(k,s)-Riemann-Liouville fractional integral and applications

Mehmet Zeki SARIKAYA*† , Zoubir DAHMANI‡, Mehmet Eyüp KIRIS§ and Farooq AHMAD¶

Abstract

In this paper, we introduce a new approach on fractional integration, which generalizes the Riemann-Liouville fractional integral. We prove some properties for this new approach. We also establish some new integral inequalities using this new fractional integration.

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1. Introduction

Fractional calculus and its widely application have recently been paid more and more attentions. For more recent development on fractional calculus, we refer the reader to [7, 12, 15, 16, 19]. There are several known forms of the fractional integrals of which two have been studied extensively for their applications [5, 10, 11, 14, 21]. The first is the Riemann-Liouville fractional integral of $\alpha \geq 0$ for a continuous function f on $[a,b]$ which is defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \geq 0, \; a < x \leq b.$$

This integral is motivated by the well known Cauchy formula:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 ... \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt, n \in \mathbb{N}^*.$$

The second is the Hadamard fractional integral introduced by Hadamard [9]. It is given by:

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad \alpha > 0, \; x > a.$$

*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, Email: sarikayamz@gmail.com

†Corresponding Author.

‡Laboratory of Pure and Applied Mathematics, UMAB, University of Mostaganem, Algeria, Email: zzdahmani@yahoo.fr

§Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon-TURKEY, Email: mkiris@gmail.com, kiris@aku.edu.tr

¶Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, 60800, Pakistan, Email: farooqgujar@gmail.com

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1}\int_a^{t_1}\frac{dt_2}{t_2}...\int_a^{t_{n-1}}\frac{f(t_n)}{t_n}dt_n = \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-1} f(t)\frac{dt}{t}$$

for $n \in \mathbb{N}^*$.
In [10], Katugampola gave a new fractional integration which generalizes both the Riemann-Liouville and Hadamard fractional integrals into a single form. This generalization is based on the observation that, for $n \in \mathbb{N}^*$,

$$\int_a^x t_1^s dt_1 \int_a^{t_1} t_2^s dt_2 ... \int_a^{t_{n-1}} t_n^s f(t_n)dt_n = \frac{(s+1)^{1-n}}{\Gamma(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{n-1} t^s f(t)dt,$$

which gives the following fractional version

$${}^sJ_a^\alpha f(x) = \frac{(s+1)^{1-n}}{\Gamma(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\alpha-1} t^s f(t)dt,$$

where α and $s \neq -1$ are real numbers.
Recently, in [6], Diaz and Pariguan have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ k>0,$$

where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt, x>0.$$

Clearly, $\Gamma(x) = \lim_{k\to 1}\Gamma_k(x)$, $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})$ and $\Gamma_k(x+k) = x\Gamma_k(x)$. Furthermore, k-beta function is defined as follows

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt,$$

so that $B_k(x,y) = \frac{1}{k}B(\frac{x}{k},\frac{y}{k})$ and $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.
Later, under the above definitions, in [13], Mubeen and Habibullah have introduced the k-fractional integral of the Riemann-Liouville type as follows:

$${}_kJ^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt,\quad \alpha>0,\ x>0.$$

Note that when $k\to 1$, then it reduces to the classical Riemann-liouville fractional integral.

2. (k,s)-Riemann-Liouville fractional integral

In this section, we present the (k,s) fractional integration which generalizes all of the above Riemann-Liouville fractional integrals as follows:

2.1. Definition. Let f be a continuous function on on a the finite real interval $[a,b]$. Then (k,s)-Riemann-Liouville fractional integral of f of order $\alpha > 0$ is defined by:

$$ {}_k^sJ_a^\alpha f(x) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s f(t)dt, \quad x \in [a,b], \tag{2.1}$$

where $k > 0, s \in \mathbb{R}\backslash\{-1\}$.

