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New bounds for the Ostrowski-type inequalities via conformable fractional calculus

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Abstract In this paper, we have introduced the new upper bounds for Ostrowski-type integral inequalities by using conformable fractional integral. In accordance with this purpose, we have benefited from the Taylor expansion for conformable fractional derivatives which was introduced by Anderson.

Mathematics Subject Classification 26D15 · 26A33 · 41A58 · 41A55 · 65D30

المخلص

في هذا المقال، قدمنا حدوداً عظمى جديدة لمراجعات تكاملية من نوع أوستروفسكي وذلك باستخدام التكامل الكسري المريح. وبالتوافق مع هذا الغرض، فقد استفدنا من نشر تايلور للمشتقات الكسرية المريحة والتي قُدمت من طرف د. ر. أندرسون.

1 Introduction

In the history of development calculus, integral inequalities have been thought of as a key factor in the theory of differential and integral equations. The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of Ostrowski [13] is the following classical integral inequality associated with the differentiable mappings.

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Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \quad (1.1)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Moreover, fractional derivative and integration have a number of fields of application such as control theory, computational analysis and engineering [12], see also [14]. Thus, a number of new definitions have been introduced to provide the best method for fractional calculus. For instance, recently, a new local limit-based definition of a conformable derivative has been introduced in [1, 10], with several follow-up papers [3–7, 9, 11, 15–20].

Some authors have argued that conformable derivatives are not considered as fractional derivatives in the fractional calculus community; it is an interesting derivative that enables to derive with respect to arbitrary order but without memory effect. This question seems today to still be open, and perhaps, it is a philosophical issue. Such derivative makes it possible to generalize many mathematical concepts depending on ordinary derivatives. For instance, it contributes in generalizing certain mathematical inequalities [2]. It also contributed to of general form of Sturm–Liouville problems [21]. Conformable local-type derivatives also make it possible to obtain generalized-type fractional derivatives by iterating their corresponding integrals [22, 23]. Conformable (fractional) derivatives have the drawback that the limiting case $\alpha \rightarrow 0$ does not give us the function itself. To improve this drawback, Anderson [24] made use of proportional calculus to define better well-behaved derivatives in the limiting case, and therefore, he improved conformable (fractional) derivatives.

In this study, we present new Ostrowski-type conformable fractional integral inequalities using the rules of conformable fractional calculus and Taylor formula for conformable fractional derivatives.

This work is organized as follows: in Sect. 2, the conformable fractional derivatives and integrals are summarised. In Sect. 3, the new upper bounds for Ostrowski-type inequalities with the help of conformable fractional calculus are introduced. Application to numerical integration is given in Sect. 4, while some conclusions and further directions of research are discussed in Sect. 5.

2 Definitions and properties of conformable fractional derivative and integral

The following definitions and theorems with respect to conformable fractional derivative and integral [1, 6, 11] are summarised.

Definition 2.1 [11] (*Conformable fractional derivative*) “Given a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then, the conformable fractional derivative” of f of order α is defined by

$$D_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \quad (2.1)$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $a > 0$, $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t). \quad (2.2)$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivative of f of order α . In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α -differentiable.

Theorem 2.2 [11] Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $t > 0$. Then

- i. $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$, for all $a, b \in \mathbb{R}$,
- ii. $D_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,
- iii. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$,
- iv. $D_\alpha\left(\frac{f}{g}\right) = \frac{fD_\alpha(g) - gD_\alpha(f)}{g^2}$.



If f is differentiable, then

$$D_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \quad (2.3)$$

Definition 2.3 [11] (Conformable fractional integral) Let $\alpha \in (0, 1]$ and $0 \leq a < b$. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -fractional integrable on $[a, b]$ if the integral

$$\int_a^b f(x) d_{\alpha}x := \int_a^b f(x) x^{\alpha-1} dx \quad (2.4)$$

exists and is finite. The set of all α -integrable functions on $[a, b]$ is indicated by $L_{\alpha}^1([a, b])$.

