

On a New Ostrowski-Type Inequality and Related Results

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ABSTRACT. We provide a new Ostrowski-type inequality involving functions of two independent variables, as well as some related results.

1. Introduction

The well-known Ostrowski inequality (see [3]) states that

$$(1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) M,$$

where $f : [a, b] \rightarrow \mathbb{R}$ is a differentiable function such that $|f'| \leq M$ for all $x \in [a, b]$.

This article is greatly motivated and inspired also by the following results.

Theorem 1.([1]) *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I, a < b$. $f : I \rightarrow \mathbb{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f'(x) \leq \Gamma, x \in [a, b]$. Then we have*

$$(2) \quad \left| \frac{1}{2} f(x) - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)} (\Gamma - \gamma)$$

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for all $x \in [a, b]$.

Theorem 2. ([10]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (a, b) and suppose that $\gamma \leq f''(t) \leq \Gamma$ for all $t \in (a, b)$. Then we have the double inequality

$$(3) \quad \frac{3S - \Gamma}{24} (b - a)^2 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{3S - \gamma}{24} (b - a)^2$$

where $S = \frac{f'(b) - f'(a)}{b - a}$.

Theorem 3. ([2]) Under the assumptions of Theorem 2, we have

$$(4) \quad \begin{aligned} & \frac{\Gamma \left[(x - a)^3 + (b - x)^3 \right]}{12(b - a)} + \frac{1}{8} \left[\frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right]^2 (S - \Gamma) \\ & \leq \frac{1}{2} \left[f(x) + \frac{(x - a)f(a) + (b - x)f(b)}{b - a} \right] - \frac{1}{b - a} \int_a^b f(t) dt \\ & \leq \frac{\gamma \left[(x - a)^3 + (b - x)^3 \right]}{12(b - a)} + \frac{1}{8} \left[\frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right]^2 (S - \gamma), \end{aligned}$$

for all $x \in [a, b]$, where $S = \frac{f'(b) - f'(a)}{b - a}$.

Theorem 4. ([7]) Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be an continuous function such that the partial derivative of order 2 exists and supposes that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq \frac{\partial^2 f(t, s)}{\partial t \partial s} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then, we have

$$(5) \quad \begin{aligned} & \left| \frac{1}{4} f(x, y) + \frac{1}{4} H(x, y) - \frac{1}{2(b - a)} \int_a^b f(t, y) dt - \frac{1}{2(d - c)} \int_c^d f(x, s) ds \right. \\ & \quad \left. - \frac{1}{2(b - a)(d - c)} \int_a^b [(y - c)f(t, c) + (d - y)f(t, d)] dt \right. \\ & \quad \left. - \frac{1}{2(b - a)(d - c)} \int_c^d [(x - a)f(a, s) + (b - x)f(b, s)] ds \right. \\ & \quad \left. + \frac{1}{2(b - a)(d - c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq \frac{[(x - a)^2 + (b - x)^2] [(y - c)^2 + (d - y)^2]}{32(b - a)(d - c)} (\Gamma - \gamma) \end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$ where

$$\begin{aligned}
 & H(x, y) \\
 = & \frac{(x-a)[(y-c)f(a, c) + (d-y)f(a, d)]}{(b-a)(d-c)} \\
 & + \frac{(b-x)[(y-c)f(b, c) + (d-y)f(b, d)]}{(b-a)(d-c)} \\
 & + \frac{(x-a)f(a, y) + (b-x)f(b, y)}{b-a} \\
 & + \frac{(y-c)f(x, c) + (d-y)f(x, d)}{d-c}
 \end{aligned}$$

Theorem 5.([5]) Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and suppose that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq \frac{\partial^2 f(t, s)}{\partial t \partial s} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then, we have

$$\begin{aligned}
 & \left| (1-\lambda)^2 f(x, y) + \frac{\lambda}{2}(1-\lambda)[f(a, y) + f(b, y) + f(x, c) + f(x, d)] \right. \\
 & + \left. \left(\frac{\lambda}{2}\right)^2 [f(a, c) + f(b, c) + f(a, d) + f(b, d)] \right. \\
 & - \frac{1}{b-a} \left\{ (1-\lambda) \int_a^b f(t, y) dt + \frac{\lambda}{2} \int_a^b [f(t, c) + f(t, d)] dt \right\} \\
 & - \frac{1}{d-c} \left\{ (1-\lambda) \int_c^d f(x, s) ds + \frac{\lambda}{2} \int_c^d [f(a, s) + f(b, s)] ds \right\} \\
 (6) \quad & - \frac{\Gamma + \gamma}{2} (1-\lambda)^2 \left(x - \frac{a+b}{2}\right) \left(y - \frac{c+d}{2}\right) \\
 & + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \Big| \\
 & \leq \frac{\Gamma - \gamma}{2} \frac{1}{(b-a)(d-c)} \left[(\lambda^2 + (1-\lambda)^2) \frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2 \right] \\
 & \times \left[(\lambda^2 + (1-\lambda)^2) \frac{(d-c)^2}{4} + \left(y - \frac{c+d}{2}\right)^2 \right]
 \end{aligned}$$

for all $(x, y) \in [a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}] \times [c + \lambda \frac{d-c}{2}, d - \lambda \frac{d-c}{2}]$ and $\lambda \in [0, 1]$.

