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*Research Article*

# $q$-Hermite–Hadamard Inequalities for Generalized Exponentially $(s, m; \eta)$-Preinvex Functions

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In this article, we introduce a new extension of classical convexity which is called generalized exponentially $(s, m; \eta)$-preinvex functions. Also, it is seen that the new definition of generalized exponentially $(s, m; \eta)$-preinvex functions describes different new classes as special cases. To prove our main results, we derive a new ${}^{m\kappa_2}q$-integral identity for the twice ${}^{m\kappa_2}q$-differentiable function. By using this identity, we show essential new results for Hermite–Hadamard-type inequalities for the ${}^{m\kappa_2}q$-integral by utilizing differentiable exponentially $(s, m; \eta)$-preinvex functions. The results presented in this article are unification and generalization of the comparable results in the literature.

## 1. Introduction and Preliminaries

In mathematics, quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus," and literally, it means how much, in Swedish "Kvant"). It has two major branches: $q$-calculus and $h$-calculus. And both of them were worked out by Cheung and Kac [1] in the early twentieth century. In the same era, Jackson started to work on quantum calculus or $q$-calculus, but Euler and Jacobi had already figured out this type of calculus. A number of studies have recently been widely used in the field of $q$-analysis, beginning with Euler, due to the vast necessity for mathematics that models of quantum computing $q$-calculus exist in the framework between physics and mathematics. In 2013, Tariboon and Ntouyas introduced the ${}_{\kappa_1}Dq$-difference operator [2, 3]. This inspired other researchers, and as a consequence, numerous novel results concerning quantum analogues of classical mathematical results have already been launched in the literature. In various mathematical fields, it has many applications, such as theory of numbers, combinations, orthogonal polynomials, basic hypergeometric functions and other subjects, quantum mechanics, physics, and the principle of relativity. Many important aspects of quantum calculus are covered in the articles by Humaira et al. [4–7]. The quantum calculus is currently a subfield of the more general scientific field of time-scale calculus. New developments have recently been made in the research and methodology of dynamic derivatives on time scales. The research offers a consolidation and application of traditional differential and difference equations. Moreover, it is a unification of the discrete theory with the continuous theory, from the theoretical perspective. Recently, in 2020, Bermudo et al. introduced the notion of the ${}^{\kappa_2}Dq$-derivative and integral [8]. For more details, see [9–15] and references cited therein.

The discussion and application of convex functions has become a very rich source of motivational material in pure and applied science. This vision not only promoted new and profound results in many branches of mathematical and engineering sciences but also provided a comprehensive framework for the study of many problems. Many scholars have studied various classes of convex sets and convex functions; see [16, 17]. The concept of convexity has been extended in several directions, since these generalized versions have significant applications in different fields of pure and applied sciences. One of the convincing examples on extensions of convexity is the introduction of invex function, which was introduced by Hanson [18] Weir and Mond [19] explored the idea of preinvex functions and actualized it to the foundation of adequate optimality conditions and duality in nonlinear programming.

The Hermite–Hadamard inequality was introduced by Hermite and Hadamard; see [20]. It is one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows.

Let $\digamma: \mathcal{Q}\subseteq\mathcal{R} \longrightarrow \mathcal{R}$ be a convex mapping and $\kappa_1, \kappa_2 \in \mathcal{Q}$ with $\kappa_1 < \kappa_2$. Then,

$$\digamma\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)\mathrm{d}\varkappa \leq \frac{\digamma(\kappa_1)+\digamma(\kappa_2)}{2}. \quad (1)$$

If $\digamma$ is concave, both inequalities hold in the reverse direction.

The important objective of this paper is to introduce an exponentially generalized definition of $(s, m; \eta)$-preinvex functions. Furthermore, the new ${}^{m\kappa 2}q$-integral identity is determined. By using this new identity, we proved many new estimates of bounds for it, essentially based on the concept of quantum calculus.

## 2. Preliminaries

In this section, we derive a new definition of the generalized exponentially $(s, m; \eta)$-preinvex function. Also, we present all necessary concepts related to quantum calculus.

