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Trapezoid and Midpoint Type Inequalities for Preinvex Functions via Quantum Calculus

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Abstract: In this article, we use quantum integrals to derive Hermite–Hadamard inequalities for preinvex functions and demonstrate their validity with mathematical examples. We use the q^{\varkappa_2} -quantum integral to show midpoint and trapezoidal inequalities for q^{\varkappa_2} -differentiable preinvex functions. Furthermore, we demonstrate with an example that the previously proved Hermite–Hadamard-type inequality for preinvex functions via q^{\varkappa_1} -quantum integral is not valid for preinvex functions, and we present its proper form. We use q^{\varkappa_1} -quantum integrals to show midpoint inequalities for q^{\varkappa_1} -differentiable preinvex functions. It is also demonstrated that by considering the limit $q \rightarrow 1^-$ and $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) = \varkappa_2 - \varkappa_1$ in the newly derived results, the newly proved findings can be turned into certain known results.

Keywords: Hermite–Hadamard inequality; q -integral; quantum calculus; preinvex function; trapezoid inequalities; midpoint inequalities



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1. Introduction

C. Hermite and J. Hadamard are the founders of the well-known inequality, which is called the Hermite–Hadamard inequality (see [1,2], p. 137). In the theory of convexity, the Hermite–Hadamard inequality is a well-established inequality with many applications and geometrical interpretations. This inequality states that if a function $\phi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex, then for $\varkappa_1, \varkappa_2 \in I$ with $\varkappa_1 < \varkappa_2$, we have the following:

$$\phi\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \phi(x) dx \leq \frac{\phi(\varkappa_1) + \phi(\varkappa_2)}{2}. \quad (1)$$

If the given function is concave, then the above inequality holds in reversed direction. This inequality can be easily captured by using Jensen's inequality for convex functions. In recent years, several generalizations and extensions have been provided for classical convex functions. The invex function introduced by Hanson in [3] is a significant generalization of convex functions. The concept of preinvex functions was given by Weir and Mond in [4] and is used in optimization theory in different ways. The concept of prequasiinvex functions, which is the generalization of the invex functions introduced by Pini in [5]. After

that, the authors considered some basic properties of the generalized preinvex functions in [6]. In [7–9], Noor proved Hermite–Hadamard integral inequalities for the preinvex functions. In [10,11], the authors gave the left and right bounds of the Hermite–Hadamard inequalities for preinvex functions, using the ordinary and fractional integrals. For more recent results about the integral inequalities for different kinds of preinvexities, one can read [12–21].

On the other hand, several research studies were recently carried out on the subject of q -analysis, beginning with Euler, due to a large need for mathematics that models quantum computing q -calculus, occurring for the interaction between physics and mathematics. It has a wide range of applications in mathematics, including combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other disciplines, as well as mechanics, relativity theory, and quantum theory [22–25]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton’s work on infinite series, he used the q parameter. Jackson [24] was the first to present q -calculus that knew, without limits, calculus in a logical fashion. Jackson [24] defined the general form of the q -integral and q -difference operator in 1908–1909. Agarwal [26] defined the q -fractional derivative for the first time in 1969. Al-Salam [27] introduced a q -analog of the q -fractional integral and q -Riemann–Liouville fractional in 1966–1967. Rajkovic defined the Riemann-type q -integral in 2004, which was later generalized to the Jackson q -integral. The ${}_{\varkappa_1}D_q$ -difference operator was first presented in [28] by Tariboon and Ntouyas in 2013.

Many integral inequalities have been studied, using quantum integrals for various types of functions. For example, in [29–37], the authors used quantum integrals to prove Hermite–Hadamard integral inequalities and their left–right estimates for convex, coordinated convex functions and some other classes of functions. In [38], Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [39]. Khan et al. proved quantum Hermite–Hadamard inequality, using the green function in [40]. Budak et al. [41], Vivas-Cortez et al. [42] and Ali et al. [43] developed new quantum Simpson’s and quantum Newton’s type inequalities for convex and coordinated convex functions. In [44], Deng et al. proved the generalized version of Simpson’s inequalities for quantum integrals. For Ostrowski’s inequalities via quantum integrals, one can consult [45–48].

This work has a general structure with seven main sections, including an introduction. In Section 2, we provide some essential notations for the concept of q -calculus, as well as a list of relevant literature. In Section 3, we prove the Hermite–Hadamard inequalities for preinvex functions, using the q_{\varkappa_1} and q^{\varkappa_2} -quantum integrals. In Sections 4 and 5, we provide trapezoid and midpoint-type inequalities for q^{\varkappa_2} -differentiable preinvex functions through q^{\varkappa_2} -quantum integrals, respectively. Some new midpoint inequalities for q_{\varkappa_1} -differentiable preinvex functions via q_{\varkappa_1} -quantum integrals are proved in Section 6. We also look at the relationship between our findings and the inequities discussed in previous research. Finally, some findings and future research options are explored in Section 7. We believe that our work’s viewpoint and methodology may stimulate additional study in this field.

2. Preliminaries of q -Calculus and Some Inequalities

This section reviews the fundamental concepts and findings that are needed in the next sections to prove our critical findings.

Definition 1 ([4,6]). A set $\omega \subseteq \mathbb{R}^n$ is considered to be invex with respect to a given $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ if

$$\varkappa + t\eta(\gamma, \varkappa) \in \omega, \forall \varkappa, \gamma \in \omega, t \in [0, 1].$$

The invex set ω is more commonly referred to as η -connected set.

Definition 2 ([4,6]). Let $\omega \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. A mapping $\phi : \omega \rightarrow \mathbb{R}$ is called preinvex, if

$$\phi(\varkappa + t\eta(\gamma, \varkappa)) \leq t\phi(\gamma) + (1 - t)\phi(\varkappa), \forall \varkappa, \gamma \in \omega, t \in [0, 1]. \tag{2}$$

The mapping ϕ is called preconcave if $-\phi$ is preinvex.

Remark 1. If we set $\eta(\gamma, \varkappa) = \gamma - \varkappa$ in Definition 2, then the Definition 2 shall becomes the definition of convex functions given below:

$$\phi(\varkappa + t(\gamma - \varkappa)) \leq t\phi(\gamma) + (1 - t)\phi(\varkappa), \forall \varkappa, \gamma \in \omega, t \in [0, 1].$$

Condition C. Ref. [6] The function η satisfies the following condition if

$$\begin{aligned} \eta(\gamma, \gamma + t\eta(\varkappa, \gamma)) &= -t\eta(\varkappa, \gamma), \\ \eta(\varkappa, \gamma + \eta(\varkappa, \gamma)) &= (1 - t)\eta(\varkappa, \gamma) \end{aligned} \tag{3}$$

for every $\varkappa, \gamma \in \omega$ and any $t \in [0, 1]$. Note that for every $\varkappa, \gamma \in \omega, t_1, t_2 \in [0, 1]$, and from Condition C, we have the following:

$$\eta(\gamma + t_2\eta(\varkappa, \gamma), \gamma + t_1\eta(\varkappa, \gamma)) = (t_2 - t_1)\eta(\varkappa, \gamma).$$

Theorem 1 ([49]). (Jensen’s inequality for preinvex functions) Let $\phi : \omega \rightarrow \mathbb{R}$ be a preinvex function. Let $\gamma_1, \gamma_2, \dots, \gamma_n \in [0, 1]$ be the coefficients such that $\sum_{i=0}^n \gamma_i = 1$, and let $t_1, t_2, \dots, t_n \in [0, 1]$ be the coefficients. Then, the inequality

$$\phi\left(\sum_{i=1}^n \gamma_i(\varkappa + t_i\eta(\gamma, \varkappa))\right) \leq \sum_{i=1}^n \gamma_i\phi(\varkappa + t_i\eta(\gamma, \varkappa)) \tag{4}$$

holds for all $\varkappa, \gamma \in \omega$.