For the other recent Ostrowski type results, see [4],[6],[8] and [9].

The main purpose of this article is to establish a new inequality similar to the inequalities (2)-(6) for higher-order derivatives of f involving functions of two independent variables.

2. Main Results

Theorem 6. Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be an continuous function such that the partial derivative of order 4 exists and supposes that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \leq \Gamma$ for all $(t, s) \in [a, b] \times [c, d]$. Then, we have

$$\begin{aligned} & \left| E(x, y) - \frac{[(x-a)^3 + (b-x)^3][(y-c)^3 + (d-y)^3]}{288} (\Gamma + \gamma) \right| \\ & \leq \frac{[(x-a)^3 + (b-x)^3][(y-c)^3 + (d-y)^3]}{288} (\Gamma - \gamma) \end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$\begin{aligned} E(x, y) &= \frac{(x-a)(y-c)}{4} [f(x, y) + f(x, c) + f(a, y) + f(a, c)] \\ &+ \frac{(d-y)(x-a)}{4} [f(x, y) + f(x, d) + f(a, y) + f(a, d)] \\ &+ \frac{(y-c)(b-x)}{4} [f(x, y) + f(x, c) + f(b, y) + f(b, c)] \\ &+ \frac{(d-y)(b-x)}{4} [f(x, y) + f(x, d) + f(b, y) + f(b, d)] \\ &- \frac{y-c}{2} \int_a^b [f(t, y) + f(t, c)] dt - \frac{d-y}{2} \int_a^b [f(t, d) + f(t, y)] dt \\ &- \frac{x-a}{2} \int_c^d [f(x, s) + f(a, s)] ds - \frac{b-x}{2} \int_c^d [f(b, s) + f(x, s)] ds \\ &+ \int_a^b \int_c^d f(t, s) ds dt. \end{aligned}$$

Proof. We first define the following kernel functions: $p : [a, b] \times [a, b] \rightarrow \mathbb{R}$ and

$q : [c, d] \times [c, d] \rightarrow \mathbb{R}$ given by

$$p(x, t) = \begin{cases} \frac{(x-t)(t-a)}{2}, & a \leq t \leq x \\ \frac{(x-t)(t-b)}{2}, & x < t \leq b \end{cases}$$

and

$$q(y, s) = \begin{cases} \frac{(y-s)(s-c)}{2}, & c \leq s \leq y \\ \frac{(y-s)(s-d)}{2}, & y < s \leq d \end{cases}$$

By the definitions of these kernel functions, we can write

$$\begin{aligned} & \int_a^b \int_c^d p(x, t)q(y, s) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt \\ (1.1) \quad &= \frac{1}{4} \int_a^x \int_c^y (x-t)(t-a)(y-s)(s-c) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt \\ &+ \frac{1}{4} \int_a^x \int_y^d (x-t)(t-a)(y-s)(s-d) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt \\ &+ \frac{1}{4} \int_x^b \int_c^y (x-t)(t-b)(y-s)(s-c) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt \\ &+ \frac{1}{4} \int_x^b \int_y^d (x-t)(t-b)(y-s)(s-d) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt. \end{aligned}$$

Integrating by parts in the right hand side of (1.1), we have

$$\begin{aligned} & \frac{1}{4} \int_a^x \int_c^y (x-t)(t-a)(y-s)(s-c) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt \\ (1.2) \quad &= \frac{(x-a)(y-c)}{4} [f(x, y) + f(x, c) + f(a, y) + f(a, c)] \\ &- \frac{y-c}{2} \int_a^x [f(t, y) + f(t, c)] dt - \frac{x-a}{2} \int_c^y [f(x, s) + f(a, s)] ds \\ &\int_a^x \int_c^y f(t, s) ds dt. \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{4} \int_a^x \int_y^d (x-t)(t-a)(y-s)(s-d) \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt \\
 (1.3) \quad &= \frac{(d-y)(x-a)}{4} [f(x,y) + f(x,d) + f(a,y) + f(a,d)] \\
 & - \frac{d-y}{2} \int_a^x [f(t,d) + f(t,y)] dt - \frac{x-a}{2} \int_y^d [f(x,s) + f(a,s)] ds \\
 & + \int_a^x \int_y^d f(t,s) ds dt.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{4} \int_x^b \int_c^y (x-t)(t-b)(y-s)(s-c) \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt \\
 (1.4) \quad &= \frac{(y-c)(b-x)}{4} [f(x,y) + f(x,c) + f(b,y) + f(b,c)] \\
 & - \frac{y-c}{2} \int_x^b [f(t,y) + f(t,c)] dt - \frac{b-x}{2} \int_c^y [f(b,s) + f(x,s)] ds \\
 & + \int_x^b \int_c^y f(t,s) ds dt.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{4} \int_x^b \int_y^d (x-t)(t-b)(y-s)(s-d) \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt \\
 (1.5) \quad &= \frac{(d-y)(b-x)}{4} [f(x,y) + f(x,d) + f(b,y) + f(b,d)] \\
 & - \frac{d-y}{2} \int_x^b [f(t,d) + f(t,y)] dt - \frac{b-x}{2} \int_y^d [f(b,s) + f(x,s)] ds \\
 & + \int_x^b \int_y^d f(t,s) ds dt.
 \end{aligned}$$