First of all, let $\mathcal{Q} \subset \mathcal{R}^n$ be a nonempty set, $\digamma: \mathcal{Q} \longrightarrow \mathcal{R}$ be a continuous function, and $\eta: \mathcal{Q}\times\mathcal{Q}\times(0,1] \longrightarrow \mathcal{R}\setminus\{0\}$ and $\vartheta: \mathcal{Q}\times\mathcal{Q}\times(0,1] \longrightarrow \mathcal{R}^n$ be two continuous functions.

*Definition 1.* A set $\mathcal{Q}\subseteq\mathcal{R}^n$ is supposed to be $\eta$-invex concerning $\eta(\cdot,\cdot,\cdot)$ and $\vartheta(\cdot,\cdot,\cdot)$ with some fixed $m \in (0,1]$ if

$$m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m) \in \mathcal{Q}, \quad (2)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$.

If $\eta(\kappa_1,\kappa_2,m) = 1$, the above equation is called the convex set, and $\vartheta(\kappa_1,\kappa_2,m) = \kappa_1 - m\kappa_2$ is an invex set; however, the reverse is not possible.

*Example 1.* Consider $\mathcal{Q} = [-3,-2] \cup [-1,2]$ and

$$\vartheta(\kappa_1,\kappa_2,m) = \begin{cases} \kappa_1 - m\kappa_2 & \text{if } 2 \geq \kappa_2 \geq -1, 2 \geq \kappa_1 \geq -1, \\ \kappa_1 - m\kappa_2 & \text{if } -3 \leq \kappa_2 \leq -2, -3 \leq \kappa_1 \leq -2, \\ -1 - m\kappa_2 & \text{if } -3 \leq \kappa_2 \leq -2, -1 \leq \kappa_1 \leq 2, \\ -3 - m\kappa_2 & \text{if } -1 \leq \kappa_2 \leq 2, -3 \leq \kappa_1 \leq -2. \end{cases} \quad (3)$$

As one can see, $\mathcal{Q}$ is also an invex set for $\vartheta$, but not a convex set.

*Definition 2.* A function $\digamma: \mathcal{Q} \longrightarrow \mathcal{R}$ is said to be a generalized exponentially $(s, m; \eta)$-preinvex function if there exist $\eta(\cdot,\cdot,\cdot)$ and $\vartheta(\cdot,\cdot,\cdot)$, $\chi \geq 1$, and nonpositive $\alpha$ such that

$$\begin{aligned}&\digamma(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) \\ &\quad \leq k^s\frac{\digamma(\kappa_1)}{\chi^{\alpha\kappa_1}} + (1-k)^s\frac{\digamma(m\kappa_2)}{\chi^{\alpha m\kappa_2}},\end{aligned} \quad (4)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$ and for some fixed $s, m \in (0,1]$.

*Remark 1.* In Definition 2,

(1) If we choose $\alpha = 0$ or $\chi = 1$, then the definition of the generalized exponentially $(s, m; \eta)$-preinvex function is converted into the definition of the generalized $(s, m; \eta)$-preinvex function

(2) If we choose $\alpha = 0$ and $\eta(\kappa_1,\kappa_2,m) = 1$, then we get the definition of $(s, m)$-preinvexity

(3) If we choose $\alpha = 0$, $\eta(\kappa_1,\kappa_2,m) = 1$, and $\vartheta(\kappa_1,\kappa_2,m) = \kappa_1 - m\kappa_2$, then we get the definition of $(s, m)$-convexity

(4) If we choose $m = 1$ and $\vartheta(\kappa_1,\kappa_2,m) = 1$, we get the definition in [21]

(4) If we choose $\chi = e$, then we have the definition of exponentially $(s, m; \eta)$-preinvex functions, stated as follows

*Definition 3.* A function $\digamma: \mathcal{Q} \longrightarrow \mathcal{R}$ is called exponentially $(s, m; \eta)$-preinvex if there exist $\eta(\cdot,\cdot,\cdot)$, $\vartheta(\cdot,\cdot,\cdot)$, and nonpositive $\alpha$ such that

$$\begin{aligned}&\digamma(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)) \\ &\quad \leq k^s\frac{\digamma(\kappa_1)}{e^{\alpha\kappa_1}} + (1-k)^s\frac{\digamma(m\kappa_2)}{e^{\alpha m\kappa_2}},\end{aligned} \quad (5)$$

for all $\kappa_1, \kappa_2 \in \mathcal{Q}$ and $k \in [0,1]$ and for some fixed $s, m \in (0,1]$.