We use the notation

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}, \quad q \in (0, 1)$$

for $n \in \mathbb{R}$.

In [24], Jackson gave the q -integral from 0 to \varkappa_2 , namely, the q -Jackson integral for $0 < q < 1$ and for the function $\phi : [0, \varkappa_2] \rightarrow \mathbb{R}$ in the following way:

$$\int_0^{\varkappa_2} \phi(\varkappa) d_q \varkappa = (1 - q)\varkappa_2 \sum_{n=0}^{\infty} q^n \phi(\varkappa_2 q^n) \tag{5}$$

provided the sum converge absolutely.

Moreover, he gave the q -Jackson integral in a general interval $[\varkappa_1, \varkappa_2]$ as follows:

$$\int_{\varkappa_1}^{\varkappa_2} \phi(\varkappa) d_q \varkappa = \int_0^{\varkappa_2} \phi(\varkappa) d_q \varkappa - \int_0^{\varkappa_1} \phi(\varkappa) d_q \varkappa.$$

Definition 3 ([28]). Let $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q_{\varkappa_1} -derivative of ϕ at $\varkappa \in [\varkappa_1, \varkappa_2]$ is defined in the following way:

$${}_{\varkappa_1}D_q\phi(\varkappa) = \frac{\phi(\varkappa) - \phi(q\varkappa + (1 - q)\varkappa_1)}{(1 - q)(\varkappa - \varkappa_1)}, \quad \varkappa \neq \varkappa_1. \tag{6}$$

In view of the fact that the function $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ is continuous, thus we state the following:

$${}_{\varkappa_1}D_q\phi(\varkappa_1) = \lim_{\varkappa \rightarrow \varkappa_1} {}_{\varkappa_1}D_q\phi(\varkappa).$$

If ${}_{\varkappa_1}D_q\phi(\varkappa)$ exists for all $\varkappa \in [\varkappa_1, \varkappa_2]$, then function ϕ is said to be q_{\varkappa_1} -differentiable on $[\varkappa_1, \varkappa_2]$. If we assume that $\varkappa_1 = 0$ in (6), then ${}_0D_q\phi(\varkappa) = D_q\phi(\varkappa)$, where $D_q\phi(\varkappa)$ is the familiar q -derivative of ϕ at $\varkappa \in [\varkappa_1, \varkappa_2]$ defined by the following expression (see [25]):

$$D_q\phi(\varkappa) = \frac{\phi(\varkappa) - \phi(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0.$$

Definition 4 ([50]). Let $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q^{\varkappa_2} -derivative of ϕ at $\varkappa \in [\varkappa_1, \varkappa_2]$ is defined in the following way:

$${}_{\varkappa_2}D_q\phi(\varkappa) = \frac{\phi(q\varkappa + (1-q)\varkappa_2) - \phi(\varkappa)}{(1-q)(\varkappa_2 - \varkappa)}, \quad \varkappa \neq \varkappa_2. \tag{7}$$

Since a function $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ is continuous, thus we state the following:

$${}_{\varkappa_2}D_q\phi(\varkappa_2) = \lim_{\varkappa \rightarrow \varkappa_2} {}_{\varkappa_2}D_q\phi(\varkappa).$$

If ${}_{\varkappa_2}D_q\phi(\varkappa)$ exists for all $\varkappa \in [\varkappa_1, \varkappa_2]$, then function ϕ is said to be q^{\varkappa_2} -differentiable on $[\varkappa_1, \varkappa_2]$. If we consider $\varkappa_2 = 0$ in (7), then ${}_0D_q\phi(\varkappa) = D_q\phi(\varkappa)$, where $D_q\phi(\varkappa)$ is the familiar q -derivative of ϕ at $\varkappa \in [\varkappa_1, \varkappa_2]$ defined by the expression (see [25]):

$$D_q\phi(\varkappa) = \frac{\phi(\varkappa) - \phi(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0.$$

Definition 5 ([28]). Let $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q_{\varkappa_1} -definite integral on $[\varkappa_1, \varkappa_2]$ is defined as follows:

$$\begin{aligned} \int_{\varkappa_1}^{\varkappa_2} \phi(\varkappa) {}_{\varkappa_1}d_q\varkappa &= (1-q)(\varkappa_2 - \varkappa_1) \sum_{n=0}^{\infty} q^n \phi(q^n \varkappa_2 + (1-q^n)\varkappa_1) \\ &= (\varkappa_2 - \varkappa_1) \int_0^1 \phi((1-t)\varkappa_1 + t\varkappa_2) d_q t. \end{aligned}$$

Lemma 1 ([28]). We have the following equality

$$\int_{\varkappa_1}^{\varkappa_2} (\varkappa - \varkappa_1)^\alpha {}_{\varkappa_1}d_q\varkappa = \frac{(\varkappa_2 - \varkappa_1)^{\alpha+1}}{[\alpha + 1]_q}$$

for $\alpha \in \mathbb{R} \setminus \{-1\}$.

In [29], Alp et al. established the succeeding quantum integral inequality of the Hermite–Hadamard type for convex functions in q -calculus:

Theorem 2. (q_{\varkappa_1} -Hermite–Hadamard inequality) Assume that $\phi : [\varkappa_1, \varkappa_2] \rightarrow \mathbb{R}$ is convex differentiable function on $[\varkappa_1, \varkappa_2]$ and $0 < q < 1$. Then, we have the succeeding inequality:

$$\phi\left(\frac{q\varkappa_1 + \varkappa_2}{[2]_q}\right) \leq \frac{1}{\varkappa_2 - \varkappa_1} \int_{\varkappa_1}^{\varkappa_2} \phi(\varkappa) {}_{\varkappa_1}d_q\varkappa \leq \frac{q\phi(\varkappa_1) + \phi(\varkappa_2)}{[2]_q}. \tag{8}$$

In [29,34], the authors offered some estimates for the right and left hand sides of the inequality (8).

