

# A Numerical Study of a First Order Modular Grad-Div Stabilization for the Magnetohydrodynamics Equations

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**Abstract.** This paper proposes a stabilization method to approximate analytical solutions of magnetohydrodynamics (MHD) equations. The method adds two modular grad-div steps into fully-discrete finite element MHD solver. The main idea in these intrusive steps is to penalize the divergence of the velocity/magnetic fields both in  $L^2$  and  $H^1$ -norms. The paper confirms the optimal convergence of the method, and gives numerical experiments which reveal positive effect of the method as in the usual grad-div stabilization.

Keywords: Modular grad-div, mixed finite element method, magnetohydrodynamics.

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## INTRODUCTION

This report considers the incompressible magnetohydrodynamics (MHD) equations. The set of equations which describe MHD is a combination of the Navier-Stokes equations of fluid dynamics and Maxwells equations of electro-magnetism through Ohms law : [1]:

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - s(\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla p = \mathbf{f}, \quad (1)$$

$$\mathbf{B}_t - \nu_m \Delta \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = \nabla \times \mathbf{g}, \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad (3)$$

in  $\Omega \times (0, T)$ , where  $\Omega$  is the domain of the fluid,  $\mathbf{u} := (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$  is the velocity of the fluid,  $p(\mathbf{x}, t)$  is the pressure,  $\mathbf{B} := (B_1(\mathbf{x}, t), B_2(\mathbf{x}, t), B_3(\mathbf{x}, t))$  is the magnetic field,  $\nu$  the kinematic viscosity,  $\nu_m$  magnetic resistivity,  $s$  the coupling number, and  $\mathbf{f}$  is the body force,  $\nabla \times \mathbf{g}$  is the forcing on the magnetic field.

Most classical, conforming mixed finite element method enforce the divergence constraint only weakly. This weak enforcement leads to errors depending on the continuous pressure scaled by the Reynolds number, and inaccurate computed solutions for many flow problems [2, 3, 4]

To overcome this issue in conforming mixed finite element methods, this paper proposes a variant of grad-div stabilization which was first introduced in [5]. The method is more attractive from an implementation standpoint and is resistant to solver breakdown as the stabilization parameters increase.

## NUMERICAL SCHEME

Assume that  $\Omega$  in  $\mathbb{R}^d$  ( $d = 2, 3$ ) is a polygonal or polyhedral domain with the boundary  $\partial\Omega$ . We consider the classical function spaces  $\mathbf{X} := (H_0^1(\Omega))^d$ ,  $Q = L_0^2(\Omega)$ ,  $= H_0^1(\Omega)$ . The skew-symmetric trilinear forms are defined by  $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2} ((\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}))$ ,  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$ . The algorithm is given below.

**Algorithm 0.1** Let forcing terms  $\mathbf{f}$ ,  $\nabla \times \mathbf{g}$  and initial conditions  $\mathbf{u}_h^0$ ,  $\mathbf{B}_h^0$  be given. Choose an end time  $T$  and a time step  $\Delta t$  such that  $T = N \Delta t$ . For  $n = 0, 1, 2, \dots, N$ , denote the discrete solutions at time levels  $t^n := n \Delta t$  by  $\mathbf{u}_h^n := \mathbf{u}_h(t^n)$ ,  $\mathbf{B}_h^n := \mathbf{B}_h(t^n)$ ,  $p_h^n := p_h(t^n)$ ,  $\lambda_h^n := \lambda_h(t^n)$ , Find  $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{B}_h^{n+1}, \lambda_h^{n+1})$  via the following two steps:

**Step 1:** Compute  $(\tilde{\mathbf{u}}_h^{n+1}, p_h^{n+1}, \tilde{\mathbf{B}}_h^{n+1}, \lambda_h^{n+1}) \in \mathbf{X}_h \times Q_h \times \mathbf{X}_h \times Q_h$  such that for all  $(\mathbf{v}_h, q_h, \mathbf{w}_h, r_h) \in \mathbf{X}_h \times Q_h \times \mathbf{X}_h \times Q_h$  it

holds

$$\left( \frac{\tilde{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu (\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla \mathbf{v}_h) + b(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) - sb(\mathbf{B}_h^n, \tilde{\mathbf{B}}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \quad (4)$$

$$\left( \frac{\tilde{\mathbf{B}}_h^{n+1} - \mathbf{B}_h^n}{\Delta t}, \mathbf{w}_h \right) + \nu_m (\nabla \tilde{\mathbf{B}}_h^{n+1}, \nabla \mathbf{w}_h) + b(\mathbf{u}_h^n, \tilde{\mathbf{B}}_h^{n+1}, \mathbf{w}_h) - b(\mathbf{B}_h^n, \tilde{\mathbf{u}}_h^{n+1}, \mathbf{w}_h) + (\lambda_h^{n+1}, \nabla \cdot \mathbf{w}_h) = (\nabla \times \mathbf{g}^{n+1}, \mathbf{w}_h), \quad (5)$$

$$(\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, q_h) = 0, \quad (\nabla \cdot \tilde{\mathbf{B}}_h^{n+1}, r_h) = 0. \quad (6)$$

