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New Variants of Quantum Midpoint-Type Inequalities

Saad Ihsan Butt [1], Hüseyin Budak [2] and Kamsing Nonlaopon [3,*]

[1] Department of Mathematics, Lahore Campus, COMSATS University Islamabad, Islamabad 54000, Pakistan
[2] Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey
[3] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
* Correspondence: nkamsi@kku.ac.th; Tel.: +66-8-6642-1582

Abstract: Recently, there has been a strong push toward creating and expanding quadrature inequalities in quantum calculus. In order to investigate various avenues for quantum inquiry, a number of quantum extensions of midpoint estimations are studied. The goal of this research article is to discover novel quantum midpoint-type inequalities that are twice $\mathfrak{q}^{\xi_2}$-differentiable for (α, m)-convex functions. Firstly, we obtain novel identity for $\mathfrak{q}^{\xi_2}$-integral by employing quantum calculus tools. Then by using the auxiliary identity, we formulate new bounds by taking into account the known quantum Hölder and Power mean inequalities. An example is provided with a graphical representation to show the validity of obtaining results. The outcomes of this study clarify and expand earlier research on midpoint-type inequalities. Analytic inequalities of this type as well as particularly related strategies have applications for various fields where symmetry plays an important role.

Keywords: quantum derivative; quantum integral; quantum midpoint inequalities; (α, m)-convex functions

MSC: 26D07; 26D10; 26D15; 26A33

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1. Introduction

Calculus without limits, or quantum calculus, is an infinitesimal calculus that has a number of applications. Due to the vast number of applications in areas of mathematics such as fundamental hypergeometric functions, number theory, combinatorics, orthogonal polynomials, physics, relativity, and quantum science, $\mathfrak{q}$-analysis has recently been the topic of a lot of research [1–4]. It is believed that Euler invented this branch of mathematics by utilizing the $\mathfrak{q}$-parameter in Newton's infinitive series work. Jackson was the one who first introduced the $\mathfrak{q}$-calculus [1]. Jackson [3] introduced $\mathfrak{q}$-definite integrals in the nineteenth century and started his work in a symmetrical fashion. Agarwal [5] initially presented the $\mathfrak{q}$-fractional derivative in 1969.

The science of differential equations heavily relies on integral inequalities. Numerous academics have investigated applications of integral inequality in both conventional and quantum calculus. Tariboon et al. in [6] uses ${}_{\xi_1}D_{\mathfrak{q}}$-difference operator and provided applications to impulsive difference equations on finite quantum intervals. In light of the historical knowledge of the importance of mathematical inequalities. Tariboon et al. in [7] introduced the quantum variants of well know integral inequalities such as Hermite-Hadamard, Jensen, Ostrowski, Cebysev and Grüss type inequalities as they performed an essential role in quantum calculus.

The connection between inequalities and convex functions has been discovered to be extremely useful in developing new integral inequalities along with their significant applications [8–10]. A convex function plays a notable role in both theoretical and applied sciences. Convexity also has the finest effect on our daily life through countless implementations in medicine, industry, and business. Due to the wide range of implementations, it is among the most advanced branches of mathematical modelling. In literature, serval kinds of convexities are introduced depending on their useful strengths and general nature. However, our study requires the classical convex functions and (α, m)-convex functions defined as:

Definition 1 ([8]). *If $\mathfrak{G} : [\xi_1, \xi_2] \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then the inequality*

$$\mathfrak{G}(\lambda\mathfrak{x} + (1-\lambda)y) \le \lambda\mathfrak{G}(\mathfrak{x}) + (1-\lambda)\mathfrak{G}(y)$$

is valid for all $\mathfrak{x}, y \in [\xi_1, \xi_2]$ and $\lambda \in [0,1]$.

Definition 2 ([11]). *If $\mathfrak{G} : [0, \xi_2) \to \mathbb{R}$ is called (α, m)-convex, then the inequality*

$$\mathfrak{G}(\lambda\mathfrak{x} + m(1-\lambda)y) \le \lambda^{\alpha}\mathfrak{G}(\mathfrak{x}) + m(1-\lambda^{\alpha})\mathfrak{G}(y)$$

holds for all $\mathfrak{x}, y \in [0, \xi_2), \lambda \in [0,1], (\alpha, m) \in (0,1]$.

In order to move further we required some basic notions and background of quantum calculus.

2. Description of Quantum Calculus

Some basic known definitions of q-calculus are presented below:

Definition 3 ([4,7]). *If $\mathfrak{G} : [\xi_1, \xi_2] \to mathbbR$, the $\mathfrak{q}_{\xi_1}$-derivative of $\mathfrak{G}$ at $\in [\xi_1, \xi_2]$ is defined as:*

$$_{\xi_1}D_{\mathfrak{q}}\mathfrak{G}(\mathfrak{x}) = \frac{\mathfrak{G}(\mathfrak{x}) - \mathfrak{G}(\mathfrak{q}\mathfrak{x} + (1-\mathfrak{q})\xi_1)}{(1-\mathfrak{q})(\mathfrak{x} - \xi_1)}, \quad \mathfrak{x} \ne \xi_1. \tag{1}$$

If $\mathfrak{x} = \xi_1$, we define $_{\xi_1}D_{\mathfrak{q}}\phi(\xi_1) = \lim_{\mathfrak{x}\to\xi_1} {}_{\xi_1}D_{\mathfrak{q}}\phi(\mathfrak{x})$ if it exists.

Rajkovic [12] defined the Riemann $\mathfrak{q}$-integral which was then generalised to Jackson $\mathfrak{q}$-integral on $[\xi_1, \xi_2]$:

$$\int_{\imath}^{\mathfrak{x}} \mathfrak{G}(\lambda)_{\imath} d_{\mathfrak{q}}\lambda = (1-\mathfrak{q})(\mathfrak{x} - \xi_1) \sum_{\mathfrak{n}=0}^{\infty} \mathfrak{q}^{\mathfrak{n}}\mathfrak{G}(\mathfrak{q}^{\mathfrak{n}}\mathfrak{x} + (1-\mathfrak{q}^{\mathfrak{n}})\xi_1), \quad \mathfrak{x} \in [\xi_1, \xi_2]. \tag{2}$$

Definition 4. *If $\xi_1 = 0$ in (2), then*

$$\int_0^{\mathfrak{x}} \mathfrak{G}(\lambda)_0 d_{\mathfrak{q}}\lambda = \int_0^{\mathfrak{x}} \mathfrak{G}(\lambda) d_{\mathfrak{q}}\lambda,$$

where $\int_0^{\mathfrak{x}} \mathfrak{G}(\lambda) d_{\mathfrak{q}}\lambda$ is $\mathfrak{q}$-definite integral on $[0, \mathfrak{x}]$ and defined as [4]:

$$\int_0^{\mathfrak{x}} \mathfrak{G}(\lambda)_0 d_{\mathfrak{q}}\lambda = \int_0^{\mathfrak{x}} \mathfrak{G}(\lambda) d_{\mathfrak{q}}\lambda = (1-\mathfrak{q})\mathfrak{x} \sum_{\mathfrak{n}=0}^{\infty} \mathfrak{q}^{\mathfrak{n}}\mathfrak{G}(\mathfrak{q}^{\mathfrak{n}}\mathfrak{x}). \tag{3}$$

If $c \in (\xi_1, \mathfrak{x})$, then the $\mathfrak{q}$-definite integral on $[c, \mathfrak{x}]$ is expressed as:

$$\int_c^{\mathfrak{x}} \mathfrak{G}(\lambda)_{\xi_1} d_{\mathfrak{q}}\lambda = \int_{\xi_1}^{\mathfrak{x}} \mathfrak{G}(\lambda)_{\xi_1} d_{\mathfrak{q}}\lambda - \int_{\xi_1}^{c} \mathfrak{G}(\lambda)_{\xi_1} d_{\mathfrak{q}}\lambda. \tag{4}$$

In [13], $\mathfrak{q}$-Hermite–Hadamard inequality was established by Alp et al. and stated as:

Theorem 1. *Let $\mathfrak{G} : [\xi_1, \xi_2] \to \mathbb{R}$ be a convex function on $[\xi_1, \xi_2]$ and $\mathfrak{q} \in (0,1)$, we have*

$$\mathfrak{G}\left(\frac{\mathfrak{q}\xi_1 + \xi_2}{1+\mathfrak{q}}\right) \leq \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\mathfrak{x})_{\xi_1} d_{\mathfrak{q}}\mathfrak{x} \leq \frac{\mathfrak{q}\mathfrak{G}(\xi_1) + \mathfrak{G}(\xi_2)}{1+\mathfrak{q}}.$$