On the other hand, a new definition of quantum integral and related integral inequalities is given by Bermudo et al. in the following way:

Definition 6 ([50]). Let $\phi : [x_1, x_2] \rightarrow \mathbb{R}$ be a continuous function. Then, the q^{x_2} -definite integral on $[x_1, x_2]$ is defined as follows:

$$\begin{aligned} \int_{x_1}^{x_2} \phi(x) \, {}_{x_2}d_q x &= (1-q)(x_2 - x_1) \sum_{n=0}^{\infty} q^n \phi(q^n x_1 + (1-q^n)x_2) \\ &= (x_2 - x_1) \int_0^1 \phi(tx_1 + (1-t)x_2) \, d_q t. \end{aligned}$$

Theorem 3 ([50]). (q^{x_2} -Hermite–Hadamard inequality) Assume that $\phi : [x_1, x_2] \rightarrow \mathbb{R}$ is convex differentiable function on $[x_1, x_2]$ and $0 < q < 1$. Then, we have the succeeding inequality:

$$\phi\left(\frac{x_1 + q x_2}{[2]_q}\right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \phi(x) \, {}_{x_2}d_q x \leq \frac{\phi(x_1) + q\phi(x_2)}{[2]_q}. \tag{9}$$

In [51], Budak offered some estimates for the right and left hand sides of the inequality (9).

Noor and Awan [38] proved the following q -Hermite-Hadamard type inequalities for preinvex functions using the q_{x_1} -integral:

Theorem 4. If $\phi : [x_1 + \eta(x_2, x_1), x_2] \rightarrow \mathbb{R}$ is an integrable and preinvex function, then we obtain the succeeding inequality

$$\phi\left(\frac{2x_1 + \eta(x_2, x_1)}{2}\right) \leq \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) \, {}_{x_1}d_q x \leq \frac{q\phi(x_1) + \phi(x_2)}{2} \tag{10}$$

where $q \in (0, 1)$ and $\eta(x_2, x_1) > 0$.

We observed that the inequality (10) is not valid for preinvex functions. For the explanation, we give the following example.

Example 1. A function $\phi(x) = -|x|$ is a preinvex function with respect to the following bifunction

$$\eta(x, \gamma) = \begin{cases} x - \gamma, & \text{if } x\gamma \geq 0 \\ \gamma - x, & \text{if } x\gamma < 0. \end{cases}$$

Then, from the inequality (10) the succeeding inequality should be held for $x_1 < x_2, x_1, x_2 > 0$ and $q \in (0, 1)$

$$\begin{aligned} \phi\left(\frac{2x_1 + \eta(x_2, x_1)}{2}\right) &\leq \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) \, {}_{x_1}d_q x \\ -\frac{x_1 + x_2}{2} &\leq -\frac{q x_1 + x_2}{[2]_q} \\ \frac{x_1 + x_2}{2} &\geq \frac{q x_1 + x_2}{[2]_q}. \end{aligned}$$

Thus, we have the following:

$$(q - 1)x_1 + (1 - q)x_2 \leq 0$$

which shows that the left side of the inequality (10) is not valid for the preinvex functions.

The main objective of this paper is to prove the Hermite–Hadamard inequality for q^{\varkappa_2} -integrals and find its left and right side estimates. We also give the correct version of the inequality (10) and its left hand side estimates. For the right estimates of the correct version of the inequality (10) that given in the next section, one can read [38].

3. q -Hermite–Hadamard Inequalities

Theorem 5. Let $\phi : I = [\varkappa_2 + \eta(\varkappa_1, \varkappa_2), \varkappa_2] \rightarrow \mathbb{R}$ be a preinvex function on I . Then, we obtain the succeeding inequality:

$$\begin{aligned} \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q \varkappa_2}{[2]_q}\right) &\leq \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \\ &\leq \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q} \end{aligned} \tag{11}$$

where $q \in (0, 1)$ and $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) > 0$.

Proof. Notice the following:

$$\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q \varkappa_2}{[2]_q} = \sum_{k=0}^{\infty} (1-q)q^k \left(q^k(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1-q^k)\varkappa_2 \right)$$

where $\sum_{k=0}^{\infty} (1-q)q^k = 1$. Now, from Jensen’s inequality for preinvex functions (4), we have the following:

$$\begin{aligned} \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q \varkappa_2}{[2]_q}\right) &\leq \sum_{k=0}^{\infty} (1-q)q^k \phi\left(q^k(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1-q^k)\varkappa_2 \right) \\ &= \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa. \end{aligned}$$

Thus, the first inequality in (11) is proved. For the proof of the second inequality in (11), we note that ϕ is a preinvex function on I and from the inequality (3), and we have the following:

$$\phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2)) \leq t\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1-t)\phi(\varkappa_2). \tag{12}$$

q^{\varkappa_2} -integrating (12) with respect to t over $[0, 1]$, we have the following:

$$\frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \leq \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q}.$$

Thus, the proof is accomplished. \square

Example 2. Let $\phi(\varkappa) = -|\varkappa|$. Then ϕ is a preinvex function with respect to the following bifunction:

$$\eta(\varkappa, \gamma) = \begin{cases} \varkappa - \gamma, & \text{if } \varkappa\gamma \geq 0, \\ \gamma - \varkappa, & \text{if } \varkappa\gamma \leq 0. \end{cases}$$

(i) Let us consider $\varkappa_1, \varkappa_2 > 0$.

Then, we have $\eta(\varkappa_1, \varkappa_2) = \varkappa_1 - \varkappa_2$,

$$\phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q \varkappa_2}{[2]_q}\right) = -\frac{\varkappa_1 + q\varkappa_2}{[2]_q},$$

$$\frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}_{x_2}d_q x = -\frac{x_1 + qx_2}{[2]_q}$$

and

$$\frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} = -\frac{x_1 + qx_2}{[2]_q}.$$

(ii) Let $x_1, x_2 < 0$. Then $\eta(x_1, x_2) = x_1 - x_2$

$$\phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) = \frac{x_1 + qx_2}{[2]_q},$$

$$\frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}_{x_2}d_q x = \frac{x_1 + qx_2}{[2]_q}$$

and

$$\frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} = \frac{x_1 + qx_2}{[2]_q}.$$

(iii) Finally, let us consider $x_1 < 0 < x_2$. Then, we obtain $\eta(x_1, x_2) = x_2 - x_1$

$$\phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) = -\frac{(2+q)x_2 - x_1}{[2]_q},$$

$$\frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}_{x_2}d_q x = -\frac{(2+q)x_2 - x_1}{[2]_q}$$

and

$$\frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} = -\frac{(2+q)x_2 - x_1}{[2]_q}.$$

These show that Theorem 5 is valid for the function ϕ .

Remark 2. If we set $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Theorem 5, then Theorem 5 becomes Theorem 12 of [50].

Corollary 1. In Theorem 5, if we consider the limit $q \rightarrow 1^-$, then inequality (11) reduces to the succeeding inequality:

$$\begin{aligned} \phi\left(\frac{\eta(x_1, x_2) + 2x_2}{2}\right) &\leq \frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) dx \\ &\leq \frac{\phi(x_2 + \eta(x_1, x_2)) + \phi(x_2)}{2} \leq \frac{\phi(x_1) + \phi(x_2)}{2}. \end{aligned}$$

Now, we give the correct version of the inequality (10) and we can show its validation with a mathematical example using the techniques used in Example 2.