In the following theorem, we prove that the (k,s) fractional integral is well defined:

2.2. Theorem. *Let $f \in L_1[a,b], s \in \mathbb{R}\backslash\{-1\}$ and $k > 0$. Then ${}_k^sJ_a^\alpha f(x)$ exists for any $x \in [a,b], \alpha > 0$.*

Proof. Let $\Delta := [a,b] \times [a,b]$ and $P : \Delta \to \mathbb{R}$; $P(x,t) = \left[\left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s\right]$. It clear to see that $P = P_+ + P_-$, where

$$P_+(x,t) := \begin{cases} \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s & , a \leq t \leq x \leq b \\ 0 & , a \leq x \leq t \leq b \end{cases}$$

and

$$P_-(x,t) := \begin{cases} \left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s & , a \leq t \leq x \leq b \\ 0 & , a \leq x \leq t \leq b. \end{cases}$$

Since P is measurable on Δ, then we can write

$$\int_a^b P(x,t)dt = \int_a^x P(x,t)dt = \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s dt = \frac{k}{\alpha}\left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}}.$$

By using the repeated integral, we obtain

$$\begin{aligned} \int_a^b \left(\int_a^b P(x,t)\left|f(x)\right| dt\right) dx &= \int_a^b \left|f(x)\right| \left(\int_a^b P(x,t)dt\right) dx \\ &= \frac{k}{\alpha}\int_a^b \left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} \left|f(x)\right| dx \\ &\leq \frac{k}{\alpha}\left(b^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} \int_a^b \left|f(x)\right| dx. \end{aligned}$$

That is

$$\begin{aligned} \int_a^b \left(\int_a^b P(x,t)\left|f(x)\right| dt\right) dx &= \int_a^b \left|f(x)\right| \left(\int_a^b P(x,t)dt\right) dx \\ &\leq \frac{k}{\alpha}\left(b^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} \left\|f(x)\right\|_{L_1[a,b]} < \infty. \end{aligned}$$

Therefore, the function $Q : \Delta \to \mathbb{R}$; $Q(x,t) := P(x,t)f(x)$ is integrable over Δ by Tonelli's theorem. Hence, by Fubini's theorem $\int_a^b P(x,t)f(x)dx$ is an integrable function on $[a,b]$, as a function of $t \in [a,b]$. That is, ${}_k^sJ_a^\alpha f(x)$ exists. □

Now, we prove the commutativity and the semigroup properties of the (k,s)-Riemann-Liouville fractional integral. We have:

2.3. Theorem. *Let f be continuous on $[a,b], k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then,*

$$ {}^s_kJ^{\alpha}_a\left[\ {}^s_kJ^{\beta}_a f(x)\right] = {}^s_kJ^{\alpha+\beta}_a f(x) = {}^s_kJ^{\beta}_a\left[\ {}^s_kJ^{\alpha}_a f(x)\right], $$

for all $\alpha>0, \beta>0, x\in[a,b]$.

Proof. Thanks to Definition 1 and by Dirichlet's formula, we have

$$ {}^s_kJ^{\alpha}_a\left[\ {}^s_kJ^{\beta}_a f(x)\right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s\ {}^s_kJ^{\beta}_a f(t)dt $$

$$ = \tfrac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s\left[\tfrac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau\right]dt. $$

$$ = \tfrac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s\left[\tfrac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s f(\tau)d\tau\right]dt. $$

That is

(2.2)
$$ {}^s_kJ^{\alpha}_a\left[\ {}^s_kJ^{\beta}_a f(x)\right] = \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^x \tau^s f(\tau)\left[\int_\tau^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}dt\right]d\tau. $$

Using the change of variable $y=\left(t^{s+1}-\tau^{s+1}\right)/\left(x^{s+1}-\tau^{s+1}\right)$, we can write

(2.3)
$$ \int_\tau^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}\left(t^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}t^s dt = \tfrac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy $$

$$ = \tfrac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}\int_0^1(1-y)^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}dy \qquad = \tfrac{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{s+1}kB_k(\alpha,\beta). $$

According to the k-beta function and by (2.2) and (2.3), we obtain

$$ \begin{aligned} {}^s_kJ^{\alpha}_a\left[\ {}^s_kJ^{\beta}_a f(x)\right] &= \frac{(s+1)^{1-\frac{\alpha+\beta}{k}}}{k\Gamma_k(\alpha+\beta)}\int_a^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}\tau^s f(\tau)d\tau \\ &= {}^s_kJ^{\alpha+\beta}_a f(x). \end{aligned} $$

This completes the proof of the Theorem 2.3. □

2.4. Theorem. *Let $\alpha,\beta>0, k>0$ and $s\in\mathbb{R}\backslash\{-1\}$. Then, we have*

(2.4)
$$ {}^s_kJ^{\alpha}_a\left[\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1}\right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}\left(x^{s+1}-a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}, $$

where Γ_k denotes the k-gamma function.