Remark 2.4 [11]

$$I_{\alpha}^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 2.5 [11] Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$, we have

$$I_{\alpha}^a D_{\alpha}^a f(t) = f(t) - f(a). \quad (2.5)$$

We will also use the following important results, which can be derived from the results above.

Lemma 2.6 [1] Let the conformable differential operator D^{α} be given as in (1.1), where $\alpha \in (0, 1]$ and $t \geq 0$, and assume the functions f and g are α -differentiable as needed. Then

- i. $D^{\alpha}(\ln t) = t^{-\alpha}$ for $t > 0$
- ii. $D^{\alpha} \left[\int_a^t f(t, s) d_{\alpha}s \right] = f(t, t) + \int_a^t D^{\alpha} [f(t, s)] d_{\alpha}s$
- iii. $\int_a^b f(x) D^{\alpha}(g)(x) d_{\alpha}x = fg|_a^b - \int_a^b g(x) D^{\alpha}(f)(x) d_{\alpha}x$.

Theorem 2.7 [1] Let $f : [a, \infty) \rightarrow \mathbb{R}$, such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have

$$D_{\alpha}^a I_{\alpha}^a f(t) = f(t).$$

We can give the Hölder's inequality in conformable integral as follows:

Lemma 2.8 Let $f, g \in C[a, b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_a^b |f(x)g(x)| d_{\alpha}x \leq \left(\int_a^b |f(x)|^p d_{\alpha}x \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q d_{\alpha}x \right)^{\frac{1}{q}}.$$

Remark 2.9 If we take $p = q = 2$ in Lemma 2.8 then, we have the Cauchy–Schwartz inequality for conformable integral.

The following lemma and theorems are given by Anderson in [3].

Theorem 2.10 (Taylor Formula) [3] Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is $n+1$ times α -fractional differentiable on $[0, \infty)$, and $x_0, x \in [0, \infty)$. Then, we have

$$f(x) = \sum_{k=0}^n \frac{1}{k!} \left(\frac{x^{\alpha} - x_0^{\alpha}}{\alpha} \right)^k D_{\alpha}^k f(x_0) + \frac{1}{n!} \int_{x_0}^x \left(\frac{x^{\alpha} - \tau^{\alpha}}{\alpha} \right)^n D_{\alpha}^{n+1}(f)(\tau) d_{\alpha}\tau. \quad (2.6)$$

Using Taylor's Theorem, we define the remainder function by

$$R_{-1,f}(\cdot, s) := f(s),$$



and for $n > -1$,

$$\begin{aligned} R_{n,f}(t,s) &:= f(s) - \sum_{k=0}^n \frac{1}{k!} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right)^k D_\alpha^k f(s) \\ &= \frac{1}{n!} \int_s^t \left(\frac{t^\alpha - \tau^\alpha}{\alpha} \right)^n D_\alpha^{n+1} f(\tau) d_\alpha \tau. \end{aligned} \quad (2.7)$$

Lemma 2.11 [3] (Montgomery Identity) Let $a, b, t, x \in \mathbb{R}$ with $0 \leq a < b$, and let $f : [a, b] \rightarrow \mathbb{R}$ be α -fractional differentiable for $\alpha \in (0, 1]$. Then

$$f(x) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t + \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b p(x, t) D_\alpha f(t) d_\alpha t \quad (2.8)$$

where

$$p(x, t) = \begin{cases} \frac{t^\alpha - a^\alpha}{\alpha}, & a \leq t < x \\ \frac{t^\alpha - b^\alpha}{\alpha}, & x \leq t \leq b. \end{cases}$$

Theorem 2.12 [3] Let $a, b, x \in \mathbb{R}$ with $0 \leq a < b$ and let $f : [a, b] \rightarrow \mathbb{R}$ be an α -fractional differentiable function for $\alpha \in (0, 1]$. Then

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{M}{2\alpha (b^\alpha - a^\alpha)} \left[(x^\alpha - a^\alpha)^2 + (b^\alpha - x^\alpha)^2 \right], \quad (2.9)$$

where

$$M = \sup_{x \in (a, b)} |D_\alpha(f)(x)|.$$

This inequality is sharp in the sense that the right-hand side of (2.9) cannot be replaced by a smaller one.