Theorem 6. Let $\phi : I = [x_1, x_1 + \eta(x_2, x_1)] \rightarrow \mathbb{R}$ be a preinvex function on I . Then, we obtain the succeeding inequality:

$$\begin{aligned} \phi\left(\frac{[2]_q x_1 + \eta(x_2, x_1)}{[2]_q}\right) &\leq \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1+\eta(x_2, x_1)} \phi(x) {}_{x_1}d_q x \\ &\leq \frac{q\phi(x_1) + \phi(x_1 + \eta(x_2, x_1))}{[2]_q} \end{aligned} \tag{13}$$

where $q \in (0, 1)$ and $\eta(x_2, x_1) > 0$.

Proof. Following the arguments similar to those in the proof of Theorem 5 by considering the q_{\varkappa_1} -integral, the desirable inequality (13) can be proved. \square

Remark 3. In Theorem 6, if we set $\eta(\varkappa_2, \varkappa_1) = \varkappa_2 - \varkappa_1$, then Theorem 6 becomes Theorem 6 of [29].

Remark 4. In Theorem 6, if we assume the limit $q \rightarrow 1^-$, then inequality (13) becomes the succeeding inequality:

$$\begin{aligned} \phi\left(\frac{2\varkappa_1 + \eta(\varkappa_2, \varkappa_1)}{2}\right) &\leq \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_1}^{\varkappa_1 + \eta(\varkappa_2, \varkappa_1)} \phi(\varkappa) d\varkappa \\ &\leq \frac{\phi(\varkappa_1) + \phi(\varkappa_1 + \eta(\varkappa_2, \varkappa_1))}{2} \leq \frac{\phi(\varkappa_1) + \phi(\varkappa_2)}{2} \end{aligned}$$

which can be found in [11] (Theorem 2.1 for $\alpha = 1$) and [8,9].

Summing the results proved in Theorems 5 and 6, we obtain the succeeding corollary.

Corollary 2. For any preinvex function and $q \in (0, 1)$, we obtain the succeeding inequality:

$$\begin{aligned} &\phi\left(\frac{[2]_q \varkappa_1 + \eta(\varkappa_2, \varkappa_1)}{[2]_q}\right) + \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q \varkappa_2}{[2]_q}\right) \\ &\leq \frac{1}{\eta(\varkappa_2, \varkappa_1)} \left[\int_{\varkappa_1}^{\varkappa_1 + \eta(\varkappa_2, \varkappa_1)} \phi(\varkappa) \varkappa_1 d_q \varkappa + \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \right] \\ &\leq \frac{q\phi(\varkappa_1) + \phi(\varkappa_1 + \eta(\varkappa_2, \varkappa_1))}{[2]_q} + \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q}. \end{aligned}$$

4. New Trapezoid Type Inequalities for q^{\varkappa_2} -Integrals

In this section, the trapezoidal estimates of q -Hermite–Hadamard inequalities proved in Theorem 5 are discussed.

Let us start with the following identity that is needed to prove the key results of this section.

Lemma 2. Let $\phi : I = [\varkappa_2 + \eta(\varkappa_1, \varkappa_2), \varkappa_2] \rightarrow \mathbb{R}$ be a q^{\varkappa_2} -differentiable function on I° (interior of I) and ${}^{\varkappa_2}D_q \phi$ be a continuous and integrable function on I . Then, we have the following identity:

$$\begin{aligned} &\frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q} - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \quad (14) \\ &= \frac{q\eta(\varkappa_2, \varkappa_1)}{[2]_q} \int_0^1 (1 - [2]_q t) {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2)) d_q t \end{aligned}$$

where $0 < q < 1$ and $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) > 0$.

Proof. From Definition 4 of q^{\varkappa_2} -derivative, we obtain the following:

$${}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2)) = \frac{\phi(\varkappa_2 + qt\eta(\varkappa_1, \varkappa_2)) - \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))}{(1 - q)\eta(\varkappa_2, \varkappa_1)t}. \quad (15)$$

From (15) and the right side of identity (14), we obtain the following:

$$\int_0^1 (1 - [2]_q t) {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2)) d_q t \quad (16)$$

$$\begin{aligned}
 &= \int_0^1 (1 - [2]_q t) \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)\eta(x_2, x_1)t} d_q t \\
 &= \frac{1}{\eta(x_2, x_1)} \int_0^1 \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)t} d_q t \\
 &\quad - \frac{[2]_q}{\eta(x_2, x_1)} \int_0^1 \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)} d_q t.
 \end{aligned}$$

Computing the first integral in the right side of (16), we obtain the following:

$$\begin{aligned}
 &\frac{1}{\eta(x_2, x_1)} \int_0^1 \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)t} d_q t \tag{17} \\
 &= \frac{1}{\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi(q^{k+1}(x_2 + \eta(x_1, x_2)) + (1 - q^{k+1})x_2) \\
 &\quad - \frac{1}{\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2) \\
 &= \frac{1}{\eta(x_2, x_1)} [\phi(x_2) - \phi(x_2 + \eta(x_1, x_2))].
 \end{aligned}$$

If we similarly notice that the other integral in the right side of (16) and Definition 6, we obtain the following:

$$\begin{aligned}
 &\frac{[2]_q}{\eta(x_2, x_1)} \int_0^1 \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)} d_q t \tag{18} \\
 &= \frac{[2]_q}{\eta(x_2, x_1)} \sum_{k=0}^{\infty} q^k \phi(q^{k+1}(x_2 + \eta(x_1, x_2)) + (1 - q^{k+1})x_2) \\
 &\quad - \frac{[2]_q}{\eta(x_2, x_1)} \sum_{k=0}^{\infty} q^k \phi(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2) \\
 &= \frac{[2]_q}{q\eta(x_2, x_1)} \sum_{k=1}^{\infty} q^k \phi(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2) \\
 &\quad - \frac{[2]_q}{\eta(x_2, x_1)} \sum_{k=0}^{\infty} q^k \phi(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2) \\
 &= \frac{[2]_q(1 - q)}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} q^k \phi(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2) - \frac{[2]_q}{q\eta(x_2, x_1)} \phi(x_2 + \eta(x_1, x_2)) \\
 &= \frac{[2]_q}{q\eta^2(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) x_2 d_q x - \frac{[2]_q}{q\eta(x_2, x_1)} \phi(x_2 + \eta(x_1, x_2)).
 \end{aligned}$$

Substituting the identities (17) and (18) in (16), and later multiplying both sides of the resultant one by $\frac{q\eta(x_2, x_1)}{[2]_q}$, the identity (14) can be captured. □

Remark 5. If we consider $q \rightarrow 1^-$ and $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Lemma 2, then Lemma 2 becomes Lemma 2.1 of [52].

Corollary 3. In Lemma 2, if we take the limit $q \rightarrow 1^-$, then identity (14) reduces into the succeeding identity:

$$\begin{aligned} & \frac{\phi(x_2 + \eta(x_1, x_2)) + \phi(x_2)}{2} - \frac{1}{\eta(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) dx \\ &= \frac{\eta(x_2, x_1)}{2} \int_0^1 (1 - 2t) \phi'(x_2 + t\eta(x_1, x_2)) dt. \end{aligned}$$

Remark 6. If we consider $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Lemma 2, then Lemma 2 reduces to Lemma 1 of [51].