Proof. By Definition 1 and using the change of variable $y = \left(x^{s+1} - t^{s+1}\right) / \left(x^{s+1} - a^{s+1}\right); x \in]a, b]$, we get

$$\begin{aligned} {}_k^s J_a^\alpha \left[\left(x^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}-1} \right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s \left(t^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1} dt \\ &= \frac{(s+1)^{-\frac{\alpha}{k}} \left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy \\ &= \frac{\left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha+\beta}{k}-1}}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha)} B_k(\alpha, \beta). \end{aligned}$$

The case $a = x$ is trivial. The proof of the Theorem 2.4 is complete. □

2.5. Remark. $(i :)$ Taking $s = 0, k > 0$ in (2.4), we obtain

$$(2.5) \qquad {}_k J_a^\alpha \left[(x-a)^{\frac{\beta}{k}-1} \right] = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} (x-a)^{\frac{\alpha+\beta}{k}-1}.$$

$(ii :)$ The formula (2.4) for $s = 0, k = 1$ becomes

$$J_a^\alpha \left[(x-a)^{\beta-1} \right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}.$$

2.6. Corollary. *Let $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then the formula*

$$(2.6) \qquad {}_k^s J_a^\alpha (1) = \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} \left(x^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}-2}$$

is valid for any $\alpha > 0$.

2.7. Remark. $(a :)$ For $s = 0, k > 0$ in (2.6), we get

$$(2.7) \qquad {}_k J_a^\alpha (1) = \frac{1}{\Gamma_k(\alpha+k)} (x-a)^{\frac{\alpha}{k}-2}.$$

$(b :)$ For $s = 0, k = 1$ we have

$$J_a^\alpha (1) = \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha-2}.$$

3. Some new (k, s)-Riemann-Liouville fractional integral inequalities

Chebyshev inequalities can be represented in (k, s)-fractional integral forms as follows:

3.1. Theorem. *Let f, g be two synchronous on $[0, \infty)$, then for all $t > a \geq 0$, $\alpha > 0$, $\beta > 0$, the following inequalities for (k, s)-fractional integrals hold:*

$$(3.1) \qquad {}_k^s J_a^\alpha fg(t) \geq \frac{1}{J_a^\alpha (1)} \, {}_k^s J_a^\alpha f(t) \, {}_k^s J_a^\alpha g(t)$$

$$(3.2) \quad {}_k^s J_a^\alpha fg(t) \, {}_k^s J_a^\beta (1) + {}_k^s J_a^\beta fg(t) \, {}_k^s J_a^\alpha (1) \geq {}_k^s J_a^\alpha f(t) \, {}_k^s J_a^\beta g(t) + {}_k^s J_a^\alpha g(t) \, {}_k^s J_a^\beta f(t).$$

Proof. Since the functions f and g are synchronous on $[0,\infty)$, then for all $x, y \geq 0$, we have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Therefore

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x) \tag{3.3}$$

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s$, then integrating the resulting inequality with respest to x over (a,t), we obtain

$$\begin{aligned}
&\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s f(x)g(x)\,dx \\
&+\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s f(y)g(y)\,dx \\
\geq\; &\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s f(x)g(y)\,dx \\
&+\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - x^{s+1}\right)^{\frac{\alpha}{k}-1} x^s f(y)g(x)\,dx,
\end{aligned}$$

i.e.

$${}_k^sJ_a^\alpha fg(t) + f(y)g(y)\;{}_k^sJ_a^\alpha(1) \geq g(y)\;{}_k^sJ_a^\alpha f(t) + f(y)\;{}_k^sJ_a^\alpha g(t). \tag{3.4}$$

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1} y^s$, then integrating the resulting inequality with respest to y over (a,t), we obtain

$$\begin{aligned}
&{}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1} y^s dy \\
&+{}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1} y^s f(y)g(y)\,dy \\
\geq\; &{}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1} y^s g(y)dy \\
&+{}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t \left(t^{s+1} - y^{s+1}\right)^{\frac{\alpha}{k}-1} y^s f(y)\,dy,
\end{aligned}$$

that is

$${}_k^sJ_a^\alpha fg(t) \geq \frac{1}{J_a^\alpha(1)}\;{}_k^sJ_a^\alpha f(t)\;{}_k^sJ_a^\alpha g(t)$$

and the first inequality is proved.