Adding (1.2)-(1.5), we easily deduce

$$\begin{aligned}
 & \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt = \\
 & \frac{(x-a)(y-c)}{4} [f(x,y) + f(x,c) + f(a,y) + f(a,c)] \\
 & + \frac{(d-y)(x-a)}{4} [f(x,y) + f(x,d) + f(a,y) + f(a,d)] \\
 & + \frac{(y-c)(b-x)}{4} [f(x,y) + f(x,c) + f(b,y) + f(b,c)] \\
 (1.6) \quad & + \frac{(d-y)(b-x)}{4} [f(x,y) + f(x,d) + f(b,y) + f(b,d)] \\
 & - \frac{y-c}{2} \int_a^b [f(t,y) + f(t,c)] dt - \frac{d-y}{2} \int_a^b [f(t,d) + f(t,y)] dt \\
 & - \frac{x-a}{2} \int_c^d [f(x,s) + f(a,s)] ds - \frac{b-x}{2} \int_c^d [f(b,s) + f(x,s)] ds \\
 & + \int_a^b \int_c^d f(t,s) ds dt.
 \end{aligned}$$

We also have

$$(1.7) \quad \int_a^b \int_c^d p(x,t) q(y,s) ds dt = \frac{[(x-a)^3 + (b-x)^3] [(y-c)^3 + (d-y)^3]}{144}$$

Let $M = \frac{\Gamma+\gamma}{2}$. From (1.6) and (1.7), we can write

$$\begin{aligned}
 & \int_a^b \int_c^d p(x,t) q(y,s) \left[\frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} - M \right] ds dt \\
 (1.8) \quad & = \int_a^b \int_c^d p(x,t) q(y,s) \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} ds dt \\
 & - \frac{\Gamma + \gamma}{2} \frac{[(x-a)^3 + (b-x)^3] [(y-c)^3 + (d-y)^3]}{144}
 \end{aligned}$$

On the other hand, we get

$$(1.9) \quad \left| \int_a^b \int_c^d p(x, t) q(y, s) \left[\frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} - M \right] ds dt \right| \leq \max_{(t, s) \in [a, b] \times [c, d]} \left| \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} - M \right| \int_a^b \int_c^d |p(x, t) q(y, s)| ds dt$$

Since

$$(1.10) \quad \max_{(t, s) \in [a, b] \times [c, d]} \left| \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} - M \right| \leq \frac{\Gamma - \gamma}{2}$$

and

$$(1.11) \quad \int_a^b \int_c^d |p(x, t) q(y, s)| ds dt = \frac{[(x - a)^3 + (b - x)^3][(y - c)^3 + (d - y)^3]}{144}.$$

By using (1.10) and (1.11) in (1.9), we get

$$(1.12) \quad \left| \int_a^b \int_c^d p(x, t) q(y, s) \left[\frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} - M \right] ds dt \right| \leq \frac{[(x - a)^3 + (b - x)^3][(y - c)^3 + (d - y)^3]}{288} (\Gamma - \gamma).$$

From (1.8) and (1.12), we see that the required result holds. □

Corollaty 7. *By taking $x = a$ and $y = c$ or $x = b$ and $y = d$ in Theorem 6, we have*

$$\begin{aligned} & \left| \frac{(d - c)(b - a)}{4} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right. \\ & - \frac{d - c}{2} \int_a^b [f(t, c) + f(t, d)] dt - \frac{b - a}{2} \int_c^d [f(a, s) + f(b, s)] ds \\ & \left. + \int_a^b \int_c^d f(t, s) ds dt - \frac{(b - a)^3 (d - c)^3}{288} (\Gamma + \gamma) \right| \\ & \leq \frac{(b - a)^3 (d - c)^3}{288} (\Gamma - \gamma). \end{aligned}$$

Corollaty 8. *By taking $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 6, we have*

$$\begin{aligned}
& \left| \frac{(b-a)(d-c)}{16} \left\{ 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \right. \\
& + 2 \left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) \right] \\
& \left. + [f(a, c) + f(a, d) + f(b, c) + f(b, d)] \right\} \\
& - \frac{d-c}{4} \int_a^b \left[f(t, d) + 2f\left(t, \frac{c+d}{2}\right) + f(t, c) \right] dt \\
& - \frac{b-a}{2} \int_c^d \left[f(a, s) + 2f\left(\frac{a+b}{2}, s\right) + f(b, s) \right] ds \\
& + \int_a^b \int_c^d f(t, s) ds dt - \frac{(b-a)^3(d-c)^3}{16 \times 288} (\Gamma + \gamma) \Big| \\
& \leq \frac{(b-a)^3(d-c)^3}{16 \times 288} (\Gamma - \gamma).
\end{aligned}$$

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