,v,E}^{(s)}(X_0, Y)$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(TS)\right)^p < \infty.$$

Since $s = (s_n)$ is surjective, we have

$$s_n(T) = s_n(TS) \text{ for all } T \in \mathcal{L}(X,Y), n = 1,2,\dots \quad (2.4)$$

Hence we get

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(T)\right)^p = \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(TS)\right)^p < \infty.$$

Thus $T \in L_{u,v,E}^{(s)}(X,Y)$ and we have from (2.4)

$$\|TS\|_{u,v,E}^{(s)} = \frac{\left(\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(TS)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1}$$

$$= \frac{\left(\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(T)\right)^p\right)^{\frac{1}{p}}}{\left(\sum_{n=1}^{\infty}(u_n)^p\right)^{\frac{1}{p}} v_1} = \|T\|_{u,v,E}^{(s)}.$$

Hence the operator ideal $\left[L_{u,v,E}^{(s)}, \|T\|_{u,v,E}^{(s)}\right]$ is surjective.

Corollary 2. Since the number sequences $(d_n(T))$ and $(y_n(T))$ are surjective , the quasi-Banach operator ideals $\left[L_{u,v,E}^{(d)}, \|T\|_{u,v,E}^{(d)}\right]$ and $\left[L_{u,v,E}^{(y)}, \|T\|_{u,v,E}^{(y)}\right]$ are surjective [3].

Theorem 6. Let $1 < p < \infty$. Then the following inclusion relations hold:

i. $L_{u,v,E}^{(a)} \subseteq L_{u,v,E}^{(c)} \subseteq L_{u,v,E}^{(x)} \subseteq L_{u,v,E}^{(h)}$

ii. $L_{u,v,E}^{(a)} \subseteq L_{u,v,E}^{(d)} \subseteq L_{u,v,E}^{(y)} \subseteq L_{u,v,E}^{(h)}$.

Proof. Let $T \in L_{u,v,E}^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j s_j(T)\right)^p < \infty$$

where $1 < p < \infty$. And from Proposition 3, we have;

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j h_j(T)\right)^p \le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j x_j(T)\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j c_j(T)\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j a_j(T)\right)^p$$

$$< \infty$$

and

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j h_j(T)\right)^p \le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j y_j(T)\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j d_j(T)\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j a_j(T)\right)^p$$

$$< \infty.$$

So it is shown that the inclusion relations are satisfied.

Theorem 7. The operator ideal $L_{u,v,E}^{(a)}$ is symmetric and the operator ideal $L_{u,v,E}^{(h)}$ is completely symmetric for $1 < p < \infty$.

Proof. Let $1 < p < \infty$.

Firstly, we prove that the inclusion $L_{u,v,E}^{(a)} \subseteq \left(L_{u,v,E}^{(a)}\right)'$ holds. Let $T \in L_{u,v,E}^{(a)}$. Then

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j a_j(T)\right)^p < \infty.$$

It follows from [2] $a_n(T') \le a_n(T)$ for $T \in \mathcal{L}$. Hence we get

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j a_j(T')\right)^p \le \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j a_j(T)\right)^p < \infty.$$

Therefore $T \in \left(L_{u,v,E}^{(a)}\right)'$. Thus $L_{u,v,E}^{(a)}$ is symmetric.

Now we prove that the equation $L_{u,v,E}^{(h)} = \left(L_{u,v,E}^{(h)}\right)'$ holds. It follows from [3] that $h_n(T') = h_n(T)$ for $T \in \mathcal{L}$. Then we can write

$$\sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j h_j(T')\right)^p = \sum_{n=1}^{\infty}\left(u_n \sum_{j\in E_n} v_j h_j(T)\right)^p.$$

Hence $L_{u,v,E}^{(h)}$ is completely symmetric.

Theorem 8 Let $1 < p < \infty$. The equation $L_{u,v,E}^{(c)} = \left(L_{u,v,E}^{(d)}\right)'$ and the inclusion relation $L_{u,v,E}^{(d)} \subseteq \left(L_{u,v,E}^{(c)}\right)'$ holds. Also, the equation $L_{u,v,E}^{(d)} = \left(L_{u,v,E}^{(c)}\right)'$ holds for any compact operators.

Proof. Let $1 < p < \infty$. For $T \in \mathcal{L}$ we have from [3] that $c_n(T) = d_n(T')$ and $c_n(T') \le d_n(T)$. Also, if T is a compact operator, then the equality $c_n(T') = d_n(T)$ holds. Thus the proof is clear.

Theorem 9 $L_{u,v,E}^{(x)} = \left(L_{u,v,E}^{(y)}\right)'$ and $L_{u,v,E}^{(y)} = \left(L_{u,v,E}^{(x)}\right)'$ hold.

Proof. Let $1 < p < \infty$. For $T \in \mathcal{L}$ we have from [3] that $x_n(T) = y_n(T')$ and $y_n(T) = x_n(T')$. Thus the proof is clear.

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