In [14], authors presented some new definitions of quantum calculus and also presented a novel variant of Hermite-Hadamard inequality:

Definition 5 ([14])**.** *If $\mathfrak{G} : [\xi_1, \xi_2] \to \mathbb{R}$ be a function, then $\mathfrak{q}^{\xi_2}$-definite integral on $[\xi_1, \xi_2]$ is expressed as:*

$$\begin{aligned} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\mathfrak{x})^{\xi_2} d_{\mathfrak{q}}\mathfrak{x} &= (1-\mathfrak{q})(\xi_2 - \xi_1) \sum_{\mathfrak{n}=0}^{\infty} \mathfrak{q}^{\mathfrak{n}} \mathfrak{G}(\mathfrak{q}^{\mathfrak{n}}\xi_1 + (1-\mathfrak{q}^{\mathfrak{n}})\xi_2) \\ &= (\xi_2 - \xi_1) \int_0^1 \mathfrak{G}(\lambda\xi_1 + (1-\lambda)\xi_2) d_{\mathfrak{q}}\lambda. \end{aligned} \quad (5)$$

Definition 6 ([14])**.** *If $\mathfrak{G} : [\xi_1, \xi_2] \to \mathbb{R}$ be a function, then $\mathfrak{q}^{\xi_2}$-derivative of $\mathfrak{G}$ at $\mathfrak{x} \in [\xi_1, \xi_2]$ is expressed as:*

$$^{\xi_2}D_{\mathfrak{q}}\mathfrak{G}(\mathfrak{x}) = \frac{\mathfrak{G}(\mathfrak{q}\mathfrak{x} + (1-\mathfrak{q})\xi_2) - \mathfrak{G}(\mathfrak{x})}{(1-\mathfrak{q})(\xi_2 - \mathfrak{x})}, \quad \mathfrak{x} \neq \xi_2.$$

Theorem 2 ([14])**.** *Let $\mathfrak{G} : [\xi_1, \xi_2] \to \mathbb{R}$ is a convex function on $[\xi_1, \xi_2]$ and $\mathfrak{q} \in (0,1)$, then $\mathfrak{q}$-Hermite–Hadamard inequalities are given bellow:*

$$\mathfrak{G}\left(\frac{\xi_1 + \mathfrak{q}\xi_2}{1+\mathfrak{q}}\right) \leq \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\mathfrak{x})^{\xi_2} d_{\mathfrak{q}}\mathfrak{x} \leq \frac{\mathfrak{G}(\xi_1) + \mathfrak{q}\mathfrak{G}(\xi_2)}{1+\mathfrak{q}}. \quad (6)$$

The following notations were frequently used:

$$[\mathfrak{n}]_{\mathfrak{q}} = \sum_{i=0}^{\mathfrak{n}-1} \mathfrak{q}^i$$

and

$$(1-\lambda)^{\mathfrak{n}}_{\mathfrak{q}} = (\lambda, \mathfrak{q})_{\mathfrak{n}} = \prod_{i=0}^{\mathfrak{n}-1} (1 - \mathfrak{q}^i \lambda), \quad (7)$$

where $\mathfrak{q} \in (0,1)$.

Lemma 1 ([13])**.** *Following equality holds:*

$$\int_{\xi_1}^{\mathfrak{x}} (\lambda - \xi_1)^{\alpha}{}_{\xi_1} d_{\mathfrak{q}}\lambda = \frac{(\mathfrak{x} - \xi_1)^{\alpha+1}}{[\alpha+1]_{\mathfrak{q}}}, \quad (8)$$

for $\alpha \in \mathbb{R} \backslash \{-1\}$.

Lemma 2 ([15]). *Following equality holds:*

$$\int_{1/[2]_q}^{1} (1-q\lambda)_q^n d_q\lambda = \frac{(1-\frac{1}{[2]_q})_q^{n+1}}{[n+1]_q}.$$

The q-analogues of integral inequalities are a subject of utmost interest while concentrating on the exhilaration and enchantment of the emergence of q-calculus and its applicability in mathematical physics. Taking into account ${}_{\xi_1}D_q$-derivative, q_{ξ_1}-integral, ${}^{\xi_2}D_q$-derivative and q^{ξ_2}-integral, quantum variants of notable integral inequalities have been investigated pertaining various kinds of convexity (see [7,16–19]). Some quantum inequalities for coordinate convex functions can be seen in [20,21]. However, the notable bounds via q-midpoint inequalities can be observed in [13,15,22].

The compelling goal of this study, which was inspired by the current trend, is to set new limits for q-midpoint inequalities. Firstly, we established an auxiliary identity pertaining to twice different quantum functions. Then, by employing (α, m)-convexity to this identity, we gain some new results for quantum midpoint inequalities for twice q-differentiable functions. By choosing $\alpha = 1$, $m = 1$ and taking $q \to 1^-$, we can also recapture the findings in a classical sense.

3. New Quantum Midpoint Type Identity for Twice q-Differentiable Functions

We will demonstrate equality in this section which help us in achieving our main goals.

Lemma 3. *Let $\mathfrak{G} : [\xi_1, \xi_2] \to \Re$ be a q-differentiable function on (ξ_1, ξ_2). If ${}^{\xi_2}D_q^2\mathfrak{G}$ is continuous and integrable on $[\xi_1, \xi_2]$, we attain the identity:*

$$\frac{(m\xi_2-\xi_1)^2}{[2]_q}\left[\int_0^{1/[2]_q} q^3\lambda^2\, {}^{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)d_q\lambda \right.$$
$$\left. + \int_{1/[2]_q}^{1} (1-q\lambda)_q^2\, {}^{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)d_q\lambda\right]$$
$$= \frac{1}{(m\xi_2-\xi_1)}\int_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\, {}^{m\xi_2}d_q\lambda - \mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right).$$

Proof. By using Definition 6, we attain the following equality

$$\begin{aligned}
&{}^{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\\
&= {}^{\xi_2}D_q({}^{\xi_2}D_q(\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)))\\
&= {}^{\xi_2}D_q\left(\frac{\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)-\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)}{(1-q)(\xi_2-\xi_1)\lambda}\right)\\
&= \frac{1}{(1-q)(\xi_2-\xi_1)\lambda}\left[\frac{\mathfrak{G}(q^2\lambda\xi_1+m(1-\lambda q^2)\xi_2)-\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)}{(1-q)q(\xi_2-\xi_1)\lambda}\right.\\
&\quad\left.-\frac{\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)-\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)}{(1-q)(\xi_2-\xi_1)\lambda}\right]\\
&= \frac{\mathfrak{G}(q^2\lambda\xi_1+m(1-\lambda q^2)\xi_2)-\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)}{(1-q)^2q(\xi_2-\xi_1)^2\lambda^2}\\
&\quad-\frac{\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)-\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)}{(1-q)^2(\xi_2-\xi_1)^2\lambda^2}
\end{aligned} \tag{9}$$

$$= \frac{\mathfrak{G}(\mathfrak{q}^2\lambda\xi_1 + m(1-\lambda\mathfrak{q}^2)\xi_2) - (1+\mathfrak{q})\mathfrak{G}(\mathfrak{q}\lambda\xi_1 + m(1-\mathfrak{q}\lambda)\xi_2) + \mathfrak{q}\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)m\xi_2)}{(1-\mathfrak{q})^2\mathfrak{q}(\xi_2-\xi_1)^2\lambda^2}.$$

From (9) and using the fundamental properties of $\mathfrak{q}$-integral, we have

$$\int_0^{1/[2]_\mathfrak{q}} \mathfrak{q}^3\lambda^{2\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda + \int_{1/[2]_\mathfrak{q}}^{1} (1-\mathfrak{q}\lambda)_\mathfrak{q}^{2\,\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda$$

$$= \int_0^{1/[2]_\mathfrak{q}} \mathfrak{q}^3\lambda^{2\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda + \int_0^1 (1-\mathfrak{q}\lambda)_\mathfrak{q}^{2\,\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda$$

$$- \int_0^{1/[2]_\mathfrak{q}} (1-\mathfrak{q}\lambda)_\mathfrak{q}^{2\,\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda$$

$$= \int_0^1 (1-\mathfrak{q}\lambda)_\mathfrak{q}^{2\,\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda$$

$$+ \int_0^{1/[2]_\mathfrak{q}} (\mathfrak{q}^3\lambda^2 - (1-\mathfrak{q}\lambda)_\mathfrak{q}^2)^{\xi_2} D_\mathfrak{q}^2\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)d_\mathfrak{q}\lambda$$