Theorem 7. Assume that the conditions of Lemma 2 hold. If $|{}^{x_2}D_q\phi|$ is the preinvex function on I , then we obtain the succeeding inequality:

$$\begin{aligned} & \left| \frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} - \frac{1}{\eta(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x \right| \\ & \leq \eta(x_2, x_1) \left[|{}^{x_2}D_q\phi(x_1)| \frac{q^2(1 + 4q + q^2)}{[3]_q [2]_q^4} + |{}^{x_2}D_q\phi(x_2)| \frac{q^2(1 + 3q^2 + 2q^3)}{[3]_q [2]_q^4} \right] \end{aligned}$$

where $0 < q < 1$.

Proof. On taking modulus in Lemma 2 and using properties of the modulus, we have the following:

$$\begin{aligned} & \left| \frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} - \frac{1}{\eta(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x \right| \tag{19} \\ & \leq \frac{q\eta(x_2, x_1)}{[2]_q} \int_0^1 |1 - [2]_q t| |{}^{x_2}D_q \phi(x_2 + t\eta(x_1, x_2))| d_q t. \end{aligned}$$

Using preinvexity of $|{}^{x_2}D_q\phi|$, we obtain the following:

$$\begin{aligned} & \left| \frac{\phi(x_2 + \eta(x_1, x_2)) + q\phi(x_2)}{[2]_q} - \frac{1}{\eta(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x \right| \\ & \leq \frac{q\eta(x_2, x_1)}{[2]_q} \int_0^1 |1 - [2]_q t| [t |{}^{x_2}D_q\phi(x_1)| + (1 - t) |{}^{x_2}D_q\phi(x_2)|] d_q t \\ & = \frac{q\eta(x_2, x_1)}{[2]_q} \left[|{}^{x_2}D_q\phi(x_1)| \int_0^1 |1 - [2]_q t| t d_q t + |{}^{x_2}D_q\phi(x_2)| \int_0^1 |1 - [2]_q t| (1 - t) d_q t \right] \\ & = \frac{q\eta(x_2, x_1)}{[2]_q} \left[|{}^{x_2}D_q\phi(x_1)| \frac{q(1 + 4q + q^2)}{[3]_q [2]_q^3} + |{}^{x_2}D_q\phi(x_2)| \frac{q(1 + 3q^2 + 2q^3)}{[3]_q [2]_q^3} \right] \end{aligned}$$

which ends the proof. \square

Remark 7. If we consider $q \rightarrow 1^-$ and $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Theorem 7, then Theorem 7 reduces to Theorem 2.2 of [52].

Corollary 4. In Theorem 7, if we take the limit $q \rightarrow 1^-$, then we obtain the succeeding inequality:

$$\left| \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + \phi(\varkappa_2)}{2} - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) d\varkappa \right| \leq \frac{\eta(\varkappa_2, \varkappa_1)}{8} [|\phi'(\varkappa_1)| + |\phi'(\varkappa_2)|].$$

Remark 8. In Theorem 7, if we consider $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) = \varkappa_2 - \varkappa_1$, then Theorem 7 transforms into Theorem 3 of [51].

Theorem 8. Assume that the conditions of Lemma 2 hold. If $|\varkappa_2 D_q \phi|^{p_1}$, $p_1 \geq 1$, is a preinvex function on I , then we obtain the succeeding inequality:

$$\begin{aligned} & \left| \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q} - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \right| \\ & \leq \frac{q\eta(\varkappa_2, \varkappa_1)}{[2]_q} \left(\frac{q(2 + q + q^3)}{[2]_q^3} \right)^{1 - \frac{1}{p_1}} \\ & \quad \times \left(\frac{q(1 + 4q + q^2)}{[3]_q [2]_q^3} |\varkappa_2 D_q \phi(\varkappa_1)|^{p_1} + \frac{q^2(1 + 3q^2 + 2q^3)}{[3]_q [2]_q^3} |\varkappa_2 D_q \phi(\varkappa_2)|^{p_1} \right)^{\frac{1}{p_1}} \end{aligned}$$

where $0 < q < 1$.

Proof. From the integrals in the right side of inequality (19) and considering the quantum integral inequality of power mean, we have the following:

$$\begin{aligned} & \left| \frac{\phi(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + q\phi(\varkappa_2)}{[2]_q} - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa_2 d_q \varkappa \right| \\ & \leq \frac{q\eta(\varkappa_2, \varkappa_1)}{[2]_q} \left(\int_0^1 |1 - [2]_q t| d_q t \right)^{1 - \frac{1}{p_1}} \left(\int_0^1 |1 - [2]_q t| |\varkappa_2 D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1} d_q t \right)^{\frac{1}{p_1}}. \end{aligned}$$

Since $|\varkappa_2 D_q \phi|^{p_1}$ is preinvex function, we have the following:

$$\begin{aligned} & \int_0^1 |1 - [2]_q t| |\varkappa_2 D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1} d_q t \\ & \leq \int_0^1 |1 - [2]_q t| \left[t |\varkappa_2 D_q \phi(\varkappa_1)|^{p_1} + (1 - t) |\varkappa_2 D_q \phi(\varkappa_2)|^{p_1} \right] d_q t \\ & = \frac{q(1 + 4q + q^2)}{[3]_q [2]_q^3} |\varkappa_2 D_q \phi(\varkappa_1)|^{p_1} + \frac{q^2(1 + 3q^2 + 2q^3)}{[3]_q [2]_q^3} |\varkappa_2 D_q \phi(\varkappa_2)|^{p_1}. \end{aligned}$$

We also can observe the following:

$$\int_0^1 |1 - [2]_q t| d_q t = \frac{q(2 + q + q^3)}{[2]_q^3}.$$

This ends the proof. \square

Remark 9. If we consider $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) = \varkappa_2 - \varkappa_1$ in Theorem 8, then Theorem 8 becomes Theorem 4 of [51].

5. New Midpoint Type Inequalities for q^{\varkappa_2} -Integrals

In this section, the midpoint estimates of q -Hermite–Hadamard inequalities proved in Theorem 5 are discussed.

Let us proceed with the succeeding identity, which is needed to establish the key results of this section.

Lemma 3. Let $\phi : I = [\varkappa_2 + \eta(\varkappa_1, \varkappa_2), \varkappa_2] \rightarrow \mathbb{R}$ be a q^{\varkappa_2} -differentiable function on I° and ${}^{\varkappa_2}D_q\phi$ be a continuous and integrable function on I . Then, we obtain the succeeding identity:

$$\begin{aligned}
 & q\eta(\varkappa_2, \varkappa_1) \left[\int_0^{\frac{1}{[2]_q}} t \, {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{1}{q}\right) {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt \right] \quad (20) \\
 &= \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(x) \, {}^{\varkappa_2}d_qx - \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q\varkappa_2}{[2]_q}\right)
 \end{aligned}$$

where $0 < q < 1$.