Many researchers proved several results about the importance and development in the theory of exponentially convex functions and their applications. For more details, see [22–25] and references cited therein.

Jackson derived the $q$-Jackson integral in [12] from 0 to $\kappa_2$ for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} \digamma(\varkappa)\,\mathrm{d}_q\varkappa = (1-q)\kappa_2\sum_{n=0}^{\infty} q^n \digamma(\kappa_2 q^n), \quad (6)$$

provided the sum converges absolutely.

The $q$-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ was given by Jackson in [12] and defined as follows:

$$\int_{\kappa_1}^{\kappa_2} \digamma(\varkappa)\,d_q\varkappa = \int_0^{\kappa_2} \digamma(\varkappa)\,d_q\varkappa - \int_0^{\kappa_1} \digamma(\varkappa)\,d_q\varkappa. \quad (7)$$

*Definition 4* (see [3]). We suppose that $\digamma: [\kappa_1, \kappa_2] \longrightarrow \mathcal{R}$ is an arbitrary function. Then, the $q_{\kappa_1}$-derivative of $\digamma$ at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$$_{\kappa_1}D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa + (1-q)\kappa_1)}{(1-q)(\varkappa - \kappa_1)}, \quad \varkappa \neq \kappa_1. \tag{8}$$

Since $F$ is an arbitrary function from $[\kappa_1, \kappa_2]$ to $\mathscr{R}$, $_{\kappa_1}D_q F(\kappa_1) = \lim_{\varkappa \longrightarrow \kappa_1}\, _{\kappa_1}D_q F(\varkappa)$. The function $F$ is said to be $q$-differentiable on $[\kappa_1, \kappa_2]$ if $_{\kappa_1}D_q F(t)$ exists for all $\varkappa \in [\kappa_1, \kappa_2]$. If $\kappa_1 = 0$ in (3), then $_0D_q F(\varkappa) = D_q F(\varkappa)$, where $D_q F(\varkappa)$ is a familiar $q$-derivative of $F$ at $\varkappa \in [\kappa_1, \kappa_2]$ defined by the following expression (see [1]):

$$D_q F(\varkappa) = \frac{F(\varkappa) - F(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0. \tag{9}$$

*Definition 5* (see [8]). We suppose that $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ is an arbitrary function; then, the $q^{\kappa_2}$-derivative of $F$ at $\varkappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$$^{\kappa_2}D_q F(\varkappa) = \frac{F(q\varkappa + (1-q)\kappa_2) - F(\varkappa)}{(1-q)(\kappa_2 - \varkappa)}, \quad \varkappa \neq \kappa_2. \tag{10}$$

*Definition 6* (see [3]). We suppose that $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ is an arbitrary function; then, the $q_{\kappa_1}$-definite integral on $[\kappa_1, \kappa_2]$ is defined as follows:

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa &= (1-q)(\kappa_2 - \kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_2 + (1-q^n)\kappa_1) \\ &= (\kappa_2 - \kappa_1)\int_0^1 F((1-t)\kappa_1 + t\kappa_2) d_q t.\end{aligned} \tag{11}$$

In [10], Alp et al. established the $q_{\kappa_1}$-Hermite–Hadamard inequalities for convexity, which are defined as follows.

**Theorem 1.** *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$F\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa \leq \frac{qF(\kappa_1) + F(\kappa_2)}{[2]_q}. \tag{12}$$

On the contrary, the following new description and related Hermite–Hadamard-form inequalities were given by Bermudo et al.