Multiplying both sides of (3.3) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respest to y over (a,t), we obtain

$$\begin{aligned} & {}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s dy \\ & +{}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,dy \\ \geq\ & {}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)dy \\ & +{}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,dy, \end{aligned}$$

that is

$${}_k^sJ_a^\alpha fg(t)\,{}_k^sJ_a^\beta(1)+{}_k^sJ_a^\beta fg(t)\,{}_k^sJ_a^\alpha(1)\geq{}_k^sJ_a^\alpha f(t)\,{}_k^sJ_a^\beta g(t)+{}_k^sJ_a^\alpha g(t)\,{}_k^sJ_a^\beta f(t)$$

and the second inequality is proved. The proof is completed. □

3.2. Theorem. *Let f,g be two synchronous on $[0,\infty)$, $h\geq 0$, then for all $t>a\geq 0$, $\alpha>0$, $\beta>0$, the following inequalities for (k,s)-fractional integrals hold:*

$$\begin{aligned} & \frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_k^sJ_a^\alpha fgh(t) \\ & +\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^\beta fgh(t) \\ \geq\ & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\beta g(t)+\ {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\beta f(t)-\ {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\beta fg(t)-\ {}_k^sJ_a^\alpha fg(t)\ {}_k^sJ_a^\beta h(t) \\ & +\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\beta gh(t)+\ {}_k^sJ_a^\alpha g(t)\ {}_k^sJ_a^\beta fh(t). \end{aligned}$$

Proof. Since the functions f and g are synchronous on $[0,\infty)$ and $h\geq 0$, then for all $x,y\geq 0$, we have

$$(f(x)-f(y))\,(g(x)-g(y))\,(h(x)+h(y))\geq 0.$$

By opening the above, we get

$$\begin{aligned} & f(x)\,g(x)\,h(x)+f(y)\,g(y)\,h(y) \\ \geq\ & f(x)\,g(y)\,h(x)+f(y)\,g(x)\,h(x)-f(y)\,g(y)\,h(x) \\ & -f(x)\,g(x)\,h(y)+f(x)\,g(y)\,h(y)+f(y)\,g(x)\,h(y). \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s$, then integrating the resulting inequality with respest to x over (a,t), we obtain

$$\begin{aligned}
& \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(x)\,h(x)\,dx \\
& +f(y)\,g(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s dx \\
\geq\; & g(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,h(x)\,dx \\
& +f(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s g(x)\,h(x)\,dx \\
& -f(y)\,g(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s h(x)\,dx \\
& -h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,g(x)\,dx \\
& +g(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s f(x)\,dx \\
& +f(y)\,h(y)\,\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^t\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s g(x)\,dx.
\end{aligned}$$

i.e,

$$\begin{aligned}
(3.6)\qquad & {}^s_kJ^\alpha_a fgh(t)+f(y)\,g(y)\,h(y)\;{}^s_kJ^\alpha_a(1) \\
\geq\; & g(y)\;{}^s_kJ^\alpha_a fh(t)+f(y)\;{}^s_kJ^\alpha_a gh(t)-f(y)\,g(y)\;{}^s_kJ^\alpha_a h(t)-h(y)\;{}^s_kJ^\alpha_a fg(t) \\
& +g(y)\,h(y)\;{}^s_kJ^\alpha_a f(t)+f(y)\,h(y)\;{}^s_kJ^\alpha_a g(t).
\end{aligned}$$

Multiplying both sides of (3.6) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respest to y over (a,t), we obtain

$$\begin{aligned}
& {}_k^sJ_a^\alpha fgh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s dy \\
& +{}_k^sJ_a^\alpha(1)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,h(y)dy \\
\geq\ & {}_k^sJ_a^\alpha fh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)\,dy \\
& +{}_k^sJ_a^\alpha gh(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,dy \\
& -{}_k^sJ_a^\alpha h(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,g(y)\,dy \\
& -{}_k^sJ_a^\alpha fg(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s h(y)dy \\
& +{}_k^sJ_a^\alpha f(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s g(y)\,h(y)\,dy \\
& +{}_k^sJ_a^\alpha g(t)\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^t\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s f(y)\,h(y)\,dy,
\end{aligned}$$

that is

$$\begin{aligned}
{}_k^sJ_a^\alpha fgh(t){}_k^sJ_a^\beta(1)+{}_k^s J_a^\alpha(1){}_k^sJ_a^\beta fgh(t) \geq\ & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\beta g(t)+\ {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\beta f(t) \\
& -\ {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\beta fg(t)-\ {}_k^sJ_a^\alpha fg(t)\ {}_k^sJ_a^\beta h(t) \\
& +\ {}_k^sJ_a^\alpha f(t)\ {}_k^sJ_a^\beta gh(t)+\ {}_k^sJ_a^\alpha g(t)\ {}_k^sJ_a^\beta fh(t)
\end{aligned}$$

