

Some generalization of integral inequalities for twice differentiable mappings involving fractional integrals

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Abstract. In this paper, a general integral identity involving Riemann-Liouville fractional integrals is derived. By use this identity, we establish new some generalized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.

1 Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

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Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications (see, e.g., [12, p.137], [6]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [6, 8, 9, 12], [14]-[16], [22], [23]) and the references cited therein.

In [16], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

Lemma 1 *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° , $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{2} \int_0^1 m(t) [f''(ta + (1-t)b) + f''(tb + (1-t)a)] dt, \end{aligned} \quad (2)$$

where

$$m(t) := \begin{cases} t^2, & t \in [0, \frac{1}{2}] \\ (1-t)^2, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Also, the main inequalities in [16], pointed out as follows:

Theorem 1 *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a, b]$. If $|f''|$ is convex on $[a, b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\frac{|f''(a)| + |f''(b)|}{2} \right]. \quad (3)$$

Theorem 2 *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{8(2p+1)^{1/p}} \left[\frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^{1/q} \quad (4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7, 10, 11, 13].

Definition 1 Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Meanwhile, Sarikaya et al. [19] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2 Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)_+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)_-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \\ & = \frac{b-a}{4} \left\{ \int_0^1 t^\alpha f' \left(\frac{t}{2}a + \frac{2-t}{2}b \right) dt - \int_0^1 t^\alpha f' \left(\frac{2-t}{2}a + \frac{t}{2}b \right) dt \right\} \end{aligned} \quad (5)$$

with $\alpha > 0$.

It is remarkable that Sarikaya et al. [19] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3 Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^\alpha} \left[J_{\left(\frac{a+b}{2}\right)_+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)_-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2} \quad (6)$$

with $\alpha > 0$.

For some recent results connected with fractional integral inequalities see ([1, 2, 3, 4, 5], [17], [18], [20], [21], [24])

In this paper, we expand the Lemma 2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals and some other integral inequalities using the generalized identity is obtained for fractional integrals.

2 Main results

For our results, we give the following important fractional integral identity:

Lemma 3 *Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable mapping on (a, b) with $0 \leq a < b$. If $f'' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}
 & (\alpha + 1) (1 - \lambda)^{\alpha} \lambda^{\alpha} f(\lambda a + (1 - \lambda)b) \\
 & - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(b - a)^{\alpha}} \left[\lambda^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^-}^{\alpha} f(a) + (1 - \lambda)^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^+}^{\alpha} f(b) \right] \\
 & = - (b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1 - \lambda) \int_0^1 t^{\alpha+1} f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \right. \\
 & \quad \left. + \lambda \int_0^1 (1 - t)^{\alpha+1} f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)] dt \right\} \tag{7}
 \end{aligned}$$

where $\lambda \in (0, 1)$ and $\alpha > 0$.

Proof. Integrating by parts

$$\begin{aligned}
 & \int_0^1 t^{\alpha+1} f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \\
 & = \frac{t^{\alpha+1} f' [t(\lambda a + (1 - \lambda)b) + (1 - t)a]}{(1 - \lambda)(b - a)} \Big|_0^1 \\
 & \quad - \frac{\alpha + 1}{(1 - \lambda)(b - a)} \int_0^1 t^{\alpha} f' [t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \\
 & = \frac{f'(\lambda a + (1 - \lambda)b)}{(1 - \lambda)(b - a)} - \frac{\alpha + 1}{(1 - \lambda)(b - a)}
 \end{aligned}$$