*Definition 7* (see [8]). Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be an arbitrary function. Then, the $q^{\kappa_2}$-definite integral on $[\kappa_1, \kappa_2]$ is defined as

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa &= (1-q)(\kappa_2 - \kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_1 + (1-q^n)\kappa_2) \\ &= (\kappa_2 - \kappa_1)\int_0^1 F(t\kappa_1 + (1-t)\kappa_2) d_q t.\end{aligned} \tag{13}$$

**Theorem 2** (see [8]). *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite–Hadamard inequalities are as follows:*

$$F\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa \leq \frac{F(\kappa_1) + qF(\kappa_2)}{[2]_q}. \tag{14}$$

From Theorems 1 and 2, one can achieve the following inequalities.

**Corollary 1** (see [8]). *For any convex function $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ and $0 < q < 1$, we have*

$$F\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) + F\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\left\{\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa\right\} \leq F(\kappa_1) + F(\kappa_2), \tag{15}$$

*and*

$$\begin{aligned}F\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{2(\kappa_2 - \kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} F(\varkappa)_{\kappa_1} d_q\varkappa + \int_{\kappa_1}^{\kappa_2} F(\varkappa)^{\kappa_2} d_q\varkappa\right\} \\ &\leq \frac{F(\kappa_1) + F(\kappa_2)}{2}.\end{aligned} \tag{16}$$

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which we will call as $_{\kappa_1}T_q$-integral.

*Definition 8* (see [11]). Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be an arbitrary function. For $\varkappa \in [\kappa_1, \kappa_2]$,

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\xi)_{\kappa_1} d_q^T &= \frac{(1-q)(\kappa_2 - \kappa_1)}{2q} \\ &\cdot \left[[2]_q \sum_{n=0}^{\infty} q^n F(q^n\kappa_2 + (1-q^n)\kappa_1) - F(\kappa_2)\right],\end{aligned} \tag{17}$$

where $0 < q < 1$.

**Theorem 3** ($\overline{q}$-Hermite–Hadamard; see [11]). *Let $F\colon [\kappa_1, \kappa_2] \longrightarrow \mathscr{R}$ be a convex continuous function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, we have*

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(\varkappa)_{\kappa_1}d_q^T\varkappa\leq\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{18}$$

*Definition 9* (see [11]). For any real number $n$, the $q$ analogue of $n$ is defined as

$$[n]_q=\frac{1-q^n}{1-q}. \tag{19}$$

*Definition 10* (see [11]). Let $k,p>0$. Then, $\mathbb{B}_q(k,p)$ is defined by

$$\mathbb{B}_q(k,p)=\int_0^1 x^{k-1}(1-qx)_q^{p-1}d_qx. \tag{20}$$

## 3. A New ${}^{m\kappa_2}q$-Integral Identity

In this section, we present a new ${}^{m\kappa_2}q$-integral identity.

**Lemma 1.** *For $m\in(0,1]$ with $0<q<1$, let there be an arbitrary function $\mathcal{F}\colon\mathcal{Q}\longrightarrow\mathcal{R}$ such that ${}^{m\kappa_2}D_2^q\mathcal{F}$ is ${}^{m\kappa_2}q$-integrable on $\mathcal{Q}$. Then, one has*

$$\begin{aligned}{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)&=\frac{\mathcal{F}(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))+q\mathcal{F}(m\kappa_2)}{[2]_q}\\&\quad-\frac{1}{\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2}\mathcal{F}(x)^{m\kappa_2}d_qx\\&=\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2\mathcal{F}(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk.\end{aligned} \tag{21}$$