$$= \frac{1}{(1-\mathfrak{q})^2(\mathfrak{m}_2-\xi_1)^2}\int_0^1 \frac{(1-\mathfrak{q}\lambda)_\mathfrak{q}^2}{\lambda^2}\bigg(\frac{1}{\mathfrak{q}}\mathfrak{G}((\mathfrak{q}^2\lambda\xi_1 + m(1-\mathfrak{q}^2\lambda)\xi_2))$$

$$-\frac{1+\mathfrak{q}}{\mathfrak{q}}\mathfrak{G}(\mathfrak{q}\lambda\xi_1 + m(1-\mathfrak{q}\lambda)\xi_2) + \mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)\bigg)d_\mathfrak{q}\lambda$$

$$+ \frac{1}{(1-\mathfrak{q})^2(m\xi_2-\xi_1)^2}\int_0^{1/[2]_\mathfrak{q}} \frac{(\mathfrak{q}^3\lambda^2 - (1-\mathfrak{q}\lambda)_\mathfrak{q}^2)}{\lambda^2}\bigg(\frac{1}{\mathfrak{q}}\mathfrak{G}((\mathfrak{q}^2\lambda\xi_1 + m(1-\mathfrak{q}^2\lambda)\xi_2))$$

$$-\frac{1+\mathfrak{q}}{\mathfrak{q}}\mathfrak{G}(\mathfrak{q}\lambda\xi_1 + m(1-\mathfrak{q}\lambda)\xi_2) + \mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2)\bigg)d_\mathfrak{q}\lambda$$

$$= \frac{[I_1 + I_2]}{(1-\mathfrak{q})^2(m\xi_2-\xi_1)^2}.$$

Calculate the values of integrals I_1 and I_2 in the following way:

$$I_1 = \int_0^1 \frac{(1-\mathfrak{q}\lambda)_\mathfrak{q}^2}{\lambda^2}\bigg(\frac{1}{\mathfrak{q}}\mathfrak{G}\Big(\mathfrak{q}^2\lambda\xi_1 + m(1-\mathfrak{q}^2\lambda)\xi_2\Big) - \frac{1+\mathfrak{q}}{\mathfrak{q}}\mathfrak{G}(\mathfrak{q}\lambda\xi_1 + m(1-\mathfrak{q}\lambda)\xi_2)$$

$$+\mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2))d_\mathfrak{q}\lambda$$

$$= (1-\mathfrak{q})\sum_{\mathfrak{n}=0}^{\infty}\mathfrak{q}^\mathfrak{n}\frac{(1-\mathfrak{q}\mathfrak{q}^\mathfrak{n})_\mathfrak{q}^2}{\mathfrak{q}^{2\mathfrak{n}}}\bigg\{\frac{1}{\mathfrak{q}}\mathfrak{G}\Big(\mathfrak{q}^2\mathfrak{q}^\mathfrak{n}\xi_1 + m(1-\mathfrak{q}^2\mathfrak{q}^\mathfrak{n})\xi_2\Big)$$

$$-\frac{1+\mathfrak{q}}{\mathfrak{q}}\mathfrak{G}(\mathfrak{q}\mathfrak{q}^\mathfrak{n}\xi_1 + m(1-\mathfrak{q}\mathfrak{q}^\mathfrak{n})\xi_2) + \mathfrak{G}(\mathfrak{q}^\mathfrak{n}\xi_1 + m(1-\mathfrak{q}^\mathfrak{n})\xi_2)\bigg\}$$

$$= (1-\mathfrak{q})\bigg\{\frac{1}{\mathfrak{q}}\sum_{\mathfrak{n}=0}^{\infty}\frac{(1-\mathfrak{q}\mathfrak{q}^\mathfrak{n})_\mathfrak{q}^2}{\mathfrak{q}^\mathfrak{n}}\mathfrak{G}((\mathfrak{q}^{\mathfrak{n}+2}\xi_1 + mb(1-\mathfrak{q}^{\mathfrak{n}+2})\xi_2))$$

$$-\frac{1+\mathfrak{q}}{\mathfrak{q}}\sum_{\mathfrak{n}=0}^{\infty}\frac{(1-\mathfrak{q}\mathfrak{q}^\mathfrak{n})_\mathfrak{q}^2}{\mathfrak{q}^\mathfrak{n}}\mathfrak{G}(\mathfrak{q}^{\mathfrak{n}+1}\xi_1 + m(1-\mathfrak{q}^{\mathfrak{n}+1})\xi_2)$$

$$+\sum_{\mathfrak{n}=0}^{\infty}\frac{(1-\mathfrak{q}\mathfrak{q}^\mathfrak{n})_\mathfrak{q}^2}{\mathfrak{q}^\mathfrak{n}}\mathfrak{G}(\mathfrak{q}^\mathfrak{n}\xi_1 + m(1-\mathfrak{q}^\mathfrak{n})\xi_2)\bigg\}$$

$$
\begin{aligned}
&= \frac{(1-q)}{q}\sum_{n=2}^{\infty}\frac{(1-qq^{n-2})_q^2}{q^{n-2}}\mathfrak{G}((q^n\xi_1+m(1-q^n)\xi_2))\\
&- \frac{(1-q)(1+q)}{q}\sum_{n=1}^{\infty}\frac{(1-qq^{n-1})_q^2}{q^{n-1}}\mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2)\\
&+ (1-q)\sum_{n=0}^{\infty}\frac{(1-qq^{n})_q^2}{q^{n}}\mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2)\\
&= \frac{(1-q)}{q}\sum_{n=0}^{\infty}\frac{(1-qq^{n-2})_q^2}{q^{n-2}}\mathfrak{G}((q^n\xi_1+m(1-q^n)\xi_2))\\
&- \frac{(1-q)(1+q)}{q}\sum_{n=0}^{\infty}\frac{(1-qq^{n-1})_q^2}{q^{n-1}}\mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2)\\
&+ (1-q)\sum_{n=0}^{\infty}\frac{(1-qq^{n})_q^2}{q^{n}}\mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2) + \frac{(1-q)(1+q)}{q}\frac{(1-qq^{-1})_q^2}{q^{-1}}\mathfrak{G}(\xi_1)\\
&- \frac{(1-q)}{q}\frac{(1-qq^{-2})_q^2}{q^{-2}}\mathfrak{G}(\xi_1) - \frac{(1-q)}{q}\frac{(1-qq^{-1})_q^2}{q^{-1}}\mathfrak{G}((q\xi_1+m(1-q)\xi_2))\\
&= (1-q)\sum_{n=0}^{\infty}\left\{\frac{(1-qq^{n-2})_q^2}{q^{n-1}} - \frac{(1+q)}{q}\frac{(1-qq^{n-1})_q^2}{q^{n-1}} + \frac{(1-qq^{n})_q^2}{q^{n}}\right\}\\
&\times \mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2)\\
&+ \left[\left(1-q^2\right)(1-qq^{-1})_q^2 - q(1-q)(1-qq^{-2})_q^2\right]\mathfrak{G}(\xi_1)\\
&- (1-q)(1-qq^{-1})_q^2\mathfrak{G}(q\xi_1+m(1-q)\xi_2)\\
&= (1+q)(1-q)^2(1-q)\sum_{n=0}^{\infty}q^n\mathfrak{G}(q^n\xi_1+m(1-q^n)\xi_2)\\
&= \frac{(1+q)(1-q)^2}{(m\xi_2-\xi_1)}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)^{m\xi_2}d_q\lambda.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_0^{1/[2]_q}\frac{(q^3\lambda^2-(1-q\lambda)_q^2)}{\lambda^2}\left(\frac{1}{q}\mathfrak{G}((q^2\lambda\xi_1+m(1-q^2\lambda)\xi_2))\right.\\
&\left.-\frac{1+q}{q}\mathfrak{G}(q\lambda\xi_1+m(1-q\lambda)\xi_2)+\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\right)d_q\lambda\\
&= \frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)}{q^n}\\
&\times\left(\frac{1}{q}\mathfrak{G}\left(\frac{q^{n+2}}{[2]_q}\xi_1+m\left(1-\frac{q^{n+2}}{[2]_q}\right)\xi_2\right)-\frac{1+q}{q}\mathfrak{G}\left(\frac{q^{n+1}}{[2]_q}\xi_1+m\left(1-\frac{q^{n+1}}{[2]_q}\right)\xi_2\right)\right.\\
&\left.+\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\right)\\
&= \frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)}{q^{n+1}}\mathfrak{G}\left(\frac{q^{n+2}}{[2]_q}\xi_1+m\left(1-\frac{q^{n+2}}{[2]_q}\right)\xi_2\right)\\
&- \frac{(1-q)(1+q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)}{q^{n+1}}\mathfrak{G}\left(\frac{q^{n+1}}{[2]_q}\xi_1+m\left(1-\frac{q^{n+1}}{[2]_q}\right)\xi_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)}{q^n}\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\\
&=\frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n-4}-[2]_q^2\left(1-q\frac{q^{n-2}}{[2]_q}\right)_q^2\right)}{q^{n-1}}\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\\
&-\frac{(1-q)(1+q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n-2}-[2]_q^2\left(1-q\frac{q^{n-1}}{[2]_q}\right)_q^2\right)}{q^n}\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\\
&+\frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)}{q^n}\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\\
&-\frac{(1-q)}{[2]_q}\frac{\left(q^3q^{-4}-[2]_q^2\left(1-q\frac{q^{-2}}{[2]_q}\right)_q^2\right)}{q^{-1}}\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\\
&-\frac{(1-q)}{[2]_q}\left(q^3q^{-2}-[2]_q^2\left(1-q\frac{q^{-1}}{[2]_q}\right)_q^2\right)\mathfrak{G}\left(\frac{q\xi_1+m\xi_2}{[2]_q}\right)\\
&+\frac{(1-q)(1+q)}{[2]_q}\left(q^3q^{-2}-[2]_q^2\left(1-q\frac{q^{-1}}{[2]_q}\right)_q^2\right)\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\\
&=\frac{(1-q)}{[2]_q}\sum_{n=0}^{\infty}\frac{1}{q^n}\left\{\begin{array}{c} q\left(q^3q^{2n-4}-[2]_q^2\left(1-q\frac{q^{n-2}}{[2]_q}\right)_q^2\right)\\ -(1+q)\left(q^3q^{2n-2}-[2]_q^2\left(1-q\frac{q^{n-1}}{[2]_q}\right)_q^2\right)\\ +\left(q^3q^{2n}-[2]_q^2\left(1-q\frac{q^n}{[2]_q}\right)_q^2\right)\end{array}\right\}\\
&\times\mathfrak{G}\left(\frac{q^n}{[2]_q}\xi_1+m\left(1-\frac{q^n}{[2]_q}\right)\xi_2\right)\\
&+\frac{(1-q)}{[2]_q}\left\{\begin{array}{c}(1+q)\left(q^3q^{-2}-[2]_q^2\left(1-q\frac{q^{-1}}{[2]_q}\right)_q^2\right)\\ -q\left(q^3q^{-4}-[2]_q^2\left(1-q\frac{q^{-2}}{[2]_q}\right)_q^2\right)\end{array}\right\}\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\\
&-\frac{(1-q)}{[2]_q}\left(q^3q^{-2}-[2]_q^2\left(1-q\frac{q^{-1}}{[2]_q}\right)_q^2\right)\mathfrak{G}\left(\frac{q\xi_1+m\xi_2}{[2]_q}\right)\\
&=\frac{(1-q)}{[2]_q}\left\{\left[-1+q^2+q^3-q\right]\right\}\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\\
&=-(1-q)^2(1+q)\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_0^{1/[2]_q} q^3\lambda^2{}_{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)d_q\lambda+\int_{1/[2]_q}^{1}(1-q\lambda)_q^2{}_{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)d_q\lambda\\
&=\frac{1}{(1-q)^2(m\xi_2-\xi_1)^2}\left(\frac{(1+q)(1-q)^2}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)\,{}^{m\xi_2}d_q\lambda-(1-q)^2(1+q)\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\right)
\end{aligned}
$$