which this completes the proof. □

3.3. Corollary. *Let f, g be two synchronous on $[0,\infty)$, $h\geq 0$, then for all $t>a\geq 0$, $\alpha>0$, the following inequalities for (k,s)-fractional integrals hold:*

$$\begin{aligned}
& \frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^\alpha fgh(t) \\
\geq\ & {}_k^sJ_a^\alpha fh(t)\ {}_k^sJ_a^\alpha g(t)+\ {}_k^sJ_a^\alpha gh(t)\ {}_k^sJ_a^\alpha f(t)-\ {}_k^sJ_a^\alpha h(t)\ {}_k^sJ_a^\alpha fg(t).
\end{aligned}$$

3.4. Theorem. *Let f, g and h be three monotonic functions defined on $[0,\infty)$ satisfying the following*

$$\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\left(h(x)-h(y)\right)\geq 0$$

for all $x, y \in [a,t]$, then for all $t > a \geq 0$, $\alpha > 0$, $\beta > 0$, the following inequalities for (k,s)-fractional integrals hold:

$$\frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_k^sJ_a^{\alpha}fgh(t)$$
$$-\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^{\beta}fgh(t)$$
$$\geq\ {}_k^sJ_a^{\alpha}fh(t)\ {}_k^sJ_a^{\beta}g(t)+\ {}_k^sJ_a^{\alpha}gh(t)\ {}_k^sJ_a^{\beta}f(t)-\ {}_k^sJ_a^{\alpha}h(t)\ {}_k^sJ_a^{\beta}fg(t)+\ {}_k^sJ_a^{\alpha}fg(t)\ {}_k^sJ_a^{\beta}h(t)$$
$$-{}_k^sJ_a^{\alpha}f(t)\ {}_k^sJ_a^{\beta}gh(t)-\ {}_k^sJ_a^{\alpha}g(t)\ {}_k^sJ_a^{\beta}fh(t).$$

Proof. The proof is similar to that given in Theorem 3.2. □

3.5. Theorem. *Let f and g be two functions on $[0,\infty)$, then for all $t > a \geq 0$, $\alpha > 0$, $\beta > 0$, the following inequalities for (k,s)-fractional integrals hold:*

(3.7)
$$\frac{1}{(s+1)^{\frac{\beta}{k}}\Gamma_k(\beta+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-2}\ {}_k^sJ_a^{\alpha}f^2(t)$$
$$+\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ {}_k^sJ_a^{\beta}g^2(t)$$
$$\geq\ 2\ {}_k^sJ_a^{\alpha}f(t)\ {}_k^sJ_a^{\beta}g(t)$$

(3.8)
$${}_k^sJ_a^{\alpha}f^2(t)\ {}_k^sJ_a^{\beta}g^2(t)+\ {}_k^sJ_a^{\beta}f^2(t)\ {}_k^sJ_a^{\alpha}g^2(t)\geq 2\ {}_k^sJ_a^{\alpha}fg(t)\ {}_k^sJ_a^{\beta}fg(t).$$

Proof. Since,

$$(f(x)-g(y))^2\geq 0$$

then we have

(3.9)
$$f^2(x)+g^2(y)\geq 2f(x)g(y).$$

Multiplying both sides of (3.9) by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(t^{s+1}-x^{s+1}\right)^{\frac{\alpha}{k}-1}x^s$ and $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left(t^{s+1}-y^{s+1}\right)^{\frac{\beta}{k}-1}y^s$, then integrating the resulting inequality with respest to x and y over (a,t) respectively, we obtain (3.7).

On the other hand, since

$$(f(x)g(y)-f(y)g(x))^2\geq 0$$

then under procedures similar to the above we obtain (3.8). □

3.6. Corollary. *Let f and g be two functions on $[0,\infty)$, then for all $t > a \geq 0$, $\alpha > 0$, the following inequalities for (k,s)-fractional integrals hold:*

$$\frac{1}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\left(t^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}-2}\ \left[{}_k^sJ_a^{\alpha}f^2(t)+\ {}_k^sJ_a^{\beta}g^2(t)\right]$$
$$\geq\ 2\ {}_k^sJ_a^{\alpha}f(t)\ {}_k^sJ_a^{\alpha}g(t)$$
$${}_k^sJ_a^{\alpha}f^2(t)\ {}_k^sJ_a^{\alpha}g^2(t)\geq\ \left[{}_k^sJ_a^{\alpha}fg(t)\right]^2.$$