*Proof.* We suppose that

$$\begin{aligned}&\int_0^1 k(1-qk)^{m\kappa_2}D_q^2\mathcal{F}(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\\&=\int_0^1 k(1-qk)\left\{\frac{q\mathcal{F}(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{k^2q(1-q)^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}-\frac{[2]_q\mathcal{F}(m\kappa_2+qk\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))+\mathcal{F}\left(m\kappa_2+q^2k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{k^2q(1-q)^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}\right\}d_qk\\&=\frac{q\sum_{n=0}^{\infty}\mathcal{F}(m\kappa_2+q^n\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))-[2]_q\sum_{n=0}^{\infty}\mathcal{F}\left(m\kappa_2+q^{n+1}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}+\frac{\sum_{n=0}^{\infty}\mathcal{F}\left(m\kappa_2+q^{n+2}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}\\&\quad-q\left\{\frac{q(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty}q^n\mathcal{F}(m\kappa_2+q^n\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\right.\\&\quad-\frac{[2]_q(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty}q^{n+1}\mathcal{F}\left(m\kappa_2+q^{n+1}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q^2(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\\&\quad+\frac{(1-q)\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\sum_{n=0}^{\infty}q^{n+2}\mathcal{F}\left(m\kappa_2+q^{n+2}\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)\right)}{q^3(1-q)^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\\&=\frac{q(\mathcal{F}(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))-\mathcal{F}(m\kappa_2))-\mathcal{F}(m\kappa_2+q\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}\\&\quad+\frac{\mathcal{F}(m\kappa_2)}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}-\frac{[2]_q}{q^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2}\mathcal{F}(x)^{m\kappa_2}d_qx\\&\quad-\frac{\left(q^2+q-1\right)\mathcal{F}(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q^2(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}+\frac{\mathcal{F}(m\kappa_2+q\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))}{q(1-q)\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}\\&=\frac{\mathcal{F}(m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))+q\Lambda(m\kappa_2)}{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}-\frac{[2]_q}{q^2\eta^3(\kappa_1,\kappa_2,m)\vartheta^3(\kappa_1,\kappa_2,m)}\int_{m\kappa_2+\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m)}^{m\kappa_2}\mathcal{F}(x)^{m\kappa_2}d_qx.\end{aligned} \tag{22}$$

Multiplying both sides of the above equality by $q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)/[2]_q$, we get the required result. □

## 4. Hermite–Hadamard Inequalities for Generalized Exponentially $(s, m; \eta)$-Preinvex Functions

**Theorem 4.** *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|{}^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s, m; \eta)$-preinvex function and $\mathfrak{u} \geq 1$, then for some fixed $s, m \in (0,1]$, we have*

$$\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}} \cdot \left(\Omega_1\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \Omega_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \tag{23}$$

*where*

$$\Omega_1 = \mathbb{B}_q(s+2,\mathfrak{u}+1), \tag{24}$$

$$\Omega_2 = 2^{1-s}\mathbb{B}_q(2,\mathfrak{u}+1) - \mathbb{B}_q(s+2,\mathfrak{u}+1). \tag{25}$$

*Proof.* By utilizing conditions of Lemma 1 and the famous power mean inequality, we obtain

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|{}^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 kd_qk\right)^{1-1/\mathfrak{u}} \times \left(\int_0^1 k(1-qk)^{\mathfrak{u}}\left|{}^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\int_0^1 k(1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\int_0^1 k(1-qk)_q^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}} \\
&= \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q^{2-1/\mathfrak{u}}}\left(\Omega_1\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \Omega_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}},
\end{aligned} \tag{26}$$

where

$$\Omega_1 = \int_0^1 k^{s+1}(1-qk)_q^{\mathfrak{u}}d_qk = \mathbb{B}_q(s+2,\mathfrak{u}+1), \tag{27}$$

and

$$\begin{aligned}
\Omega_1 &= \int_0^1 k\left(2^{1-s} - k^s\right)(1-qk)_q^{\mathfrak{u}}\,d_qk \\
&= 2^{1-s}\mathbb{B}_q(2,\mathfrak{u}+1) - \mathbb{B}_q(s+2,\mathfrak{u}+1) \geq 0
\end{aligned} \tag{28}$$

due to $2^{1-s} - k^s \geq 0$ for all $k \in [0,1]$ and $s \in (0,1]$.