$$= \frac{[2]_q}{(m\xi_2 - \xi_1)^2} \left(\frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\, {}^{m\xi_2}d_q\lambda - \mathfrak{G}\left(\frac{\xi_1 + qm\xi_2}{[2]_q} \right) \right),$$

which completes the proof. □

Remark 1. *By setting* $q \to 1^-$ *and* $m = 1$*, then we have*

$$\frac{(\xi_2 - \xi_1)^2}{2} \left[\int_0^{1/2} \lambda^2 \mathfrak{G}''(\lambda\xi_1 + (1-\lambda)\xi_2) d\lambda + \int_{1/2}^{1} (1-\lambda)^2 \mathfrak{G}''(\lambda\xi_1 + (1-\lambda)\xi_2) d\lambda \right]$$
$$= \frac{1}{(\xi_2 - \xi_1)} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\lambda) d\lambda - \mathfrak{G}\left(\frac{\xi_1 + \xi_2}{2} \right),$$

which was given in [23].

Remark 2. *By choosing* $m = 1$*, we recapture ([22], Lemma 5).*

Quantum Midpoint Type Inequalities

Theorem 3. *Let the assumptions of Lemma 3 hold. If* $\left| {}^{\xi_2}D_q^2 \mathfrak{G} \right|$ *is* (α, m)*-convex on* $[\xi_1, \xi_2]$*, we have the inequality*

$$\begin{aligned}
&\left| \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\, {}^{m\xi_2}d_q\lambda - \mathfrak{G}\left(\frac{\xi_1 + qm\xi_2}{[2]_q} \right) \right| \\
&\leq \frac{q^3 (m\xi_2 - \xi_1)^2}{[2]_q} \left(\frac{[3]_q \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\xi_1) \right|}{[2]_q^{\alpha+3} [3]_q [\alpha+3]_q} + m \frac{([2]_q^{\alpha}[\alpha+3]_q - [3]_q) \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\xi_2) \right|}{[2]_q^{\alpha+3} [3]_q [\alpha+3]_q} \right) \\
&\quad + \frac{(m\xi_2 - \xi_1)^2}{[2]_q} \left(\left[\frac{[2]_q^2 [\alpha+3]_q \left([2]_q^{\alpha+1} [\alpha+2]_q - q[2]_q^{\alpha+2} [\alpha+1]_q \right)}{[2]_q^{\alpha+3} [\alpha+1]_q [\alpha+2]_q [\alpha+3]_q} \right.\right. \\
&\quad \left. + \frac{[2]_q^2 [\alpha+3]_q (q[\alpha+1]_q - [\alpha+2]_q)}{[2]_q^{\alpha+3} [\alpha+1]_q [\alpha+2]_q [\alpha+3]_q} + \frac{q^3 [\alpha+2]_q \left([2]_q^{\alpha+3} - 1 \right)}{[2]_q^{\alpha+3} [\alpha+1]_q [\alpha+2]_q [\alpha+3]_q} \right] \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\xi_1) \right| \\
&\quad + m \left[\frac{q[2]_q^{\alpha} [\alpha+1]_q - [2]_q^{\alpha+1} + 1}{[2]_q^{\alpha+1} [\alpha+1]_q} - q[2]_q \left(\frac{(2q + q^2)[2]_q^{\alpha} [\alpha+2]_q - ([2]_q^{\alpha+2} - 1)[2]_q}{[2]_q^{\alpha+3} [\alpha+2]_q} \right) \right. \\
&\quad \left.\left. - q^3 \frac{([2]_q^{\alpha+3} - 1)[3]_q - (3q + 3q^2 + q^3)[2]_q^{\alpha} [\alpha+3]_q}{[2]_q^{\alpha+3} [3]_q [\alpha+3]_q} \right] \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\xi_2) \right| \right).
\end{aligned} \tag{10}$$

Proof. Taking modulus on Lemma 3, we get

$$\begin{aligned}
&\left| \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\, {}^{m\xi_2}d_q\lambda - \mathfrak{G}\left(\frac{\xi_1 + qm\xi_2}{[2]_q} \right) \right| \\
&\leq \frac{(m\xi_2 - \xi_1)^2}{[2]_q} \left[\int_0^{1/[2]_q} q^3 \lambda^2 \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2) \right| d_q\lambda \right. \\
&\quad \left. + \int_{1/[2]_q}^{1} (1 - q\lambda)_q^2 \left| {}^{\xi_2}D_q^2 \mathfrak{G}(\lambda\xi_1 + m(1-\lambda)\xi_2) \right| d_q\lambda \right].
\end{aligned}$$

By using (α, m)-convexity of $\left|{}^{\xi_2}D_{\mathrm{q}}^2\mathfrak{G}\right|$, we can write

$$\begin{aligned}
&\left|\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)^{m\xi_2}d_{\mathrm{q}}\lambda-\mathfrak{G}\left(\frac{\xi_1+\mathrm{q}m\xi_2}{[2]_{\mathrm{q}}}\right)\right| \\
&\leq\frac{\mathrm{q}^3(m\xi_2-\xi_1)^2}{[2]_{\mathrm{q}}}\int_0^{1/[2]_{\mathrm{q}}}\left(\lambda^{\alpha+2}\left|{}^{\xi_2}D_{\mathrm{q}}^2\mathfrak{G}(\xi_1)\right|+m\left(\lambda^2-\lambda^{\alpha+2}\right)\left|{}^{\xi_2}D_{\mathrm{q}}^2\mathfrak{G}(\xi_2)\right|\right)d_{\mathrm{q}}\lambda \\
&\quad+\frac{(m\xi_2-\xi_1)^2}{[2]_{\mathrm{q}}}\int_{1/[2]_{\mathrm{q}}}^1\left(\lambda^\alpha(1-\mathrm{q}\lambda)_{\mathrm{q}}^2\left|{}^{\xi_2}D_{\mathrm{q}}^2\mathfrak{G}(\xi_1)\right|+m(1-\lambda^\alpha)(1-\mathrm{q}\lambda)_{\mathrm{q}}^2\left|{}^{\xi_2}D_{\mathrm{q}}^2\mathfrak{G}(\xi_2)\right|\right)d_{\mathrm{q}}\lambda.
\end{aligned} \tag{11}$$