$$\begin{aligned} & \times \left[\frac{f(\lambda a + (1 - \lambda)b)}{(1 - \lambda)(b - a)} - \frac{\alpha}{(1 - \lambda)(b - a)} \int_0^1 t^{\alpha-1} f[t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \right] \\ & = \frac{f'(\lambda a + (1 - \lambda)b)}{(1 - \lambda)(b - a)} - \frac{(\alpha + 1) f(\lambda a + (1 - \lambda)b)}{(1 - \lambda)^2(b - a)^2} \\ & \quad + \frac{(\alpha + 1) \alpha}{(1 - \lambda)^{\alpha+2}(b - a)^{\alpha+2}} \int_a^{\lambda a + (1 - \lambda)b} (x - a)^{\alpha-1} f(x) dx \\ & = \frac{f'(\lambda a + (1 - \lambda)b)}{(1 - \lambda)(b - a)} - \frac{(\alpha + 1) f(\lambda a + (1 - \lambda)b)}{(1 - \lambda)^2(b - a)^2} \\ & \quad + \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(1 - \lambda)^{\alpha+2}(b - a)^{\alpha+2}} J_{(\lambda a + (1 - \lambda)b)^-}^\alpha f(a) \end{aligned}$$

that is,

$$\begin{aligned} & - \int_0^1 t^{\alpha+1} f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \\ & = - \frac{f'(\lambda a + (1 - \lambda)b)}{(1 - \lambda)(b - a)} + \frac{(\alpha + 1) f(\lambda a + (1 - \lambda)b)}{(1 - \lambda)^2(b - a)^2} \\ & \quad - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(1 - \lambda)^{\alpha+2}(b - a)^{\alpha+2}} J_{(\lambda a + (1 - \lambda)b)^-}^\alpha f(a) \end{aligned} \tag{8}$$

and similarly we have

$$\begin{aligned} & - \int_0^1 (1 - t)^{\alpha+1} f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)] dt \\ & = \frac{f'(\lambda a + (1 - \lambda)b)}{\lambda(b - a)} + \frac{(\alpha + 1) f(\lambda a + (1 - \lambda)b)}{\lambda^2(b - a)^2} \\ & \quad - \frac{(\alpha + 1) \alpha}{\lambda^{\alpha+2}(b - a)^{\alpha+2}} \int_{\lambda a + (1 - \lambda)b}^b (b - x)^{\alpha-1} f(x) dx \\ & = \frac{f'(\lambda a + (1 - \lambda)b)}{\lambda(b - a)} + \frac{(\alpha + 1) f(\lambda a + (1 - \lambda)b)}{\lambda^2(b - a)^2} \\ & \quad - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{\lambda^{\alpha+2}(b - a)^{\alpha+2}} J_{(\lambda a + (1 - \lambda)b)^+}^\alpha f(b). \end{aligned} \tag{9}$$

Adding (8) and (9) we have (7). This completes the proof. □

Corollary 1 Under the assumptions Lemma 3 with $\lambda = \frac{1}{2}$, then it follows that

$$\begin{aligned} & \frac{-(b-a)^2}{8} \left\{ \int_0^1 t^{\alpha+1} f'' \left[t \left(\frac{a+b}{2} \right) + (1-t)a \right] dt \right. \\ & \quad \left. + \int_0^1 (1-t)^{\alpha+1} f'' \left[tb + (1-t) \frac{a+b}{2} \right] dt \right\} \\ & = (\alpha+1) f \left(\frac{a+b}{2} \right) - \frac{(\alpha+1) \Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right]. \end{aligned}$$

Remark 1 If we choose $\alpha = 1$ in Corollary 1, we have

$$\begin{aligned} & f \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \\ & = \frac{-(b-a)^2}{16} \left\{ \int_0^1 t^2 f'' \left[t \left(\frac{a+b}{2} \right) + (1-t)a \right] dt \right. \\ & \quad \left. + \int_0^1 (1-t)^2 f'' \left[tb + (1-t) \frac{a+b}{2} \right] dt \right\}. \end{aligned}$$