Proof. From the left side of equality (20) and fundamental properties of quantum integrals, we have the following:

$$\begin{aligned}
 & q\eta(\varkappa_2, \varkappa_1) \left[\int_0^{\frac{1}{[2]_q}} t \, {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{1}{q}\right) {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt \right] \quad (21) \\
 &= q\eta(\varkappa_2, \varkappa_1) \left[\int_0^1 t \, {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt - \frac{1}{q} \int_0^1 {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt \right. \\
 &\quad \left. + \frac{1}{q} \int_0^{\frac{1}{[2]_q}} {}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))d_qt \right] \\
 &= q\eta(\varkappa_2, \varkappa_1)[I_1 - I_2 + I_3].
 \end{aligned}$$

By the equality (15) and Definition 6, we have the following:

$$\begin{aligned}
 I_1 &= \int_0^1 \frac{\phi(\varkappa_2 + qt\eta(\varkappa_1, \varkappa_2)) - \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))}{(1 - q)\eta(\varkappa_2, \varkappa_1)}d_qt \\
 &= \frac{1}{\eta(\varkappa_2, \varkappa_1)} \sum_{k=0}^{\infty} q^k \phi\left(q^{k+1}(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1 - q^{k+1})\varkappa_2\right) \\
 &\quad - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \sum_{k=0}^{\infty} q^k \phi\left(q^k(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1 - q^k)\varkappa_2\right) \\
 &= \frac{1}{q\eta(\varkappa_2, \varkappa_1)} \sum_{k=1}^{\infty} q^k \phi\left(q^k(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1 - q^k)\varkappa_2\right) \\
 &\quad - \frac{1}{\eta(\varkappa_2, \varkappa_1)} \sum_{k=0}^{\infty} q^k \phi\left(q^k(\varkappa_2 + \eta(\varkappa_1, \varkappa_2)) + (1 - q^k)\varkappa_2\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\eta(x_2, x_1)} \left(\frac{1}{q} - 1\right) \sum_{k=1}^{\infty} q^k \phi\left(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2\right) - \frac{\phi(x_2 + \eta(x_1, x_2))}{q\eta(x_2, x_1)} \\
 &= \frac{1}{q\eta(x_2, x_1)} \int_{x_2 + \eta(x_1, x_2)}^{x_2} \phi(x) x_2 d_q t - \frac{\phi(x_2 + \eta(x_1, x_2))}{q\eta(x_2, x_1)}.
 \end{aligned}$$

Similarly, we obtain the following:

$$\begin{aligned}
 I_2 &= \frac{1}{q} \int_0^1 \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)\eta(x_2, x_1)t} d_q t \\
 &= \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi\left(q^{k+1}(x_2 + \eta(x_1, x_2)) + (1 - q^{k+1})x_2\right) \\
 &\quad - \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi\left(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2\right) \\
 &= \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \left[\begin{aligned} &\phi\left(q^{k+1}(x_2 + \eta(x_1, x_2)) + (1 - q^{k+1})x_2\right) \\ &- \phi\left(q^k(x_2 + \eta(x_1, x_2)) + (1 - q^k)x_2\right) \end{aligned} \right] \\
 &= \frac{1}{q\eta(x_2, x_1)} [\phi(x_2) - \phi(x_2 + \eta(x_1, x_2))]
 \end{aligned}$$

and

$$\begin{aligned}
 I_3 &= \frac{1}{q} \int_0^{\frac{1}{[2]_q}} \frac{\phi(x_2 + qt\eta(x_1, x_2)) - \phi(x_2 + t\eta(x_1, x_2))}{(1 - q)\eta(x_2, x_1)t} d_q t \tag{22} \\
 &= \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi\left(\frac{q^{k+1}}{[2]_q}(x_2 + \eta(x_1, x_2)) + \left(1 - \frac{q^{k+1}}{[2]_q}\right)x_2\right) \\
 &\quad - \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \phi\left(\frac{q^k}{[2]_q}(x_2 + \eta(x_1, x_2)) + \left(1 - \frac{q^k}{[2]_q}\right)x_2\right) \\
 &= \frac{1}{q\eta(x_2, x_1)} \sum_{k=0}^{\infty} \left[\begin{aligned} &\phi\left(\frac{q^{k+1}}{[2]_q}(x_2 + \eta(x_1, x_2)) + \left(1 - \frac{q^{k+1}}{[2]_q}\right)x_2\right) \\ &- \phi\left(\frac{q^k}{[2]_q}(x_2 + \eta(x_1, x_2)) + \left(1 - \frac{q^k}{[2]_q}\right)x_2\right) \end{aligned} \right] \\
 &= \frac{1}{q\eta(x_2, x_1)} \left[\phi(x_2) - \phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) \right].
 \end{aligned}$$

Substituting the computed values of $(I_1)-(I_3)$ in identity (21), and later multiplying both sides of the resultant one by $q\eta(x_2, x_1)$, the identity (20) can be captured. \square

Remark 10. If we take the limit $q \rightarrow 1^-$ and $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Lemma 3, then Lemma 3 reduces to Lemma 2.1 of [53].

Corollary 5. If we consider the limit $q \rightarrow 1^-$ in Lemma 3, then we have the following identity:

$$\eta(x_2, x_1) \left[\int_0^{\frac{1}{2}} t\phi'(x_2 + t\eta(x_1, x_2))dt + \int_{\frac{1}{2}}^1 (t - 1)\phi'(x_2 + t\eta(x_1, x_2))dt \right]$$

$$= \frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) dx - \phi\left(\frac{\eta(x_1, x_2) + 2x_2}{2}\right).$$

Remark 11. If we consider $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Lemma 3, then Lemma 3 reduces to Lemma 2 of [51].

Theorem 9. We assume that the conditions of Lemma 3 hold. If $|{}^{x_2}D_q\phi|$ is preinvex function on I , then we have the following inequality:

$$\begin{aligned} & \left| \frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x - \phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) \right| \\ & \leq q\eta(x_2, x_1) \left[|{}^{x_2}D_q\phi(x_1)| \frac{3}{[2]_q^3 [3]_q} + |{}^{x_2}D_q\phi(x_2)| \frac{-1 + 2q + 2q^2}{[2]_q^3 [3]_q} \right] \end{aligned}$$

where $0 < q < 1$.

Proof. On taking modulus in Lemma 3 and from characteristics of the modulus, we have the following:

$$\begin{aligned} & \left| \frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x - \phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) \right| \tag{23} \\ & \leq q\eta(x_2, x_1) \left[\int_0^{\frac{1}{[2]_q}} t |{}^{x_2}D_q \phi(x_2 + t\eta(x_1, x_2))| d_q t \right. \\ & \quad \left. + \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) |{}^{x_2}D_q \phi(x_2 + t\eta(x_1, x_2))| d_q t \right]. \end{aligned}$$

Using the preinvexity of $|{}^{x_2}D_q\phi|$, we obtain the following:

$$\begin{aligned} & \left| \frac{1}{\eta(x_2, x_1)} \int_{x_2+\eta(x_1, x_2)}^{x_2} \phi(x) {}^{x_2}d_q x - \phi\left(\frac{\eta(x_1, x_2) + [2]_q x_2}{[2]_q}\right) \right| \\ & \leq q\eta(x_2, x_1) \left[\int_0^{\frac{1}{[2]_q}} t [t |{}^{x_2}D_q\phi(x_1)| + (1-t) |{}^{x_2}D_q\phi(x_2)|] d_q t \right. \\ & \quad \left. + \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) [t |{}^{x_2}D_q\phi(x_1)| + (1-t) |{}^{x_2}D_q\phi(x_2)|] d_q t \right] \\ & = q\eta(x_2, x_1) \left[|{}^{x_2}D_q\phi(x_1)| \left(\int_0^{\frac{1}{[2]_q}} t^2 d_q t + \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) t d_q t \right) \right. \end{aligned}$$

$$+ |{}^{\varkappa_2}D_q\phi(\varkappa_2)| \left(\int_0^{\frac{1}{[2]_q}} t(1-t)d_qt + \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right)(1-t)d_qt \right) \Bigg].$$