Now, we present the main results:

3 Ostrowski-type inequalities for conformable fractional integral

Theorem 3.1 Let $f : [a, b] \rightarrow \mathbb{R}$ be an α -fractional differentiable function for $\alpha \in (0, 1]$, $D_\alpha(f) \in L_\alpha^1([a, b])$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $x \in [a, b]$, we have the following inequality:

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq A_\alpha(x, q) \|D_\alpha(f)\|_p$$

where

$$A_\alpha(x, q) = \frac{1}{b^\alpha - a^\alpha} \left(\frac{1}{\alpha(q+1)} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha(q+1)} \right)^{\frac{1}{q}} + \left| \frac{1}{\alpha} \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \right|^{\frac{1}{q}}.$$



Proof Denote

$$F(x) = \int_a^x f(t) d_\alpha t.$$

From (2.6), we have

$$F(b) = F(x_0) + \left(\frac{b^\alpha - x_0^\alpha}{\alpha} \right) D_\alpha F(x_0) + \int_0^{(b^\alpha - x_0^\alpha)^{\frac{1}{\alpha}}} \left(\frac{b^\alpha - x_0^\alpha - \tau^\alpha}{\alpha} \right) D_\alpha^2(F) \left((\tau^\alpha + x_0^\alpha)^{\frac{1}{\alpha}} \right) d_\alpha \tau$$

which gives

$$\begin{aligned} F(b) &= F \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \left(\frac{b^\alpha - a^\alpha}{2\alpha} \right) f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \\ &\quad + \frac{1}{\alpha} \int_0^{\left(\frac{b^\alpha - a^\alpha}{2} \right)^{\frac{1}{\alpha}}} \left(\frac{b^\alpha - a^\alpha}{2} - \tau^\alpha \right) D_\alpha(f) \left(\left(\tau^\alpha + \frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) d_\alpha \tau. \end{aligned}$$

Using the change of variable $t^\alpha = \tau^\alpha + \frac{a^\alpha + b^\alpha}{2}$, we obtain

$$F(b) = F \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \left(\frac{b^\alpha - a^\alpha}{2\alpha} \right) f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \frac{1}{\alpha} \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^b (b^\alpha - t^\alpha) D_\alpha(f)(t) d_\alpha t. \quad (3.1)$$

Similarly, from (2.6), we have

$$F(a) = F(x_0) + \left(\frac{a^\alpha - x_0^\alpha}{\alpha} \right) D_\alpha F(x_0) + \int_0^{(a^\alpha - x_0^\alpha)^{\frac{1}{\alpha}}} \left(\frac{a^\alpha - x_0^\alpha - \tau^\alpha}{\alpha} \right) D_\alpha^2(F) \left((\tau^\alpha + x_0^\alpha)^{\frac{1}{\alpha}} \right) d_\alpha \tau$$

which implies that

$$\begin{aligned} F(a) &= F \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \left(\frac{a^\alpha - b^\alpha}{2\alpha} \right) f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \\ &\quad + \frac{1}{\alpha} \int_0^{\left(\frac{a^\alpha - b^\alpha}{2} \right)^{\frac{1}{\alpha}}} \left(\frac{a^\alpha - b^\alpha}{2} - \tau^\alpha \right) D_\alpha(f) \left(\left(\tau^\alpha + \frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) d_\alpha \tau. \end{aligned}$$

Then, we get

$$F(a) = F \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \left(\frac{a^\alpha - b^\alpha}{2\alpha} \right) f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) + \frac{1}{\alpha} \int_a^{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}} (t^\alpha - a^\alpha) D_\alpha(f)(t) d_\alpha t. \quad (3.2)$$



Using the identities (3.1) and (3.2), we obtain

$$\begin{aligned} \int_a^b f(t) d_\alpha t &= F(b) - F(a) = \left(\frac{b^\alpha - a^\alpha}{\alpha} \right) f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \\ &\quad + \frac{1}{\alpha} \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^b (b^\alpha - t^\alpha) D_\alpha(f)(t) d_\alpha t \\ &\quad - \frac{1}{\alpha} \int_a^{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}} (t^\alpha - a^\alpha) D_\alpha(f)(t) d_\alpha t. \end{aligned} \quad (3.3)$$