Theorem 4 Let $f: [a, b] \rightarrow \mathbb{R}$ be twice differentiable mapping on (a, b) with $0 \leq a < b$. If $|f''|^q, q \geq 1$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| (\alpha+1)(1-\lambda)^\alpha \lambda^\alpha f(\lambda a + (1-\lambda)b) - \frac{(\alpha+1) \Gamma(\alpha+1)}{(b-a)^\alpha} \right. \\ & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{(b-a)^2 (1-\lambda)^{\alpha+1} \lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}} \left\{ (1-\lambda) \left(\frac{(\alpha+2) |f''(\lambda a + (1-\lambda)b)|^q + |f''(a)|^q}{\alpha+3} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \lambda \left(\frac{(\alpha+2) |f''(\lambda a + (1-\lambda)b)|^q + |f''(b)|^q}{\alpha+3} \right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{10}$$

where $\lambda \in (0, 1)$ and $\alpha > 0$.

Proof. Firstly, we suppose that $q = 1$. Using Lemma 3 and convexity of $|f''|^q$, we find that

$$\begin{aligned}
 & \left| (\alpha + 1) (1 - \lambda)^\alpha \lambda^\alpha f(\lambda a + (1 - \lambda)b) - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(b - a)^\alpha} \right. \\
 & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^-}^\alpha f(a) + (1 - \lambda)^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^+}^\alpha f(b) \right] \right| \\
 & \leq (b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1 - \lambda) \int_0^1 t^{\alpha+1} |f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a]| dt \right. \\
 & \quad \left. + \lambda \int_0^1 (1 - t)^{\alpha+1} |f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)]| dt \right\} \\
 & \leq (b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1 - \lambda) \int_0^1 t^{\alpha+1} [t |f''(\lambda a + (1 - \lambda)b)| + (1 - t) |f''(a)|] dt \right. \\
 & \quad \left. + \lambda \int_0^1 (1 - t)^{\alpha+1} [t |f''(b)| + (1 - t) |f''(\lambda a + (1 - \lambda)b)|] dt \right\} \\
 & = \frac{(b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1}}{\alpha + 2} \left\{ (1 - \lambda) \left(\frac{(\alpha + 2) |f''(\lambda a + (1 - \lambda)b)| + |f''(a)|}{\alpha + 3} \right) \right. \\
 & \quad \left. + \lambda \left(\frac{(\alpha + 2) |f''(\lambda a + (1 - \lambda)b)| + |f''(b)|^q}{\alpha + 3} \right) \right\}.
 \end{aligned}$$

Secondly, we suppose that $q > 1$. Using Lemma 3 and power mean inequality, we have

$$\begin{aligned}
 & \left\{ (1 - \lambda) \int_0^1 t^{\alpha+1} f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a] dt \right. \\
 & \quad \left. + \lambda \int_0^1 (1 - t)^{\alpha+1} f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)] dt \right\} \\
 & \leq (1 - \lambda) \left(\int_0^1 t^{\alpha+1} \right)^{1 - \frac{1}{q}} \left(\int_0^1 t^{\alpha+1} |f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a]|^q dt \right)^{\frac{1}{q}} \\
 & \quad + \lambda \left(\int_0^1 (1 - t)^{\alpha+1} \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)^{\alpha+1} |f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)]|^q dt \right)^{\frac{1}{q}}.
 \end{aligned} \tag{11}$$

Hence, using convexity of $|f''|^q$ and (11) we obtain

$$\begin{aligned} & \left| (\alpha + 1) (1 - \lambda)^\alpha \lambda^\alpha f(\lambda a + (1 - \lambda)b) - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(b - a)^\alpha} \right. \\ & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1 - \lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{(b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1}}{(\alpha + 2)^{1-\frac{1}{q}}} \left\{ (1 - \lambda) \left(\int_0^1 t^{\alpha+1} [t |f''(\lambda a + (1 - \lambda)b)| + (1 - t) |f''(a)|] dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \lambda \left(\int_0^1 (1 - t)^{\alpha+1} [t |f''(b)| + (1 - t) |f''(\lambda a + (1 - \lambda)b)|] dt \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{(b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1}}{(\alpha + 2)^{1-\frac{1}{q}}} \left\{ (1 - \lambda) \left(\frac{(\alpha + 2) |f''(\lambda a + (1 - \lambda)b)| + |f''(a)|}{(\alpha + 2)(\alpha + 3)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \lambda \left(\frac{(\alpha + 2) |f''(\lambda a + (1 - \lambda)b)| + |f''(b)|^q}{(\alpha + 2)(\alpha + 3)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