We proved our result. □

**Theorem 5.** *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s, m; \eta)$-preinvex function and $\mathfrak{u} > 1$ with $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$, then for some fixed $s, m \in (0, 1]$, we obtain*

$$\left|^{m\kappa_2} L_q(\kappa_1, \kappa_2, m, x)\right| \leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)^{1/\mathfrak{p}}\mathbb{B}q(2, \mathfrak{p}+1)}{[2]_q} \times \left(\frac{[2]_q\left|^{m\kappa_2}D_q^2 F(\kappa_1)/\chi^{\alpha\kappa_1}\right|^{\mathfrak{u}} + \left(2^{1-s}[s+2]_q - [2]_q\right)\left|m\kappa_2 D_q^2 F(m\kappa_2)/\chi^{\alpha m\kappa_2}\right|^{\mathfrak{u}}}{[2]_q[s+2]_q}\right)^{1/\mathfrak{u}}. \tag{29}$$

*Proof.* By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|^{m\kappa_2} L_q(\kappa_1, \kappa_2, m, x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))d_q k\right| \\
&\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\int_0^1 k(1-qk)\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))\right|d_q k \\
&\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\left(\int_0^1 k(1-qk)^{\mathfrak{p}}d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k\left|^{m\kappa_2}D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))\right|^{\mathfrak{u}}d_q k\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\left(\int_0^1 k(1-qk)_q^{\mathfrak{p}}d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k\left[k^s\left|\frac{^{m\kappa_2}D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{^{m\kappa_2}D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_q k\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)\mathbb{B}_q^{1/\mathfrak{p}}(2, \mathfrak{p}+1)}{[2]_q} \times \left(\frac{[2]_q\left|^{m\kappa_2}D_q^2 F(\kappa_1)/\chi^{\alpha\kappa_1}\right|^{\mathfrak{u}} + \left(2^{1-s}[s+2]_q - [2]_q\right)\left|^{m\kappa_2}D_q^2 F(m\kappa_2)/\chi^{\alpha m\kappa_2}\right|^{\mathfrak{u}}}{[2]_q[s+2]_q}\right)^{1/\mathfrak{u}}.
\end{aligned} \tag{30}$$

This completes the proof. □

**Theorem 6.** *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2}D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s, m; \eta)$-preinvex function and $\mathfrak{u} > 1$ with $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$, then for some fixed $s, m \in (0, 1]$, we obtain*

$$\left|^{m\kappa_2} L_q(\kappa_1, \kappa_2, m, x)\right| \leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q} \cdot \left(\mathbb{B}_q(\mathfrak{u}+s+1, \mathfrak{u}+1)\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \left(\left(2^{1-s}\mathbb{B}_q(\mathfrak{u}+1, \mathfrak{u}+1) - \mathbb{B}_q(\mathfrak{u}+s+1, \mathfrak{u}+1)\right)\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}. \tag{31}$$

*Proof.* By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right|\\
&=\left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right|\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 1d_qk\right)^{1/\mathfrak{p}}\times\left(\int_0^1 k^{\mathfrak{u}}(1-qk)^{\mathfrak{u}}\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{\frac{1}{\mathfrak{u}}}\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\times\left(\int_0^1 k^{\mathfrak{u}}(1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+(1-k)^s\left|\frac{m\kappa_2D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}}\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\left|\frac{m\kappa_2D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}\right.\\
&\quad\left.+\left(\left(2^{1-s}\mathbb{B}_q(\mathfrak{u}+1,\mathfrak{u}+1)-\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\right)\left|\frac{m\kappa_2D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)\right)^{1/\mathfrak{u}},
\end{aligned}\tag{32}$$

where

$$2^{1-s}\mathbb{B}_q(\mathfrak{u}+1,\mathfrak{u}+1)-\mathbb{B}_q(\mathfrak{u}+s+1,\mathfrak{u}+1)\geq 0.\tag{33}$$

We proved our result. □

**Theorem 7.** *We assume that the conditions of Lemma 1 with $\chi\geq 1$ and $\alpha\in\mathscr{R}$ hold. If $|{}^{m\kappa_2}D_q^2F|^{\mathfrak{u}}$ is a generalized exponentially $(s,m;\eta)$-preinvex function and $\mathfrak{u}>1$ with $\mathfrak{p}^{-1}+\mathfrak{u}^{-1}=1$, then for some fixed $s,m\in(0,1]$, we obtain*

$$\begin{aligned}
\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right|&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(\mathfrak{p}+1,\mathfrak{p}+1)}{[2]_q}\\
&\quad\times\left(\vartheta_1\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+\vartheta_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}},
\end{aligned}\tag{34}$$