We have the facts that

$$\int_0^{1/[2]_{\mathrm{q}}}\lambda^{\alpha+2}d_{\mathrm{q}}\lambda=\frac{1}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+3]_{\mathrm{q}}}, \tag{12}$$

$$\int_0^{1/[2]_{\mathrm{q}}}\left(\lambda^2-\lambda^3\right)d_{\mathrm{q}}\lambda=\frac{1}{[2]_{\mathrm{q}}^3[3]_{\mathrm{q}}}-\frac{1}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+3]_{\mathrm{q}}}, \tag{13}$$

$$\begin{aligned}
\int_{1/[2]_{\mathrm{q}}}^1\lambda^\alpha(1-\mathrm{q}\lambda)_{\mathrm{q}}^2d_{\mathrm{q}}=&\frac{[2]_{\mathrm{q}}^2[\alpha+3]_{\mathrm{q}}\left([2]_{\mathrm{q}}^{\alpha+1}[\alpha+2]_{\mathrm{q}}-\mathrm{q}[2]_{\mathrm{q}}^{\alpha+2}[\alpha+1]_{\mathrm{q}}\right)}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+1]_{\mathrm{q}}[\alpha+2]_{\mathrm{q}}[\alpha+3]_{\mathrm{q}}} \\
&+\frac{[2]_{\mathrm{q}}^2[\alpha+3]_{\mathrm{q}}(\mathrm{q}[\alpha+1]_{\mathrm{q}}-[\alpha+2]_{\mathrm{q}})}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+1]_{\mathrm{q}}[\alpha+2]_{\mathrm{q}}[\alpha+3]_{\mathrm{q}}}+\frac{\mathrm{q}^3[\alpha+2]_{\mathrm{q}}[\alpha+1]_{\mathrm{q}}\left([2]_{\mathrm{q}}^{\alpha+3}-1\right)}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+1]_{\mathrm{q}}[\alpha+2]_{\mathrm{q}}[\alpha+3]_{\mathrm{q}}}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\int_{1/[2]_{\mathrm{q}}}^1(1-\lambda^\alpha)(1-\mathrm{q}\lambda)_{\mathrm{q}}^2d_{\mathrm{q}}\lambda=&\frac{\mathrm{q}[2]_{\mathrm{q}}^\alpha[\alpha+1]_{\mathrm{q}}-[2]_{\mathrm{q}}^{\alpha+1}+1}{[2]_{\mathrm{q}}^{\alpha+1}[\alpha+1]_{\mathrm{q}}} \\
&-\mathrm{q}[2]_{\mathrm{q}}\left[\frac{(2\mathrm{q}+\mathrm{q}^2)[2]_{\mathrm{q}}^\alpha[\alpha+2]_{\mathrm{q}}-([2]_{\mathrm{q}}^{\alpha+2}-1)[2]_{\mathrm{q}}}{[2]_{\mathrm{q}}^{\alpha+3}[\alpha+2]_{\mathrm{q}}}\right] \\
&-\mathrm{q}^3\left[\frac{([2]_{\mathrm{q}}^{\alpha+3}-1)[3]_{\mathrm{q}}-(3\mathrm{q}+3\mathrm{q}^2+\mathrm{q}^3)[2]_{\mathrm{q}}^\alpha[\alpha+3]_{\mathrm{q}}}{[2]_{\mathrm{q}}^{\alpha+3}[3]_{\mathrm{q}}[\alpha+3]_{\mathrm{q}}}\right].
\end{aligned} \tag{15}$$

By putting (12)–(15) into (11), we can achieve the required results. □

Remark 3. *By taking limit* $\mathrm{q}\to 1^-$ *and* $\alpha=m=1$ *in Theorem 3, we attain*

$$\left|\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\mathfrak{G}(\lambda)d\lambda-\mathfrak{G}\left(\frac{\xi_1+\xi_2}{2}\right)\right|\leq\frac{(\xi_2-\xi_1)^2}{48}\left(\left|\mathfrak{G}''(\xi_1)\right|+\left|\mathfrak{G}''(\xi_2)\right|\right),$$

which was given in ([23], Theorem 3).

Remark 4. *By choosing* $\alpha=1$ *and* $m=1$*, we recapture ([22], Theorem 3).*

For more clarity of results obtained, we provide the following example with graphs ensuring the correctness of the bounds obtained.

Example 1. *Let consider the function* $\mathfrak{G} : [0,1] \to \mathbb{R}$ *defined by* $\mathfrak{G}(\xi) = \xi^3$ *and let* $m = \frac{1}{2}$ *and* $\alpha = 1$. *Under these assumptions, we have*

$$\int\limits_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\,{}^{m\xi_2}d_q\lambda = \int_0^{1/2} \zeta^3\ {}^{\frac{1}{2}}d_q\zeta = \frac{1}{16}\left[1 - \frac{3}{[2]_q} + \frac{3}{[3]_q} - \frac{1}{[4]_q}\right]$$

and

$$^{\xi_2}D_q^2\mathfrak{G}(\xi) = \left([4]_q + q[2]_q\right)\xi + (2+q)\left(1 - q^2\right).$$

It is clear that $\left|{}^{\xi_2}D_q^2\mathfrak{G}\right|$ *is convex on* $[0,1]$. *So we can apply Theorem 3 to the function defined by* $\mathfrak{G}(\xi) = \xi^3$. *Thus, the left hand side of the inequality (10) reduces to*

$$\left|\frac{1}{m\xi_2 - \xi_1}\int\limits_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda)\,{}^{m\xi_2}d_q\lambda - \mathfrak{G}\left(\frac{\xi_1 + qm\xi_2}{[2]_q}\right)\right| = \left|\frac{1}{8}\left[1 - \frac{3}{[2]_q} + \frac{3}{[3]_q} - \frac{1}{[4]_q}\right] - \frac{q^3}{8[2]_q^3}\right|.$$

On the other hand, since we have the facts that

$$|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_1)| = (2+q)\left(1 - q^2\right),$$

and

$$|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_2)| = 3[2]_q.$$

then the right hand side of the inequality (10) reduces to

$$\begin{aligned}
&\frac{q^3(m\xi_2 - \xi_1)^2}{[2]_q}\left(\frac{[3]_q|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_1)|}{[2]_q^{\alpha+3}[3]_q[\alpha+3]_q} + m\frac{([2]_q^{\alpha}[\alpha+3]_q - [3]_q)|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_2)|}{[2]_q^{\alpha+3}[3]_q[\alpha+3]_q}\right) \\
&\quad + \frac{(m\xi_2 - \xi_1)^2}{[2]_q}\left(\left[\frac{[2]_q^2[\alpha+3]_q\left([2]_q^{\alpha+1}[\alpha+2]_q - q[2]_q^{\alpha+2}[\alpha+1]_q\right)}{[2]_q^{\alpha+3}[\alpha+1]_q[\alpha+2]_q[\alpha+3]_q}\right.\right. \\
&\quad \left. + \frac{[2]_q^2[\alpha+3]_q(q[\alpha+1]_q - [\alpha+2]_q)}{[2]_q^{\alpha+3}[\alpha+1]_q[\alpha+2]_q[\alpha+3]_q} + \frac{q^3[\alpha+2]_q\left([2]_q^{\alpha+3} - 1\right)}{[2]_q^{\alpha+3}[\alpha+1]_q[\alpha+2]_q[\alpha+3]_q}\right]\left|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right| \\
&\quad + m\left[\frac{q[2]_q^{\alpha}[\alpha+1]_q - [2]_q^{\alpha+1} + 1}{[2]_q^{\alpha+1}[\alpha+1]_q} - q[2]_q\left(\frac{(2q+q^2)[2]_q^{\alpha}[\alpha+2]_q - ([2]_q^{\alpha+2} - 1)[2]_q}{[2]_q^{\alpha+3}[\alpha+2]_q}\right)\right. \\
&\quad \left.\left. - q^3\frac{([2]_q^{\alpha+3} - 1)[3]_q - (3q + 3q^2 + q^3)[2]_q^{\alpha}[\alpha+3]_q}{[2]_q^{\alpha+3}[3]_q[\alpha+3]_q}\right]\left|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|\right) \\
&= \frac{3q^3([2]_q[4]_q - [3]_q)}{8[2]_q^4[3]_q[4]_q} + \frac{(2+q)(1-q)q^3}{4[2]_q^4[4]_q} \\
&\quad + \frac{(2+q)(1-q)}{4}\left[\frac{[3]_q - q[2]_q^2}{[2]_q[3]_q} - \frac{1}{[2]_q^3[3]_q} + \frac{q^3([2]_q^4 - 1)}{[2]_q^5[4]_q}\right] \\
&\quad + \frac{3}{2}\left[\frac{(q-1)[2]_q^2 + 1}{[2]_q^2} - \frac{q(2q+q^2)[3]_q - [2]_q^3 + 1}{[2]_q[3]_q}\right. \\
&\quad \left. - q^3\frac{([2]_q^4 - 1)[3]_q - (3q + 3q^2 + q^3)[2]_q[4]_q}{[2]_q^3[3]_q[4]_q}\right].
\end{aligned}$$