This completes the proof. □

Corollary 2 Under assumption Theorem 4 with $\lambda = \frac{1}{2}$, we obtain

$$\begin{aligned} & \left| f\left(\frac{a + b}{2}\right) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha 2^{1-\alpha}} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{(b - a)^2}{8(\alpha + 1)(\alpha + 2)^{1-\frac{1}{q}}} \left\{ \left(\frac{(\alpha + 2) |f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q}{\alpha + 3} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(\alpha + 2) |f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q}{\alpha + 3} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2 If we choose $\alpha = 1$ in Corollary 2, we have

$$\begin{aligned} & \left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b - a)^2}{16 \times 3^{1-\frac{1}{q}}} \left\{ \left(\frac{3 |f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3 |f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q}{4} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Theorem 5 Let $f: [a, b] \rightarrow \mathbb{R}$ be twice differentiable mapping on (a, b) with $0 \leq a < b$. If $|f''|^q$ is convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| (\alpha + 1) (1 - \lambda)^\alpha \lambda^\alpha f(\lambda a + (1 - \lambda)b) - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(b - a)^\alpha} \right. \\ & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^-}^\alpha f(a) + (1 - \lambda)^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{(b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1}}{(p(\alpha + 1) + 1)^{\frac{1}{p}}} \left\{ (1 - \lambda) \left(\frac{|f''(\lambda a + (1 - \lambda)b)|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \lambda \left(\frac{|f''(\lambda a + (1 - \lambda)b)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (12)$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0, 1)$ and $\alpha > 0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\begin{aligned} & \left| (\alpha + 1) (1 - \lambda)^\alpha \lambda^\alpha f(\lambda a + (1 - \lambda)b) - \frac{(\alpha + 1) \Gamma(\alpha + 1)}{(b - a)^\alpha} \right. \\ & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^-}^\alpha f(a) + (1 - \lambda)^{\alpha+1} J_{(\lambda a + (1 - \lambda)b)^+}^\alpha f(b) \right] \right| \\ & \leq (b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1 - \lambda) \left(\int_0^1 t^{p(\alpha+1)} \right)^{\frac{1}{p}} \right. \\ & \quad \left(\int_0^1 |f'' [t(\lambda a + (1 - \lambda)b) + (1 - t)a]|^q dt \right)^{\frac{1}{q}} \\ & \quad \left. + \lambda \left(\int_0^1 (1 - t)^{p(\alpha+1)} \right)^{\frac{1}{p}} \left(\int_0^1 |f'' [tb + (1 - t)(\lambda a + (1 - \lambda)b)]|^q dt \right)^{\frac{1}{q}} \right\} \\ & \leq (b - a)^2 (1 - \lambda)^{\alpha+1} \lambda^{\alpha+1} \\ & \quad \times \left\{ (1 - \lambda) \frac{1}{(p(\alpha + 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 [t |f''(\lambda a + (1 - \lambda)b)|^q + (1 - t) |f''(a)|^q] dt \right)^{\frac{1}{q}} \right. \end{aligned}$$

$$\begin{aligned}
 & +\lambda \frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}} \left(\int_0^1 [t|f''(b)|^q + (1-t)|f''(\lambda a + (1-\lambda)b)|^q] dt \right)^{\frac{1}{q}} \Bigg\} \\
 & = \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}} \left\{ (1-\lambda) \left(\frac{|f''(\lambda a + (1-\lambda)b)|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
 & \left. + \lambda \left(\frac{|f''(\lambda a + (1-\lambda)b)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