*where*

$$\begin{aligned}
\vartheta_1&=\frac{1-q}{1-q^{s+1}}=\frac{1}{[s+1]_q},\\
\vartheta_2&=(1-q)\sum_{n=0}^{\infty}q^n(1-q^n)^s.
\end{aligned}\tag{35}$$

*Proof.* By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right|\\
&=\left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right|\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k^p(1-qk)^pd_qk\right)^{1/p}\times\left(\int_0^1\left|{}^{m\kappa_2}D_q^2F(m\kappa_2+k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}}\\
&\leq\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)\mathbb{B}_q^{1/\mathfrak{p}}(\mathfrak{p}+1,\mathfrak{p}+1)}{[2]_q}\times\left(\int_0^1\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}}+(1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}}.
\end{aligned}\tag{36}$$

Applying the definition of quantum integral, we get

$$\vartheta_1 = \int_0^1 k^s \, {}_0d_qk = \frac{1-q}{1-q^{s+1}} = \frac{1}{[s+1]_q},$$
$$\vartheta_2 = \int_0^1 (1-k)^s \, {}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^n \left(1-q^n\right)^s. \tag{37}$$

This completes the proof. □

**Theorem 8.** *We assume that the conditions of Lemma 1 with* $\chi \geq 1$ *and* $\alpha \in \mathscr{R}$ *hold. If* $|{}^{m\kappa_2}D_q^2F|^{\mathfrak{u}}$ *is a generalized exponentially* $(s,m;\eta)$*-preinvex function and* $\mathfrak{u} > 1$ *with* $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$*, then for some fixed* $s, m \in (0,1]$*, we obtain*

$$\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q[\mathfrak{p}+1]_q^{1/\mathfrak{p}}} \times \left(\omega_1\left|\frac{{}_{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \omega_2\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \tag{38}$$

*where*

$$\omega_1 = (1-q)\sum_{n=0}^{\infty} q^{n(s+1)}\left(1-q^{n+1}\right)^{\mathfrak{u}}, \omega_2 = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^s\left(1-q^{n+1}\right)^{\mathfrak{u}}. \tag{39}$$

*Proof.* By utilizing conditions of Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|{}^{m\kappa_2}L_q(\kappa_1,\kappa_2,m,x)\right| \\
&= \left|\frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))d_qk\right| \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\int_0^1 k(1-qk)\left|{}^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|d_qk \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q}\left(\int_0^1 k^{\mathfrak{p}}d_qk\right)^{1/\mathfrak{p}} \times \left(\int_0^1 (1-qk)^{\mathfrak{u}}\left|{}^{m\kappa_2}D_q^2F(m\kappa_2 + k\eta(\kappa_1,\kappa_2,m)\vartheta(\kappa_1,\kappa_2,m))\right|^{\mathfrak{u}}d_qk\right)^{1/\mathfrak{u}} \\
&\leq \frac{q^2\eta^2(\kappa_1,\kappa_2,m)\vartheta^2(\kappa_1,\kappa_2,m)}{[2]_q[\mathfrak{p}+1]_q^{1/\mathfrak{p}}} \times \left(\int_0^1 (1-qk)^{\mathfrak{u}}\left[k^s\left|\frac{{}^{m\kappa_2}D_q^2F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1-k)^s\left|\frac{{}^{m\kappa_2}D_q^2F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right]d_qk\right)^{1/\mathfrak{u}}.
\end{aligned} \tag{40}$$

Applying the definition of quantum integral, we get

$$\omega_1 = \int_0^1 k^s(1-qk)^{\mathfrak{u}} \, {}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^{n(s+1)}\left(1-q^{n+1}\right)^{\mathfrak{u}}, \omega_2 = \int_0^1 (1-k)^s(1-qk)^{\mathfrak{u}} \, {}_0d_qk = (1-q)\sum_{n=0}^{\infty} q^n\left(1-q^n\right)^s\left(1-q^{n+1}\right)^{\mathfrak{u}}. \tag{41}$$