Using the change of variable $u^\alpha = a^\alpha + b^\alpha - t^\alpha$, we have

$$\int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^b (b^\alpha - u^\alpha) D_\alpha(f)(u) d_\alpha u = \int_a^{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}} (t^\alpha - a^\alpha) D_\alpha(f) \left(\left(a^\alpha + b^\alpha - t^\alpha \right)^{\frac{1}{\alpha}} \right) d_\alpha t. \quad (3.4)$$

Moreover, we have

$$f(x) - f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) = \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x D_\alpha(f)(t) d_\alpha t. \quad (3.5)$$

Thus, putting the identities (3.4) and (3.5) in (3.3), we deduce

$$\begin{aligned} f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t &= \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x D_\alpha(f)(t) d_\alpha t \\ &\quad - \frac{1}{b^\alpha - a^\alpha} \int_a^{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}} (t^\alpha - a^\alpha) \left[D_\alpha(f) \left(\left(a^\alpha + b^\alpha - t^\alpha \right)^{\frac{1}{\alpha}} \right) - D_\alpha(f)(t) \right] d_\alpha t. \end{aligned}$$

That is,

$$\begin{aligned} \left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| &\leq \left| \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x D_\alpha(f)(t) d_\alpha t \right| \\ &\quad + \frac{1}{b^\alpha - a^\alpha} \left| \int_a^{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}} (t^\alpha - a^\alpha) \left[D_\alpha(f) \left(\left(a^\alpha + b^\alpha - t^\alpha \right)^{\frac{1}{\alpha}} \right) - D_\alpha(f)(t) \right] d_\alpha t \right|. \end{aligned}$$



Using Hölder's inequality, we have

$$\begin{aligned}
 & \left| \left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \int_a \left(t^\alpha - a^\alpha \right) \left[D_\alpha(f) \left((a^\alpha + b^\alpha - t^\alpha)^{\frac{1}{\alpha}} \right) - D_\alpha(f)(t) \right] d_\alpha t \right| \\
 & \leq \int_a \left(t^\alpha - a^\alpha \right) \left| D_\alpha(f) \left((a^\alpha + b^\alpha - t^\alpha)^{\frac{1}{\alpha}} \right) \right| d_\alpha t \\
 & \quad + \int_a \left(t^\alpha - a^\alpha \right) |D_\alpha(f)(t)| d_\alpha t \\
 & \leq \left(\int_a \left(t^\alpha - a^\alpha \right)^q d_\alpha t \right)^{\frac{1}{q}} \left(\int_a \left| D_\alpha(f) \left((a^\alpha + b^\alpha - t^\alpha)^{\frac{1}{\alpha}} \right) \right|^p d_\alpha t \right)^{\frac{1}{p}} \\
 & \quad + \left(\int_a \left(t^\alpha - a^\alpha \right)^q d_\alpha t \right)^{\frac{1}{q}} \left(\int_a |D_\alpha(f)(t)|^p d_\alpha t \right)^{\frac{1}{p}} \\
 & \leq \|D_\alpha(f)\|_p \left(\frac{1}{\alpha(q+1)} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha(q+1)} \right)^{\frac{1}{q}}
 \end{aligned}$$

and similarly

$$\begin{aligned}
 & \left| \int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x D_\alpha(f)(t) d_\alpha t \right| \leq \left(\int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x 1^q d_\alpha t \right)^{\frac{1}{q}} \left(\int_{\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}}^x |D_\alpha(f)(t)|^p d_\alpha t \right)^{\frac{1}{p}} \\
 & \leq \left| \frac{1}{\alpha} \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \right|^{\frac{1}{q}} \left(\int_a^b |D_\alpha(f)(t)|^p d_\alpha t \right)^{\frac{1}{p}} \\
 & = \left| \frac{1}{\alpha} \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \right|^{\frac{1}{q}} \|D_\alpha(f)\|_p.
 \end{aligned}$$