□

Corollary 3 Under assumption Theorem 5 with $\lambda = \frac{1}{2}$, we obtain

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}} \left[J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] \right| \\
 & \leq \frac{(b-a)^2}{8(\alpha+1)(p(\alpha+1)+1)^{\frac{1}{p}}} \left\{ \left(\frac{|f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
 & \left. + \left(\frac{|f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Remark 3 If we choose $\alpha = 1$ in Corollary 3, we have

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}} \left\{ \left(\frac{|f''\left(\frac{a+b}{2}\right)|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f''\left(\frac{a+b}{2}\right)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Theorem 6 Let $f:[a, b] \rightarrow \mathbb{R}$ be twice differentiable mapping on (a, b) with $0 \leq a < b$. If $|f''|^q$ is convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
 & \left| (\alpha+1)(1-\lambda)^\alpha \lambda^\alpha f(\lambda a + (1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha} \right. \\
 & \quad \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \right| \\
 & \leq (b-a)^2 (1-\lambda)^{\alpha+1} \lambda^{\alpha+1}
 \end{aligned}$$

$$\left\{ (1-\lambda) \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right. \\ \left. + \lambda \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right\}. \tag{13}$$

where $\lambda \in (0, 1)$ and $\alpha > 0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder’s inequality, we have

$$\left| (\alpha+1)(1-\lambda)^\alpha \lambda^\alpha f(\lambda a+(1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^\alpha} \right. \\ \left. \times \left[\lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \right| \\ \leq (b-a)^2 (1-\lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1-\lambda) \left(\int_0^1 1^p \right)^{\frac{1}{p}} \left(\int_0^1 t^{q(\alpha+1)} |f''[t(\lambda a+(1-\lambda)b)+(1-t)a]|^q dt \right)^{\frac{1}{q}} \right. \\ \left. + \lambda \left(\int_0^1 1^p \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{q(\alpha+1)} |f''[tb+(1-t)(\lambda a+(1-\lambda)b)]|^q dt \right)^{\frac{1}{q}} \right\} \\ \leq (b-a)^2 (1-\lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1-\lambda) \left(\int_0^1 t^{q(\alpha+1)} [t|f''(\lambda a+(1-\lambda)b)|^q + (1-t)|f''(a)|^q] dt \right)^{\frac{1}{q}} \right. \\ \left. + \lambda \left(\int_0^1 (1-t)^{q(\alpha+1)} [t|f''(b)|^q + (1-t)|f''(\lambda a+(1-\lambda)b)|^q] dt \right)^{\frac{1}{q}} \right\} \\ = (b-a)^2 (1-\lambda)^{\alpha+1} \lambda^{\alpha+1} \left\{ (1-\lambda) \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right. \\ \left. + \lambda \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right\}.$$

□

Corollary 4 Under assumption Theorem 6 with $\lambda = \frac{1}{2}$, we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha 2^{1-\alpha}} \left[J_{(\frac{a+b}{2})^-}^\alpha f(a) + J_{(\frac{a+b}{2})^+}^\alpha f(b) \right] \right| \\ \leq \frac{(b-a)^2}{8(\alpha+1)} \left\{ \left(\frac{(q(\alpha+1)+1)|f''(\frac{a+b}{2})|^q + |f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right.$$

$$+ \left\{ \left(\frac{(q(\alpha+1)+1) |f''(\frac{a+b}{2})|^q + |f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right\}.$$

Remark 4 If we choose $\alpha = 1$ in Corollary 4, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left\{ \left(\frac{(2q+1) |f''(\frac{a+b}{2})|^q + |f''(a)|^q}{(2q+1)(2q+2)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(2q+1) |f''(\frac{a+b}{2})|^q + |f''(b)|^q}{(2q+1)(2q+2)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

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