This completes the proof. □

**Theorem 9.** *We assume that the conditions of Lemma 1 with $\chi \geq 1$ and $\alpha \in \mathscr{R}$ hold. If $|^{m\kappa_2} D_q^2 F|^{\mathfrak{u}}$ is a generalized exponentially $(s, m; \eta)$-preinvex function and $\mathfrak{u} > 1$ with $\mathfrak{p}^{-1} + \mathfrak{u}^{-1} = 1$, then for some fixed $s, m \in (0, 1]$, we obtain*

$$\left|^{m\kappa_2} L_q(\kappa_1, \kappa_2, m, x)\right| \leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)\mathbb{B}_q^{1/\mathfrak{p}}(1, \mathfrak{p} + 1)}{[2]_q} \times \left(\sigma_1\left|\frac{m\kappa_2 D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + \sigma_2\left|\frac{m\kappa_2 D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right)^{1/\mathfrak{u}}, \tag{42}$$

*where*

$$\begin{aligned}\sigma_1 &= \frac{1}{[s + \mathfrak{u} + 1]_q},\\ \sigma_2 &= (1 - q)\sum_{n=0}^{\infty} q^{n(1+\mathfrak{u})}(1 - q^n)^s.\end{aligned} \tag{43}$$

*Proof.* By utilizing conditions of Lemma 1 and the famous Hölder inequality, we obtain

$$\begin{aligned}&\left|^{m\kappa_2} L_q(\kappa_1, \kappa_2, m, x)\right|\\ &= \left|\frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\int_0^1 k(1 - qk)^{m\kappa_2} D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))\mathrm{d}_q k\right|\\ &\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\int_0^1 k(1 - qk)\left|^{m\kappa_2} D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))\right| d_q k\\ &\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)}{[2]_q}\left(\int_0^1 (1 - qk)^{\mathfrak{p}} d_q k\right)^{1/\mathfrak{p}} \times \left(\int_0^1 k^{\mathfrak{u}}\left|^{m\kappa_2} D_q^2 F(m\kappa_2 + k\eta(\kappa_1, \kappa_2, m)\vartheta(\kappa_1, \kappa_2, m))\right|^{\mathfrak{u}} d_q k\right)^{1/\mathfrak{u}}\\ &\leq \frac{q^2\eta^2(\kappa_1, \kappa_2, m)\vartheta^2(\kappa_1, \kappa_2, m)\mathbb{B}_q^{1/\mathfrak{p}}(1, \mathfrak{p} + 1)}{[2]_q} \times \left(\int_0^1 k^{\mathfrak{u}}\left[k^s\left|\frac{^{m\kappa_2} D_q^2 F(\kappa_1)}{\chi^{\alpha\kappa_1}}\right|^{\mathfrak{u}} + (1 - k)^s\left|\frac{^{m\kappa_2} D_q^2 F(m\kappa_2)}{\chi^{\alpha m\kappa_2}}\right|^{\mathfrak{u}}\right] d_q k\right)^{1/\mathfrak{u}}.\end{aligned} \tag{44}$$

Applying the definition of quantum integral, we get

$$\begin{aligned}\sigma_1 &= \int_0^1 k^{s+\mathfrak{u}}\,{}_0d_q k = \frac{1}{[s + \mathfrak{u} + 1]_q},\\ \sigma_2 &= \int_0^1 k^{\mathfrak{u}}(1 - k)^s\,{}_0d_q k = (1 - q)\sum_{n=0}^{\infty} q^{n(1+\mathfrak{u})}(1 - q^n)^s.\end{aligned} \tag{45}$$

This completes the proof. □

## 5. Conclusion

In this article, we established the new definition of generalized exponentially $(s, m; \eta)$-preinvex functions and proved a new modified $^{m\kappa_2}q$-integral identity. Using this new identity, we have been able to obtain new estimates of the quantum bounds applying the concept of generalized exponentially $(s, m; \eta)$-preinvex functions. It is worth to mention here that if we take $\chi = e$, then all of the main results reduce to the results for exponentially $(s, m; \eta)$-preinvex functions. For further research, we could expand the inequality-based analysis to other fields, including the inequality-based theory, quantum calculus, machine learning, robotics, weather forecasting, and optimizations.

## Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

This study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

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