



# Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities



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## ABSTRACT

In the present note, first we have established Hermite–Hadamard’s inequalities for fractional integrals. Second, an integral identity and some Hermite–Hadamard type integral inequalities for the fractional integrals are obtained and these results have some relationships with [S.S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11 (5) (1998), 91–95].

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## 1. Introduction

The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature (see, e.g., [1, p. 137], [2]). These inequalities state that if  $f: I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if  $f$  is concave. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example, [3–6, 2, 7–11, 1, 12, 13] and the references cited therein.

The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function  $f: [a, b] \rightarrow \mathbb{R}$ .

**Definition 1.** The function  $f: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ , is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

In [7], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

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**Lemma 1.** Let  $f: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $f' \in L[a, b]$ , then the following equality holds:

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt. \tag{1.2}$$

**Theorem 1.** Let  $f: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{8} (|f'(a)| + |f'(b)|). \tag{1.3}$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [14–16].

**Definition 2.** Let  $f \in L_1[a, b]$ . The Riemann–Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here,  $\Gamma(\alpha)$  is the Gamma function and  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

For some recent results connected with fractional integral inequalities, see [17–23].

The aim of this paper is to establish Hermite–Hadamard’s inequalities for Riemann–Liouville fractional integral and some other integral inequalities using the identity is obtained for fractional integrals.

## 2. Hermite–Hadamard’s inequalities for fractional integrals

Hermite–Hadamard’s inequalities can be represented in fractional integral forms as follows.

**Theorem 2.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq a < b$  and  $f \in L_1[a, b]$ . If  $f$  is a convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2} \tag{2.1}$$

with  $\alpha > 0$ .

**Proof.** Since  $f$  is a convex function on  $[a, b]$ , we have for  $x, y \in [a, b]$  with  $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} \tag{2.2}$$

i.e., with  $x = ta + (1-t)b, y = (1-t)a + tb$ ,

$$2f\left(\frac{a+b}{2}\right) \leq f(ta + (1-t)b) + f((1-t)a + tb). \tag{2.3}$$

Multiplying both sides of (2.3) by  $t^{\alpha-1}$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} \frac{2}{\alpha} f\left(\frac{a+b}{2}\right) &\leq \int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} f((1-t)a + tb) dt \\ &= \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u) \frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v) \frac{dv}{b-a} \\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \end{aligned}$$

i.e.

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)]$$

and the first inequality is proved.

For the proof of the second inequality in (2.2) we first note that if  $f$  is a convex function, then, for  $\lambda \in [0, 1]$ , it yields

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

and

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

By adding these inequalities we have

$$f(ta + (1-t)b) + f((1-t)a + tb) \leq tf(a) + (1-t)f(b) + (1-t)f(a) + tf(b). \quad (2.4)$$

Then multiplying both sides of (2.4) by  $t^{\alpha-1}$  and integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\int_0^1 t^{\alpha-1} f(ta + (1-t)b) dt + \int_0^1 t^{\alpha-1} f((1-t)a + tb) dt \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} dt$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{\alpha}.$$

The proof is completed.  $\square$

**Remark 1.** If in Theorem 2, we let  $\alpha = 1$ , then inequality (2.1) become inequality (1.1) of Theorem 1.

### 3. Hermite–Hadamard type inequalities for fractional integrals

We need the following lemma.

**Lemma 2.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] = \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt. \quad (3.1)$$

**Proof.** It suffices to note that

$$\begin{aligned} I &= \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt \\ &= \left[ \int_0^1 (1-t)^\alpha f'(ta + (1-t)b) dt \right] + \left[ - \int_0^1 t^\alpha f'(ta + (1-t)b) dt \right] \\ &= I_1 + I_2. \end{aligned} \quad (3.2)$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^\alpha f'(ta + (1-t)b) dt \\ &= (1-t)^\alpha \frac{f(ta + (1-t)b)}{a-b} \Big|_0^1 + \int_0^1 \alpha (1-t)^{\alpha-1} \frac{f(ta + (1-t)b)}{a-b} dt \\ &= \frac{f(b)}{b-a} - \frac{\alpha}{b-a} \int_b^a \left(\frac{a-x}{a-b}\right)^{\alpha-1} \frac{f(x)}{a-b} dx \\ &= \frac{f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{b-}^\alpha f(a) \end{aligned} \quad (3.3)$$

and similarly we get,

$$\begin{aligned}
 I_2 &= - \int_0^1 t^\alpha f'(ta + (1-t)b) dt \\
 &= - \frac{t^\alpha f'(ta + (1-t)b)}{a-b} \Big|_0^1 + \alpha \int_0^1 t^{\alpha-1} \frac{f'(ta + (1-t)b)}{a-b} dt \\
 &= \frac{f(a)}{b-a} - \frac{\alpha}{b-a} \int_b^a \left(\frac{b-x}{b-a}\right)^{\alpha-1} \frac{f(x)}{a-b} dx \\
 &= \frac{f(a)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J_{a+}^\alpha f(b).
 \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), it follows that

$$I = \frac{f(a) + f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)].$$

Thus, by multiplying both sides by  $\frac{b-a}{2}$ , we have conclusion (3.1).  $\square$

**Remark 2.** If in Lemma 2, we let  $\alpha = 1$ , then equality (3.1) becomes equality (1.2) of Lemma 1.

Using this lemma, we can obtain the following fractional integral inequality.

**Theorem 3.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality for fractional integrals holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha}\right) [f'(a) + f'(b)]. \tag{3.5}$$

**Proof.** Using Lemma 2 and the convexity of  $|f'|$ , we find

$$\begin{aligned}
 &\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\
 &\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| dt \\
 &\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| [t |f'(a)| + (1-t) |f'(b)|] dt \\
 &= \frac{b-a}{2} \left\{ \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] [t |f'(a)| + (1-t) |f'(b)|] dt \right. \\
 &\quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] [t |f'(a)| + (1-t) |f'(b)|] dt \right\} \\
 &= \frac{b-a}{2} (K_1 + K_2).
 \end{aligned} \tag{3.6}$$

Calculating  $K_1$  and  $K_2$ , we have

$$\begin{aligned}
 K_1 &= |f'(a)| \left[ \int_0^{\frac{1}{2}} t(1-t)^\alpha dt - \int_0^{\frac{1}{2}} t^{\alpha+1} dt \right] + |f'(b)| \left[ \int_0^{\frac{1}{2}} (1-t)^{\alpha+1} dt - \int_0^{\frac{1}{2}} (1-t)t^\alpha dt \right] \\
 &= |f'(a)| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right] + |f'(b)| \left[ \frac{1}{(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right]
 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
 K_2 &= |f'(a)| \left[ \int_{\frac{1}{2}}^1 t^{\alpha+1} dt - \int_{\frac{1}{2}}^1 t(1-t)^\alpha dt \right] + |f'(b)| \left[ \int_{\frac{1}{2}}^1 (1-t)t^\alpha dt - \int_{\frac{1}{2}}^1 (1-t)^{\alpha+1} dt \right] \\
 &= |f'(a)| \left[ \frac{1}{(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right] + |f'(b)| \left[ \frac{1}{(\alpha+1)(\alpha+2)} - \frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right].
 \end{aligned} \tag{3.8}$$

Thus if we use (3.7) and (3.8) in (3.6), we obtain the inequality of (3.5). This completes the proof.  $\square$

**Remark 3.** If we take  $\alpha = 1$  in Theorem 3, then inequality (3.5) becomes inequality (1.3) of Theorem 1.

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