Thus, we obtain the inequality

$$\begin{aligned}
 & \left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \\
 & \leq \left\{ \frac{1}{b^\alpha - a^\alpha} \left(\frac{1}{\alpha(q+1)} \left(\frac{b^\alpha - a^\alpha}{2} \right)^{\alpha(q+1)} \right)^{\frac{1}{q}} + \left| \frac{1}{\alpha} \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \right|^{\frac{1}{q}} \right\} \|D_\alpha(f)\|_p
 \end{aligned}$$

which completes the proof. \square



Corollary 3.2 If $q \rightarrow \infty$, then $A_\alpha(x, q) \rightarrow \frac{1}{2^\alpha}(b^\alpha - a^\alpha)^{\alpha-1} + 1$ for each $x \in [a, b]$. Thus, we have

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \left[\frac{1}{2^\alpha}(b^\alpha - a^\alpha)^{\alpha-1} + 1 \right] \|D_\alpha(f)\|_1.$$

Remark 3.3 If $\alpha = 1$, then Theorem 3.1 reduces to Theorem 2 obtained by Huy and Ngo in [8].

Corollary 3.4 Under the assumption of Theorem 3.1 with $x = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}$, we have

$$\left| f\left(\left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}\right) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{1}{2} \left(\frac{1}{\alpha(q+1)}\right)^{\frac{1}{q}} \left(\frac{b^\alpha - a^\alpha}{2}\right)^{\alpha(1+1/q)-1} \|D_\alpha(f)\|_p.$$

Theorem 3.5 Let $\alpha \in (0, 1]$, $f : [a, b] \rightarrow \mathbb{R}$ be an twice α -fractional differentiable function, $D_\alpha^2(f) \in L_\alpha^p([a, b])$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $x \in [a, b]$, we have the following inequality

$$\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t - \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right) \frac{f(b) - f(a)}{b^\alpha - a^\alpha} \right| \leq B_\alpha(q) \|D_\alpha^2 f\|_p$$

where

$$B_\alpha(q) = \frac{3}{2\alpha} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(q+1)}}{\alpha(q+1)}\right)^{\frac{1}{q}} + \frac{1}{2\alpha(b^\alpha - a^\alpha)} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(2q+1)}}{\alpha(2q+1)}\right)^{\frac{1}{q}}.$$

Proof From (2.6), we have

$$\begin{aligned} \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t &= \frac{\alpha}{b^\alpha - a^\alpha} (F(b) - F(a)) \\ &= f(a) + \left(\frac{b^\alpha - a^\alpha}{2\alpha}\right) D_\alpha f(a) + \frac{1}{2\alpha(b^\alpha - a^\alpha)} \int_a^b (b^\alpha - t^\alpha)^2 D_\alpha^2 f(t) d_\alpha t. \end{aligned} \quad (3.6)$$

Similarly, we get

$$f(x) = f(a) + \left(\frac{x^\alpha - a^\alpha}{\alpha}\right) D_\alpha f(a) + \frac{1}{\alpha} \int_a^x (x^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t \quad (3.7)$$

and

$$\alpha \frac{f(b) - f(a)}{b^\alpha - a^\alpha} = D_\alpha f(a) + \frac{1}{b^\alpha - a^\alpha} \int_a^b (b^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t. \quad (3.8)$$

Therefore, using (3.6)–(3.8), we obtain

$$\begin{aligned} &\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t - \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right) \frac{f(b) - f(a)}{b^\alpha - a^\alpha} \right| \\ &= \left| \frac{1}{\alpha} \int_a^x (x^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t - \frac{1}{2\alpha(b^\alpha - a^\alpha)} \int_a^b (b^\alpha - t^\alpha)^2 D_\alpha^2 f(t) d_\alpha t \right. \\ &\quad \left. - \frac{x^\alpha - \frac{a^\alpha + b^\alpha}{2}}{\alpha(b^\alpha - a^\alpha)} \int_a^b (b^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t \right|. \end{aligned}$$