It can be easily shown that the following holds:

$$\int_0^{\frac{1}{[2]_q}} t^2d_qt = \frac{1}{[2]_q^3[3]_q},$$

$$\int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right)t = \frac{2}{[2]_q^3[3]_q},$$

$$\int_0^{\frac{1}{[2]_q}} t(1-t)d_qt = \frac{q}{[2]_q^2[3]_q}$$

and

$$\int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right)(1-t)d_qt = \frac{-1 + q + q^2}{[2]_q^3[3]_q}.$$

By these equalities, the proof is finished. \square

Remark 12. If we assume that the limit $q \rightarrow 1^-$ and $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) = \varkappa_2 - \varkappa_1$ in Theorem 9, then Theorem 9 reduces to Theorem 2.2 of [53].

Corollary 6. In Theorem 9, if we assume the limit $q \rightarrow 1^-$, then we obtain the succeeding inequality:

$$\left| \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa)d\varkappa - \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + 2\varkappa_2}{2}\right) \right| \leq \frac{\eta(\varkappa_2, \varkappa_1)}{8} [|\phi'(\varkappa_1)| + |\phi'(\varkappa_2)|].$$

Remark 13. If we consider $\eta(\varkappa_2, \varkappa_1) = -\eta(\varkappa_1, \varkappa_2) = \varkappa_2 - \varkappa_1$ in Theorem 9, then Theorem 9 becomes Theorem 5 of [51].

Theorem 10. Assume that the conditions of Lemma 3 hold. If $|{}^{\varkappa_2}D_q\phi|^{p_1}$, $p_1 \geq 1$, is a preinvex function on I , then we obtain the succeeding inequality:

$$\begin{aligned} & \left| \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2 + \eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) \varkappa d_q\varkappa - \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q\varkappa_2}{[2]_q}\right) \right| \\ & \leq q\eta(\varkappa_2, \varkappa_1) \left(\frac{1}{[2]_q^3}\right)^{1-\frac{1}{p_1}} \\ & \quad \times \left[\left(\frac{1}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_1)|^{p_1} + \frac{q}{[2]_q^2[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_2)|^{p_1} \right)^{\frac{1}{p_1}} \right] \end{aligned}$$

$$\left(\frac{2}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_1)|^{p_1} + \frac{-1+q+q^2}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_2)|^{p_1} \right)^{\frac{1}{p_1}}$$

where $0 < q < 1$.

Proof. From the integrals in the right side of inequality (23) and using the quantum integral inequality of power mean, we have the following:

$$\begin{aligned} & \left| \frac{1}{\eta(\varkappa_2, \varkappa_1)} \int_{\varkappa_2+\eta(\varkappa_1, \varkappa_2)}^{\varkappa_2} \phi(\varkappa) {}^{\varkappa_2}d_q\varkappa - \phi\left(\frac{\eta(\varkappa_1, \varkappa_2) + [2]_q\varkappa_2}{[2]_q}\right) \right| \\ & \leq q\eta(\varkappa_2, \varkappa_1) \left[\left(\int_0^{\frac{1}{[2]_q}} td_qt \right)^{1-\frac{1}{p_1}} \left(\int_0^{\frac{1}{[2]_q}} t |{}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1}d_qt \right)^{\frac{1}{p_1}} \right. \\ & \quad \left. \left(\int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) d_qt \right)^{1-\frac{1}{p_1}} \left(\int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) |{}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1}d_qt \right)^{\frac{1}{p_1}} \right]. \end{aligned}$$

Applying the preinvexity of $|{}^{\varkappa_2}D_q\phi|^{p_1}$, we have the following:

$$\begin{aligned} & \int_0^{\frac{1}{[2]_q}} t |{}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1}d_qt \\ & \leq \int_0^{\frac{1}{[2]_q}} t \left[t |{}^{\varkappa_2}D_q\phi(\varkappa_1)|^{p_1} + (1-t) |{}^{\varkappa_2}D_q\phi(\varkappa_2)|^{p_1} \right] d_qt \\ & = \frac{1}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_1)|^{p_1} + \frac{q}{[2]_q^2[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_2)|^{p_1}, \end{aligned}$$

and similarly, we have the following:

$$\begin{aligned} & \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) |{}^{\varkappa_2}D_q \phi(\varkappa_2 + t\eta(\varkappa_1, \varkappa_2))|^{p_1}d_qt \\ & \leq \frac{2}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_1)|^{p_1} + \frac{-1+q+q^2}{[2]_q^3[3]_q} |{}^{\varkappa_2}D_q\phi(\varkappa_2)|^{p_1}. \end{aligned}$$

Moreover, we can see that the following holds:

$$\int_0^{\frac{1}{[2]_q}} td_qt = \frac{1}{[2]_q^3} = \int_{\frac{1}{[2]_q}}^1 \left(\frac{1}{q} - t\right) d_qt.$$

This completes the proof. \square

Remark 14. If we assume $\eta(x_2, x_1) = -\eta(x_1, x_2) = x_2 - x_1$ in Theorem 10, then Theorem 10 becomes Theorem 6 of [51].

6. New Midpoint Type Inequalities for q_{x_1} -Integrals

In this section, we prove the midpoint estimates of q -Hermite–Hadamard inequalities proved in Theorem 6.

Let us begin with the succeeding identity, which is needed to offer the key results of this section.

Lemma 4. Let $\phi : I = [x_1, x_1 + \eta(x_2, x_1)] \rightarrow \mathbb{R}$ be a q_{x_1} -differentiable function on I° and ${}_{x_1}D_q\phi$ be a continuous and integrable function on I . Then, we obtain the succeeding identity:

$$\begin{aligned}
 & q\eta(x_2, x_1) \left[\int_0^{\frac{1}{[2]_q}} t {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{1}{q}\right) {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt \right] \quad (24) \\
 &= \phi\left(\frac{[2]_q x_1 + \eta(x_2, x_1)}{[2]_q}\right) - \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) {}_{x_1}d_qx
 \end{aligned}$$

where $0 < q < 1$.