By the inequality (10), we have the inequality

$$\begin{aligned}
&\left|\frac{1}{8}\left[1-\frac{3}{[2]_q}+\frac{3}{[3]_q}-\frac{1}{[4]_q}\right]-\frac{q^3}{8[2]_q^3}\right| \\
&\leq \frac{3q^3([2]_q[4]_q-[3]_q)}{8[2]_q^4[3]_q[4]_q}+\frac{(2+q)(1-q)q^3}{4[2]_q^4[4]_q} \\
&\quad+\frac{(2+q)(1-q)}{4}\left[\frac{[3]_q-q[2]_q^2}{[2]_q[3]_q}-\frac{1}{[2]_q^3[3]_q}+\frac{q^3([2]_q^4-1)}{[2]_q^5[4]_q}\right] \\
&\quad+\frac{3}{2}\left[\frac{(q-1)[2]_q^2+1}{[2]_q^2}-\frac{q(2q+q^2)[3]_q-[2]_q^3+1}{[2]_q[3]_q}\right. \\
&\quad\left.-q^3\frac{([2]_q^4-1)[3]_q-(3q+3q^2+q^3)[2]_q[4]_q}{[2]_q^3[3]_q[4]_q}\right].
\end{aligned} \tag{16}$$

One can see the validity of the inequality (16) in Figure 1.

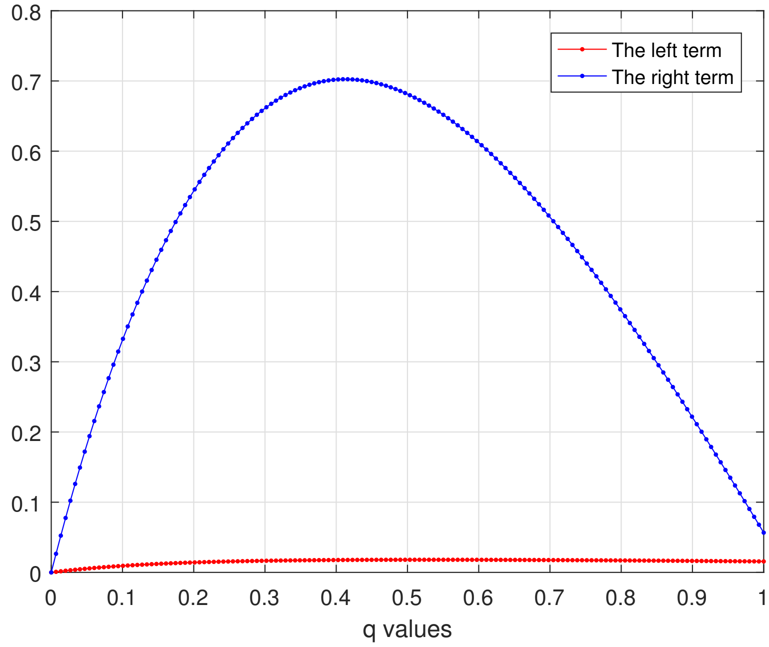

Figure 1. An example to Theorem 3, depending on q, computed and plotted with MATLAB.

Theorem 4. *With the suppositions of Lemma 3, if* $\left|{}^{\xi_2}D_q^2\mathfrak{G}\right|^{\ell_1}$, $\ell_1>1$, *is* (α,m)*-convex function on* $[\xi_1,\xi_2]$*, we have*

$$\begin{aligned}
&\left|\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)\,{}^{m\xi_2}d_q\lambda-\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\right| \\
&\leq \frac{q^3(m\xi_2-\xi_1)^2}{[2]_q^{\frac{\ell_1+\alpha+1}{\ell_1}+\frac{2\ell_2+1}{\ell_2}}[\alpha+1]_q^{\frac{1}{\ell_1}}[2\ell_2+1]_q^{\frac{1}{\ell_2}}}\left(\left|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}+m\left([2]_q^{\alpha}[\alpha+1]_q-1\right)\left|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\right)^{\frac{1}{\ell_1}} \\
&\quad+\frac{(m\xi_2-\xi_1)^2(1-\frac{1}{[2]_q})_q^{\frac{2\ell_2+1}{\ell_2}}}{[2]_q^{\frac{\ell_1+\alpha+1}{\ell_1}}[\alpha+1]_q^{\frac{1}{\ell_1}}[2\ell_2+1]_q^{\frac{1}{\ell_2}}} \\
&\quad\times\left(([2]_q^{\alpha+1}-1)|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_1)|^{\ell_1}+m\left(q[2]_q^{\alpha}[\alpha+1]_q-[2]_q^{\alpha+1}+1\right)\left|{}^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\right)^{\frac{1}{\ell_1}},
\end{aligned} \tag{17}$$

where $\frac{1}{\ell_1}+\frac{1}{\ell_2}=1$.

Proof. By implementing Hölder's inequality on Lemma 3, we attain

$$\begin{aligned}
&\left|\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)\,^{m\xi_2}d_q\lambda-\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\right|\\
&\leq\frac{(m\xi_2-\xi_1)^2}{[2]_q}q^3\left(\int_0^{1/[2]_q}\lambda^{2\ell_2}d_q\lambda\right)^{\frac{1}{\ell_2}}\left(\int_0^{1/[2]_q}\left|^{\xi_2}D_q^2\mathfrak{G}(\lambda a+m(1-\lambda)\xi_2)\right|^{\ell_1}d_q\lambda\right)^{\frac{1}{\ell_1}}\\
&\quad+\frac{(m\xi_2-\xi_1)^2}{[2]_q}\left(\int_{1/[2]_q}^{1}(1-q\lambda)_q^{2\ell_2}d_q\lambda\right)^{\frac{1}{\ell_2}}\left(\int_{1/[2]_q}^{1}\left|^{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\right|^{\ell_1}d_q\lambda\right)^{\frac{1}{\ell_1}}.
\end{aligned}$$

Since $\left|^{\xi_2}D_q^2\mathfrak{G}\right|^{\ell_1}$ is (α,m)-convex on $[\xi_1,\xi_2]$, we have

$$\left|^{\xi_2}D_q^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\right|^{\ell_1}\leq\lambda^\alpha\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}+m(1-\lambda^\alpha)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}.$$