From Hölder's inequality, we have the following inequalities:

$$\begin{aligned} \left| \int_a^x (x^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t \right| &\leq \left(\int_a^x (x^\alpha - t^\alpha)^q d_\alpha t \right)^{\frac{1}{q}} \left(\int_a^x |D_\alpha^2 f(t)|^p d_\alpha t \right)^{\frac{1}{p}} \\ &\leq \left(\frac{(b^\alpha - a^\alpha)^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} \|D_\alpha^2 f\|_p, \\ \left| \int_a^b (b^\alpha - t^\alpha)^2 D_\alpha^2 f(t) d_\alpha t \right| &\leq \left(\frac{(b^\alpha - a^\alpha)^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \|D_\alpha^2 f\|_p, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{x^\alpha - \frac{a^\alpha + b^\alpha}{2}}{\alpha(b^\alpha - a^\alpha)} \int_a^b (b^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t \right| &\leq \left| \frac{1}{2\alpha} \int_a^b (b^\alpha - t^\alpha) D_\alpha^2 f(t) d_\alpha t \right| \\ &\leq \frac{1}{2\alpha} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} \|D_\alpha^2 f\|_p. \end{aligned}$$

Thus, using these inequalities, we obtain

$$\begin{aligned} &\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t - \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \frac{f(b) - f(a)}{b^\alpha - a^\alpha} \right| \\ &\leq \left\{ \frac{3}{2\alpha} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} + \frac{1}{2\alpha(b^\alpha - a^\alpha)} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \right\} \|D_\alpha^2 f\|_p. \end{aligned}$$

which completes the proof. \square

Corollary 3.6 If $q \rightarrow \infty$, then

$$B_\alpha(q) \rightarrow \frac{3}{2\alpha} (b^\alpha - a^\alpha)^\alpha + \frac{1}{2\alpha} (b^\alpha - a^\alpha)^{2\alpha-1}$$

for each $x \in [a, b]$. Thus, we have

$$\begin{aligned} &\left| f(x) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t - \left(x^\alpha - \frac{a^\alpha + b^\alpha}{2} \right) \frac{f(b) - f(a)}{b^\alpha - a^\alpha} \right| \\ &\leq \left[\frac{3}{2\alpha} (b^\alpha - a^\alpha)^\alpha + \frac{1}{2\alpha} (b^\alpha - a^\alpha)^{2\alpha-1} \right] \|D_\alpha^2(f)\|_1. \end{aligned}$$

Remark 3.7 If $\alpha = 1$, then Theorem 3.5 reduces to Theorem 4 obtained by Huy and Ngo in [8].

Corollary 3.8 Under the assumption of Theorem 3.5 with $x = \left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned} &\left| f\left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \\ &\leq \left[\frac{3}{2\alpha} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} + \frac{1}{2\alpha(b^\alpha - a^\alpha)} \left(\frac{(b^\alpha - a^\alpha)^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \right] \|D_\alpha(f)\|_p. \end{aligned}$$



4 Applications to numerical integration

We now deal with applications of the integral inequalities involving conformable fractional integral.

Consider the partition of the interval $[a, b]$, given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \quad i = 0, \dots, n-1$$

such that $h_i = (x_{i+1}^\alpha - x_i^\alpha)$. Define the quadrature:

$$S_\alpha(f, I_n) := \frac{1}{\alpha} \sum_{i=0}^{n-1} h_i f \left(\left(\frac{x_{i+1}^\alpha + x_i^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) \quad (4.1)$$

where $i = 0, \dots, n-1$.

Theorem 4.1 Let $f : [a, b] \rightarrow \mathbb{R}$ be an α -fractional differentiable function for $\alpha \in (0, 1]$, $D_\alpha(f) \in L_\alpha^1([a, b])$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have the representation

$$\int_a^b f(t) d_\alpha t = S_\alpha(f, I_n) + R_\alpha(f, I_n)$$

where $S_\alpha(f, I_n)$ is as defined in (4.1) and the remainder satisfies the estimation:

$$|R_\alpha(f, I_n)| \leq \frac{n}{\alpha} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ \left(\frac{h_i}{2} \right)^{\alpha(1+1/q)} \right\} \|D_\alpha(f)\|_p.$$