3.7. Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ and defined by*

$$\bar{f}(x) = \int_a^x t^s f(t)dt, \quad x > a \geq 0, \ s \in \mathbb{R}\backslash\{-1\}$$

then for $\alpha \geq k > 0$

$${}_k^s J_a^\alpha f(x) = \frac{1}{k} \, {}_k^s J_a^{\alpha-k} \bar{f}(x)$$

Proof. By definition of the (k, s)-fractional integral and by using Dirichlet's formula, we have

$$\begin{aligned}
{}_k^s J_a^\alpha \bar{f}(x) &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s \int_a^t u^s f(u)dudt \\
&= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x u^s f(u) \int_u^x \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1} t^s dtdu \\
&= \frac{(s+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \int_a^x \left(x^{s+1} - u^{s+1}\right)^{\frac{\alpha}{k}} u^s f(u)du \\
&= k \, {}_k^s J_a^{\alpha+k} f(x).
\end{aligned}$$

This completes the proof of Theorem 3.7. □

We give the generalized Cauchy-Buniakovsky-Schwarz inequality as follows:

3.8. Lemma. *Let $f, g, h : [a, b] \to (0, \infty)$ be three functions $0 \leq a < b$. Then*

(3.10)
$$\left(\int_a^b g^m(t)h^x(t)f(t)dt\right)\left(\int_a^b g^n(t)h^y(t)f(t)dt\right) \geq \left(\int_a^b g^{\frac{m+n}{2}}(t)h^{\frac{x+y}{2}}(t)f(t)dt\right)^2.$$

where m, n, x, y arbitrary real numbers.

Proof.

$$\int_a^b \left[\sqrt{g^m(t)h^x(t)f(t)}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt} - \sqrt{g^n(t)h^y(t)f(t)}\sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\right]^2 dt \geq 0$$

$$\begin{aligned}
&\int_a^b \left[g^m(t)h^x(t)f(t)\int_a^b g^n(t)h^y(t)f(t)dt + g^n(t)h^y(t)f(t)\int_a^b g^m(t)h^x(t)f(t)dt\right. \\
&\left.-2\, g^{\frac{m+n}{2}}(t)h^{\frac{x+y}{2}}(t)f(t)\sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt}\right] dt \\
\geq\ & 0
\end{aligned}$$

$$2\left(\int_a^b g^m(t)h^x(t)f(t)dt\right)\left(\int_a^b g^n(t)h^y(t)f(t)dt\right)$$
$$\geq 2\left(\int_a^b g^{\frac{m+n}{2}}(t)h^{\frac{x+y}{2}}(t)f(t)dt\right)\sqrt{\int_a^b g^m(t)h^x(t)f(t)dt}\sqrt{\int_a^b g^n(t)h^y(t)f(t)dt}$$

which this give the requair inequality. □

3.9. Theorem. *Let $f \in L_1[a,b]$. Then*

$$\left({}_k^sJ_a^{m(\frac{\alpha}{k}-1)+1}f^r(x)\right)\left({}_k^sJ_a^{n(\frac{\alpha}{k}-1)+1}f^p(x)\right) \geq \left({}_k^sJ_a^{\frac{m+n}{2}(\frac{\alpha}{k}-1)+1}f^{\frac{r+p}{2}}(x)\right)^2 \quad (3.11)$$

for $k, m, n, r, p > 0$ and $\alpha > 1$.

Proof. By taking $g(t) = \left(x^{s+1} - t^{s+1}\right)^{\frac{\alpha}{k}-1}$, $f(t) = \frac{t^s(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $h(t) = f(t)$ in (3.10), we obtain

$$\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{m(\frac{\alpha}{k}-1)} t^s f^r(t)dt\right)\left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{n(\frac{\alpha}{k}-1)} t^s f^p(t)dt\right)$$
$$\geq \left(\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{m+n}{2}(\frac{\alpha}{k}-1)} t^s f^{\frac{r+p}{2}}(t)dt\right)^2$$

which can be written as (3.11). □

3.10. Remark. For $k = 1$ in (3.11), we get the following inequalities

$$\left(J_a^{m(\alpha-1)+1}f^r(x)\right)\left({}_kJ_a^{n(\alpha-1)+1}f^s(x)\right) \geq \left({}_kJ_a^{\frac{m+n}{2}(\alpha-1)+1}f^{\frac{r+s}{2}}(x)\right)^2.$$

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