By Lemma 2, we get

$$\begin{aligned}
&\left|\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)\,^{m\xi_2}d_q\lambda-\mathfrak{G}\left(\frac{\xi_1+qm\xi_2}{[2]_q}\right)\right|\\
&\leq\frac{(m\xi_2-\xi_1)^2}{[2]_q}q^3\left(\frac{1}{[2]_q^{2\ell_2+1}[2\ell_2+1]_q}\right)^{\frac{1}{\ell_2}}\\
&\quad\times\left(\int_0^{1/[2]_q}\left(\lambda^\alpha\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}+m(1-\lambda^\alpha)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\right)d_q\lambda\right)^{\frac{1}{\ell_1}}\\
&\quad+\frac{(m\xi_2-\xi_1)^2}{[2]_q}\left(\frac{(1-\frac{1}{[2]_q})_q^{2\ell_2+1}}{[2\ell_2+1]_q}\right)^{\frac{1}{\ell_2}}\\
&\quad\times\left(\int_{1/[2]_q}^{1}\left(\lambda^\alpha\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}+m(1-\lambda^\alpha)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\right)d_q\lambda\right)^{\frac{1}{\ell_1}}\\
&=\left(\frac{q^3(m\xi_2-\xi_1)^2}{[2]_q[2]_q^{\frac{2\ell_2+1}{\ell_2}}[2\ell_2+1]_q^{\frac{1}{\ell_2}}}\right)\left(\frac{\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}}{[2]_q^{\alpha+1}[\alpha+1]_q}+m\frac{\left([2]_q^\alpha[\alpha+1]_q-1\right)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}}{[2]_q^{\alpha+1}[\alpha+1]_q}\right)^{\frac{1}{\ell_1}}\\
&\quad+\frac{(m\xi_2-\xi_1)^2}{[2]_q}\frac{(1-\frac{1}{[2]_q})_q^{\frac{2\ell_2+1}{\ell_2}}}{[2\ell_2+1]_q^{\frac{1}{\ell_2}}}\\
&\quad\times\left(\frac{\left([2]_q^{\alpha+1}-1\right)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_1)\right|^{\ell_1}}{[2]_q^{\alpha+1}[\alpha+1]_q}+m\frac{\left(q[2]_q^\alpha[\alpha+1]_q-[2]_q^{\alpha+1}+1\right)\left|^{\xi_2}D_q^2\mathfrak{G}(\xi_2)\right|^{\ell_1}}{[2]_q^{\alpha+1}[\alpha+1]_q}\right)^{\frac{1}{\ell_1}}
\end{aligned}$$

$$= \left(\frac{\mathfrak{q}^3 (m\xi_2 - \xi_1)^2}{[2]_{\mathfrak{q}}^{\frac{\ell_1+\alpha+1}{\ell_1}} [2]_{\mathfrak{q}}^{\frac{2\ell_2+1}{\ell_2}} [2\ell_2+1]_{\mathfrak{q}}^{\frac{1}{\ell_2}}} \right) \left(\left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_1) \right|^{\ell_1} + ([2]_{\mathfrak{q}}^{\alpha}[\alpha+1]_{\mathfrak{q}} - 1) \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_2) \right|^{\ell_1} \right)^{\frac{1}{\ell_1}}$$
$$+ \frac{(m\xi_2 - \xi_1)^2 (1 - \frac{1}{[2]_{\mathfrak{q}}}) \mathfrak{q}^{\frac{2\ell_2+1}{\ell_2}}}{[2]_{\mathfrak{q}}^{\frac{3+\ell_1}{\ell_1}} [2\ell_2+1]_{\mathfrak{q}}^{\frac{1}{\ell_2}}}$$
$$\times \left(\left([2]_{\mathfrak{q}}^{\alpha+1} - 1 \right) \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_1) \right|^{\ell_1} + m \left(\mathfrak{q}[2]_{\mathfrak{q}}^{\alpha}[\alpha+1]_{\mathfrak{q}} - [2]_{\mathfrak{q}}^{\alpha+1} + 1 \right) \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_2) \right|^{\ell_1} \right)^{\frac{1}{\ell_1}}.$$

This completes the proof. □

Remark 5. *Taking limit $\mathfrak{q} \to 1^-$ and $\alpha = m = 1$ in Theorem 4, we attain*

$$\left| \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\lambda) d\lambda - \mathfrak{G}\left(\frac{\xi_1 + \xi_2}{2} \right) \right|$$
$$\leq \frac{(\xi_2 - \xi_1)^2}{2^{4+\frac{2}{\ell_1}} (2\ell_2+1)^{\frac{1}{\ell_2}}} \left\{ \left(3 \left| \mathfrak{G}''(\xi_1) \right|^{\ell_1} + \left| \mathfrak{G}''(\xi_2) \right|^{\ell_1} \right)^{\frac{1}{\ell_1}} + \left(\left| \mathfrak{G}''(\xi_1) \right|^{\ell_1} + 3 \left| \mathfrak{G}''(\xi_2) \right|^{\ell_1} \right)^{\frac{1}{\ell_1}} \right\},$$

which was given in ([24], Theorem 3).

Remark 6. *By choosing $\alpha = 1$ and $m = 1$, we recapture ([22], Theorem 4).*

Theorem 5. *With the assumptions of Lemma 3, if $\left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G} \right|^{\ell_1}$, $\ell_1 \geq 1$, is (α, m)-convex on $[\xi_1, \xi_2]$, then we have the inequality*

$$\left| \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} \mathfrak{G}(\lambda) \, {}^{m\xi_2}d_{\mathfrak{q}}\lambda - \mathfrak{G}\left(\frac{\xi_1 + \mathfrak{q}m\xi_2}{[2]_{\mathfrak{q}}} \right) \right|$$
$$\leq \frac{\mathfrak{q}^3 (m\xi_2 - \xi_1)^2}{[2]_{\mathfrak{q}}^{4-\frac{3}{\ell_1}} [3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}} \left(\frac{\left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_1) \right|^{\ell_1}}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+3]_{\mathfrak{q}}} + m \frac{([2]_{\mathfrak{q}}^{\alpha}[\alpha+3]_{\mathfrak{q}} - [3]_{\mathfrak{q}}) \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_2) \right|^{\ell_1}}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+3]_{\mathfrak{q}} [3]_{\mathfrak{q}}} \right)^{\frac{1}{\ell_1}}$$
$$+ \frac{(\mathfrak{q} + \mathfrak{q}^2 - \mathfrak{q}^3)^{1-\frac{1}{\ell_1}} (m\xi_2 - \xi_1)^2}{[2]_{\mathfrak{q}}^{4-\frac{3}{\ell_1}} [3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}} \left(\left[\frac{[2]_{\mathfrak{q}}^2 [\alpha+3]_{\mathfrak{q}} ([2]_{\mathfrak{q}}^{\alpha+1} [\alpha+2]_{\mathfrak{q}} - \mathfrak{q}[2]_{\mathfrak{q}}^{\alpha+2} [\alpha+1]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+1]_{\mathfrak{q}} [\alpha+2]_{\mathfrak{q}} [\alpha+3]_{\mathfrak{q}}} \right. \right.$$
$$\left. + \frac{[2]_{\mathfrak{q}}^2 [\alpha+3]_{\mathfrak{q}} (\mathfrak{q}[\alpha+1]_{\mathfrak{q}} - [\alpha+2]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+1]_{\mathfrak{q}} [\alpha+2]_{\mathfrak{q}} [\alpha+3]_{\mathfrak{q}}} + \frac{\mathfrak{q}^3 [\alpha+2]_{\mathfrak{q}} ([2]_{\mathfrak{q}}^{\alpha+3} - 1)}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+1]_{\mathfrak{q}} [\alpha+2]_{\mathfrak{q}} [\alpha+3]_{\mathfrak{q}}} \right] \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_1) \right|^{\ell_1}$$
$$+ m \left(\frac{(\mathfrak{q}[2]_{\mathfrak{q}}^{\alpha}[\alpha+1]_{\mathfrak{q}} - [2]_{\mathfrak{q}}^{\alpha+1} + 1)[2]_{\mathfrak{q}}^2 [\alpha+2]_{\mathfrak{q}}}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+1]_{\mathfrak{q}} [\alpha+2]_{\mathfrak{q}}} \right.$$
$$- \frac{\mathfrak{q}[2]_{\mathfrak{q}} [\alpha+1]_{\mathfrak{q}} ((2\mathfrak{q} + \mathfrak{q}^2)[2]_{\mathfrak{q}}^{\alpha} [\alpha+2]_{\mathfrak{q}} ([2]_{\mathfrak{q}}^{\alpha+2} - 1)[2]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3} [\alpha+1]_{\mathfrak{q}} [\alpha+2]_{\mathfrak{q}}}$$
$$\left. \left. - \mathfrak{q}^3 \frac{([2]_{\mathfrak{q}}^{\alpha+3} - 1)[3]_{\mathfrak{q}} - (3\mathfrak{q} + 3\mathfrak{q}^2 + \mathfrak{q}^3)[2]_{\mathfrak{q}} [\alpha+3]_{\mathfrak{q}}}{[2]_{\mathfrak{q}}^{\alpha+3} [3]_{\mathfrak{q}} [\alpha+3]_{\mathfrak{q}}} \right) \left| {}^{\xi_2}D_{\mathfrak{q}}^2 \mathfrak{G}(\xi_2) \right|^{\ell_1} \right)^{\frac{1}{\ell_1}}.$$