Proof Applying Corollary 3.4 on the interval $[x_i, x_{i+1}]$, we obtain

$$\left| \frac{h_i}{\alpha} f \left(\left(\frac{x_i^\alpha + x_{i+1}^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \int_{x_i}^{x_{i+1}} f(t) d_\alpha t \right| \leq \frac{1}{\alpha} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \left(\frac{h_i}{2} \right)^{\alpha(1+1/q)} \|D_\alpha(f)\|_{p, [x_i, x_{i+1}]}$$

for all $i = 0, \dots, n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality, we obtain

$$\begin{aligned} |R(f, I_n, \xi)| &\leq \frac{1}{\alpha} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \sum_{i=0}^{n-1} \left(\frac{h_i}{2} \right)^{\alpha(1+1/q)} \|D_\alpha(f)\|_{p, [x_i, x_{i+1}]} \\ &\leq \frac{n}{\alpha} \left(\frac{1}{\alpha(q+1)} \right)^{\frac{1}{q}} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ \left(\frac{h_i}{2} \right)^{\alpha(1+1/q)} \right\} \|D_\alpha(f)\|_p \end{aligned}$$

which completes the proof. \square

Theorem 4.2 Let $\alpha \in (0, 1]$, $f : [a, b] \rightarrow \mathbb{R}$ be a twice α -fractional differentiable function, $D_\alpha^2(f) \in L_\alpha^p([a, b])$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have the representation

$$\int_a^b f(t) d_\alpha t = S_\alpha(f, I_n) + R_\alpha(f, I_n)$$

where $S_\alpha(f, I_n)$ is as defined in (4.1) and the remainder satisfies the estimation:

$$\begin{aligned} |R_\alpha(f, I_n)| &\leq \left[\frac{3n}{2\alpha^\alpha} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ h_i \left(\frac{h_i^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} \right\} \right. \\ &\quad \left. + \frac{n}{2\alpha^2} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ \left(\frac{h_i^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \right\} \right] \|D_\alpha(f)\|_p. \end{aligned}$$



Proof Applying Corollary 3.8 on the interval $[x_i, x_{i+1}]$, we obtain

$$\begin{aligned} & \left| \frac{h_i}{\alpha} f \left(\left(\frac{x_i^\alpha + x_{i+1}^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) - \int_{x_i}^{x_{i+1}} f(t) d_\alpha t \right| \\ & \leq \left[\frac{3h_i}{2\alpha^\alpha} \left(\frac{h_i^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} + \frac{1}{2\alpha^2} \left(\frac{h_i^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \right] \|D_\alpha(f)\|_{p, [x_i, x_{i+1}]} \end{aligned}$$

for all $i = 0, \dots, n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality, we obtain

$$\begin{aligned} |R_\alpha(f, I_n)| & \leq \frac{3}{2\alpha^\alpha} \sum_{i=0}^{n-1} h_i \left(\frac{h_i^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} \|D_\alpha(f)\|_{p, [x_i, x_{i+1}]} \\ & \quad + \frac{1}{2\alpha^2} \sum_{i=0}^{n-1} \left(\frac{h_i^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \|D_\alpha(f)\|_{p, [x_i, x_{i+1}]} \\ & \leq \frac{3n}{2\alpha^\alpha} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ h_i \left(\frac{h_i^{\alpha(q+1)}}{\alpha(q+1)} \right)^{\frac{1}{q}} \right\} \|D_\alpha(f)\|_p \\ & \quad + \frac{n}{2\alpha^2} \max_{i \in \{0, 1, \dots, n-1\}} \left\{ \left(\frac{h_i^{\alpha(2q+1)}}{\alpha(2q+1)} \right)^{\frac{1}{q}} \right\} \|D_\alpha(f)\|_p \end{aligned}$$

which completes the proof. \square

5 Concluding remarks

New upper bounds of Ostrowski-type integral inequalities are proposed and tested in this paper. To this purpose, the rules of conformable calculus and Taylor formula for conformable derivatives are used into calculation. To verify our findings, we have presented some applications. Thus, this study should help to decide upper bounds for non-integer order of integral inequalities.

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