Proof. Considering the Definition 3 of q_{x_1} -derivative, we have the following:

$${}_{x_1}D_q\phi(x_1 + t\eta(x_2, x_1)) = \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{(1 - q)t\eta(x_2, x_1)}. \quad (25)$$

By the right side in the identity (24) and fundamental properties of the quantum integrals, we have the following:

$$\begin{aligned}
 & q\eta(x_2, x_1) \left[\int_0^{\frac{1}{[2]_q}} t {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{1}{q}\right) {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt \right] \quad (26) \\
 &= q\eta(x_2, x_1) \left[\int_0^1 t {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt - \frac{1}{q} \int_0^1 {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt \right. \\
 & \quad \left. + \frac{1}{q} \int_0^{\frac{1}{[2]_q}} {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt \right].
 \end{aligned}$$

From (25) and (26), we obtain the following:

$$\begin{aligned}
 & q\eta(x_2, x_1) \left[\int_0^{\frac{1}{[2]_q}} t {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt + \int_{\frac{1}{[2]_q}}^1 \left(t - \frac{1}{q}\right) {}_{x_1}D_q \phi(x_1 + t\eta(x_2, x_1))d_qt \right] \\
 &= q\eta(x_2, x_1) \int_0^1 t \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{(1 - q)t\eta(x_2, x_1)} d_qt
 \end{aligned}$$

$$\begin{aligned}
 & -\eta(x_2, x_1) \int_0^1 \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{(1-q)t\eta(x_2, x_1)} d_q t \\
 & + \eta(x_2, x_1) \int_0^{[\frac{1}{2}]_q} \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{(1-q)t\eta(x_2, x_1)} d_q t \\
 = & \frac{q}{1-q} \int_0^1 \phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1)) d_q t \\
 & - \frac{1}{1-q} \int_0^1 \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{t} d_q t \\
 & + \frac{1}{1-q} \int_0^{[\frac{1}{2}]_q} \frac{\phi(x_1 + t\eta(x_2, x_1)) - \phi(x_1 + qt\eta(x_2, x_1))}{t} d_q t \\
 = & I_1 - I_2 + I_3.
 \end{aligned}$$

By the equality (5) and Definition 5, we have the following:

$$\begin{aligned}
 I_1 & = q \left[\sum_{k=0}^{\infty} q^k \phi(x_1 + q^k \eta(x_2, x_1)) - \sum_{k=0}^{\infty} q^k \phi(x_1 + q^{k+1} \eta(x_2, x_1)) \right] \\
 & = q \left[\frac{1}{q} \phi(x_1 + \eta(x_2, x_1)) - \left(\frac{1}{q} - 1 \right) \sum_{k=0}^{\infty} q^k \phi(x_1 + q^k \eta(x_2, x_1)) \right] \\
 & = \phi(x_1 + \eta(x_2, x_1)) - \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) d_q x.
 \end{aligned}$$

Similarly, we have the following:

$$I_2 = \phi(x_1) - \phi(x_1 + \eta(x_2, x_1))$$

and

$$I_3 = \phi\left(\frac{[2]_q x_1 + \eta(x_2, x_1)}{[2]_q}\right) - \phi(x_1).$$

By these equalities, the proof is finished. \square

Remark 15. If we take the limit $q \rightarrow 1^-$ and $\eta(x_2, x_1) = x_2 - x_1$ in Lemma 4, then Lemma 4 transforms into Lemma 2.1 of [53].

Remark 16. In Lemma 4, if we consider the limit $q \rightarrow 1^-$, then we obtain the succeeding identity:

$$\begin{aligned}
 & \eta(x_2, x_1) \left[\int_0^{\frac{1}{2}} t \phi'(x_1 + t\eta(x_2, x_1)) dt + \int_{\frac{1}{2}}^1 (t-1) \phi'(x_1 + t\eta(x_2, x_1)) dt \right] \\
 = & \phi\left(\frac{2x_1 + \eta(x_2, x_1)}{2}\right) - \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) dx
 \end{aligned}$$

which can be found in Theorem 5 of [54].

Remark 17. If we consider $\eta(x_2, x_1) = x_2 - x_1$ in Lemma 4, then Lemma 4 reduces to Lemma 11 of [29].

Theorem 11. Assume that the conditions of Lemma 4 hold. If $|_{x_1}D_q\phi|$ is preinvex function on I , then we obtain the succeeding inequality:

$$\left| \phi\left(\frac{[2]_q x_1 + \eta(x_2, x_1)}{[2]_q}\right) - \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) _{x_1}d_q x \right| \tag{27}$$

$$\leq q\eta(x_2, x_1) \left[|_{x_1}D_q\phi(x_2)| \frac{3}{[2]_q^3 [3]_q} + |_{x_1}D_q\phi(x_1)| \frac{-1 + 2q + 2q^2}{[2]_q^3 [3]_q} \right],$$

where $0 < q < 1$.

Proof. By the strategies that were applied in the proof of Theorem 9 by considering Lemma 4, the desired inequality (27) can be proved. \square

Remark 18. If we take the limit $q \rightarrow 1^-$ and $\eta(x_2, x_1) = x_2 - x_1$ in Theorem 11, then Theorem 11 reduces to Theorem 2.2 of [53].

Remark 19. If we assume the limit $q \rightarrow 1^-$ in Theorem 11, then Theorem 11 becomes Theorem 5 of [54].

Remark 20. If we consider $\eta(x_2, x_1) = x_2 - x_1$ in Theorem 11, then Theorem 11 becomes Theorem 13 of [29].

Theorem 12. Assume that the conditions of Lemma 4 hold. If $|_{x_1}D_q\phi|^{p_1}$, $p_1 \geq 1$, is a preinvex function on I , then we obtain the succeeding inequality:

$$\left| \phi\left(\frac{[2]_q x_1 + \eta(x_2, x_1)}{[2]_q}\right) - \frac{1}{\eta(x_2, x_1)} \int_{x_1}^{x_1 + \eta(x_2, x_1)} \phi(x) _{x_1}d_q x \right|$$

$$\leq q\eta(x_2, x_1) \left(\frac{1}{[2]_q^3}\right)^{1 - \frac{1}{p_1}}$$

$$\times \left[\left(\frac{1}{[2]_q^3 [3]_q} |_{x_1}D_q\phi(x_2)|^{p_1} + \frac{q}{[2]_q^2 [3]_q} |_{x_1}D_q\phi(x_1)|^{p_1}\right)^{\frac{1}{p_1}} \right.$$

$$\left. \left(\frac{2}{[2]_q^3 [3]_q} |_{x_1}D_q\phi(x_2)|^{p_1} + \frac{-1 + q + q^2}{[2]_q^3 [3]_q} |_{x_1}D_q\phi(x_1)|^{p_1}\right)^{\frac{1}{p_1}} \right]$$

where $0 < q < 1$.

Proof. The proof follows on the same directions given in the proof of Theorem 10 by considering the Lemma 4. \square

Remark 21. If we assume $\eta(x_2, x_1) = x_2 - x_1$ in Theorem 12, then Theorem 12 becomes Theorem 16 of [29].

Remark 22. If we consider the limit $q \rightarrow 1^-$ in Theorem 12, then Theorem 12 reduces to Theorem 8 of [54].

7. Conclusions

In this research, we proved Hermite–Hadamard inequalities for preinvex functions, using quantum integrals. We derived some new inequalities of midpoint and trapezoidal types for quantum differentiable preinvex functions, using quantum integrals. Moreover, we revealed that the findings presented in this work are a strong generalization of similar conclusions in the literature. It is a very interesting and new problem for which future researchers can prove similar inequalities for different kinds of convexities in their new work.

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