Proof. Using Power-mean inequality on Lemma 3 and then also using (α, m)-convexity of $\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}\right|^{\ell_1}$, we get

$$\left|\frac{1}{m\xi_2-\xi_1}\int_{\xi_1}^{m\xi_2}\mathfrak{G}(\lambda)\,{}^{m\xi_2}d_{\mathfrak{q}}\lambda-\mathfrak{G}\left(\frac{\xi_1+\mathfrak{q}m\xi_2}{[2]_{\mathfrak{q}}}\right)\right|$$

$$\leq\frac{(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}}\mathfrak{q}^3\left(\int_0^{1/[2]_{\mathfrak{q}}}\lambda^2d_{\mathfrak{q}}\lambda\right)^{1-\frac{1}{\ell_1}}\left(\int_0^{1/[2]_{\mathfrak{q}}}\lambda^2\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\right|^{\ell_1}d_{\mathfrak{q}}\lambda\right)^{\frac{1}{\ell_1}}$$

$$+\frac{(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}}\left(\int_{1/[2]_{\mathfrak{q}}}^{1}(1-\mathfrak{q}\lambda)_{\mathfrak{q}}^2d_{\mathfrak{q}}\lambda\right)^{1-\frac{1}{\ell_1}}$$

$$\times\left(\int_{1/[2]_{\mathfrak{q}}}^{1}(1-\mathfrak{q}\lambda)_{\mathfrak{q}}^2\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\lambda\xi_1+m(1-\lambda)\xi_2)\right|^{\ell_1}d_{\mathfrak{q}}\lambda\right)^{\frac{1}{\ell_1}}$$

$$\leq\frac{(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}}\frac{\mathfrak{q}^3}{[2]_{\mathfrak{q}}^{3-\frac{3}{\ell_1}}[3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}}$$

$$\times\left(\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_1)\right|^{\ell_1}\int_0^{1/[2]_{\mathfrak{q}}}\lambda^{\alpha+2}d_{\mathfrak{q}}\lambda+m\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\int_0^{1/[2]_{\mathfrak{q}}}\left(\lambda^2-\lambda^{\alpha+2}\right)d_{\mathfrak{q}}\lambda\right)^{\frac{1}{\ell_1}}$$

$$+\frac{(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}}\frac{\left[(1-\frac{1}{[2]_{\mathfrak{q}}})_{\mathfrak{q}}^3\right]^{1-\frac{1}{\ell_1}}}{[3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}}$$

$$\times\left(\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_1)\right|^{\ell_1}\int_{1/[2]_{\mathfrak{q}}}^{1}\lambda^{\alpha}(1-\mathfrak{q}\lambda)_{\mathfrak{q}}^2d_{\mathfrak{q}}\lambda+m\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\int_{1/[2]_{\mathfrak{q}}}^{1}(1-\lambda^{\alpha})(1-\mathfrak{q}\lambda)_{\mathfrak{q}}^2d_{\mathfrak{q}}\lambda\right)^{\frac{1}{\ell_1}}$$

$$=\frac{\mathfrak{q}^3(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}^{4-\frac{3}{\ell_1}}[3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}}\left(\frac{\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_1)\right|^{\ell_1}}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+3]_{\mathfrak{q}}}+m\frac{([2]_{\mathfrak{q}}^{\alpha}[\alpha+3]_{\mathfrak{q}}-[3]_{\mathfrak{q}})\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_2)\right|^{\ell_1}}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+3]_{\mathfrak{q}}[3]_{\mathfrak{q}}}\right)^{\frac{1}{\ell_1}}$$

$$+\frac{(\mathfrak{q}+\mathfrak{q}^2-\mathfrak{q}^3)^{1-\frac{1}{\ell_1}}(m\xi_2-\xi_1)^2}{[2]_{\mathfrak{q}}^{4-\frac{3}{\ell_1}}[3]_{\mathfrak{q}}^{1-\frac{1}{\ell_1}}}\left(\left[\frac{[2]_{\mathfrak{q}}^2[\alpha+3]_{\mathfrak{q}}([2]_{\mathfrak{q}}^{\alpha+1}[\alpha+2]_{\mathfrak{q}}-\mathfrak{q}[2]_{\mathfrak{q}}^{\alpha+2}[\alpha+1]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+1]_{\mathfrak{q}}[\alpha+2]_{\mathfrak{q}}[\alpha+3]_{\mathfrak{q}}}\right.\right.$$

$$\left.+\frac{[2]_{\mathfrak{q}}^2[\alpha+3]_{\mathfrak{q}}(\mathfrak{q}[\alpha+1]_{\mathfrak{q}}-[\alpha+2]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+1]_{\mathfrak{q}}[\alpha+2]_{\mathfrak{q}}[\alpha+3]_{\mathfrak{q}}}+\frac{\mathfrak{q}^3[\alpha+2]_{\mathfrak{q}}([2]_{\mathfrak{q}}^{\alpha+3}-1)}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+1]_{\mathfrak{q}}[\alpha+2]_{\mathfrak{q}}[\alpha+3]_{\mathfrak{q}}}\right]\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_1)\right|^{\ell_1}$$

$$+m\left(\frac{(\mathfrak{q}[2]_{\mathfrak{q}}^{\alpha}[\alpha+1]_{\mathfrak{q}}-[2]_{\mathfrak{q}}^{\alpha+1}+1)[2]_{\mathfrak{q}}^2[\alpha+2]_{\mathfrak{q}}}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+1]_{\mathfrak{q}}[\alpha+2]_{\mathfrak{q}}}\right.$$

$$-\frac{\mathfrak{q}[2]_{\mathfrak{q}}[\alpha+1]_{\mathfrak{q}}((2\mathfrak{q}+\mathfrak{q}^2)[2]_{\mathfrak{q}}^{\alpha}[\alpha+2]_{\mathfrak{q}}-([2]_{\mathfrak{q}}^{\alpha+2}-1)[2]_{\mathfrak{q}})}{[2]_{\mathfrak{q}}^{\alpha+3}[\alpha+1]_{\mathfrak{q}}[\alpha+2]_{\mathfrak{q}}}$$

$$\left.\left.-\mathfrak{q}^3\frac{([2]_{\mathfrak{q}}^{\alpha+3}-1)[3]_{\mathfrak{q}}-(3\mathfrak{q}+3\mathfrak{q}^2+\mathfrak{q}^3)[2]_{\mathfrak{q}}[\alpha+3]_{\mathfrak{q}}}{[2]_{\mathfrak{q}}^{\alpha+3}[3]_{\mathfrak{q}}[\alpha+3]_{\mathfrak{q}}}\right)\left|{}^{\xi_2}D_{\mathfrak{q}}^2\mathfrak{G}(\xi_2)\right|^{\ell_1}\right)^{\frac{1}{\ell_1}}.$$

This completes the proof. □

Remark 7. *Choosing limit as* $\mathfrak{q} \to 1^-$ *in Theorem 5, we obtain*

$$\left| \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \mathfrak{G}(\lambda) d\lambda - \mathfrak{G}\left(\frac{\xi_1 + \xi_2}{2} \right) \right| \leq \frac{(\xi_2 - \xi_1)^2}{3\left(2^{4+\frac{3}{\ell_1}}\right)} \left\{ \left(5|\mathfrak{G}''(\xi_1)|^{\ell_1} + 3|\mathfrak{G}''(\xi_2)|^{\ell_1} \right)^{\frac{1}{\ell_1}} + \left(3|\mathfrak{G}''(\xi_1)|^{\ell_1} + 5|\mathfrak{G}''(\xi_2)|^{\ell_1} \right)^{\frac{1}{\ell_1}} \right\},$$

which was given in ([24], Theorem 4).

Remark 8. *By choosing* $\alpha = 1$ *and* $m = 1$, *we recapture ([22], Theorem 5).*

4. Concluding Remarks

We conclude our paper by saying that there is not much literature present while dealing with quadrature inequalities for twice quantum differentiable convexities, as they are not easy to work on. Thus, in this study, we give new extensions of quantum midpoint inequalities by employing ${}^{\xi_2}D_q$-derivative and q^{ξ_2} -integral under the influence of twice quantum differentiable (α, m)-convex functions. As a result, several fresh estimates of midpoint inequalities are achieved by utilizing quantum Holder and Power mean inequalities. As an example, we provide graphical analysis to explain the correctness of our results. As a special cases for $m = 1$ and quantum parameter $q = 1$, we recapture results in quantum and classical calculus. This idea may also be expanded to obtain other quadrature inequalities i.e Simpson and Newton type pertaining twice quantum differentiable convexities. One may also think about the case for ${}_{\xi_1}D_q$-derivative and q_{ξ_1}-integral. In future, we aim to work on such estimations of quantum midpoint inequalities by employing Mercer's scheme. It is pertinent to mention that such extensions are quite open to discovery in (p, q)-calculus and for co-ordinated convex functions in quantum calculus. A good open problem is to investigate fractional quantum midpoint-type inequalities. We think it is an intriguing and fresh subject for academics, who can produce equivalent inequalities using various convexities.

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