**Step 2:** For  $(\tilde{\mathbf{u}}_h^{n+1}, \tilde{\mathbf{B}}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{X}_h$ , find  $(\mathbf{u}_h^{n+1}, \mathbf{B}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{X}_h$  such that for all  $(\mathbf{v}_h, \mathbf{w}_h) \in \mathbf{X}_h \times \mathbf{X}_h$  it holds

$$(\mathbf{u}_h^{n+1}, \mathbf{v}_h) + (\beta + \gamma \Delta t)(\nabla \cdot \mathbf{u}_h^{n+1}, \nabla \cdot \mathbf{v}_h) = (\tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) + \beta(\nabla \cdot \mathbf{u}_h^n, \nabla \cdot \mathbf{v}_h), \quad (7)$$

$$(\mathbf{B}_h^{n+1}, \mathbf{w}_h) + (\beta + \gamma \Delta t)(\nabla \cdot \mathbf{B}_h^{n+1}, \nabla \cdot \mathbf{w}_h) = (\tilde{\mathbf{B}}_h^{n+1}, \mathbf{w}_h) + \beta(\nabla \cdot \mathbf{B}_h^n, \nabla \cdot \mathbf{w}_h). \quad (8)$$

## NUMERICAL EXPERIMENTS

To confirm convergence rates predicted by the theory, analytical solutions are chosen as

$$\mathbf{u} = \langle \cos(y), \sin(x) \rangle^T, \quad \mathbf{B} = (1 + e^t) \langle \sin(y), \cos(x) \rangle^T, \quad p := \sin(x + y), \quad \text{on } (0, 1)^2.$$

The dimensionless parameters are chosen as  $\nu = \nu_m = s = 1.0$ , and the stabilization parameters  $\gamma = 1.0 = \beta$ . The forcing terms  $\mathbf{f}$  and  $\nabla \times \mathbf{g}$  are calculated from MHD equations. To test the second order spatial convergence rates, first spatial errors are isolated by choosing an end time  $T = 0.01$  and time step  $\Delta t = 0.001$ . Then, the solutions of Algorithm 0.1 are computed with successive mesh refinements using the Taylor-Hood  $(\mathbf{P}_2, P_1)$  finite element pair. The results of this simulations are given in Table 1, which verify the second order spatial converge. Note that the deterioration of the rate for  $h = 1/64$  is due to accuracy of the linear solver.

**TABLE 1.** Spatial velocity errors and rates with  $T = 0.01$ ,  $\Delta t = 0.001$ , and  $(\mathbf{P}_2, P_1)$  elements.

$h$	$\ \mathbf{u} - \mathbf{u}_h\ _{\infty,0}$	Rate	$\ \nabla \cdot (\mathbf{u} - \mathbf{u}_h)\ _{\infty,0}$	Rate	$\ \nabla \cdot (\mathbf{u} - \mathbf{u}_h)\ _{2,0}$	Rate	$\ \nabla(\mathbf{u} - \tilde{\mathbf{u}}_h)\ _{2,0}$	Rate
1/4	$8.3364e-5$	—	$7.3523e-6$	—	$1.4855e-7$	—	$8.3364e-5$	—
1/8	$1.0428e-5$	2.9990	$2.3643e-6$	1.6367	$4.8651e-8$	1.6104	$1.8432e-5$	1.9975
1/16	$1.3037e-6$	2.9998	$4.5049e-7$	2.3919	$9.5334e-9$	2.3514	$4.6111e-6$	1.9990
1/32	$1.6300e-7$	2.9997	$6.7464e-8$	2.7393	$1.4886e-9$	2.6791	$1.1516e-6$	2.0012
1/64	$2.0714e-8$	2.9761	$1.8523e-8$	1.8650	$5.3125e-10$	1.4865	$2.8780e-7$	2.0008

### Error Comparison on a Fixed Mesh with Varying $\nu$ and $\nu_m$

We next compare the velocity, the magnetic field errors of non-stabilized, usual grad-div (with stabilization parameter 1.0) and modular grad-div methods (stabilization parameters  $\gamma = 1.0, \beta = 0.0$ ) with varying  $\nu, \nu_m$ . We use the same velocity, pressure and magnetic field solutions and setup as for 2d convergence rate verification. We fix end time and mesh size to  $h = \Delta t = 1/32$ . The computed errors from these methods are presented in Table 2. Again we observe similar error behaviour in modular and usual grad-div stabilization.

### Error Comparison for a Test Problem with Smaller $\nu$ and Larger Pressure

This numerical experiment focuses on the pressure robustness of the proposed algorithm. One important advantage of grad-div stabilization is that the stabilization parameter with the appropriate selection reduces the negative impact of the continuous pressure on the velocity error. To test that, a similar test problem and set up for the 2d-convergence rate test are used, but fixing end time and time step to  $T = 1.0$ ,  $\Delta t = 0.01$ , taking the dimensionless kinematic viscosity  $\nu$ , the magnetic resistivity  $\nu_m$  and true pressure solution as  $\nu = \nu_m = 0.01$ ,  $P(x, y) = 1000 \sin(\pi(x + 2y))$ . Then, Algorithm 0.1 is run for the stabilization parameters  $\gamma = 1.0$  and  $\beta = 0.0$  on the successively refined meshes. Errors found by Algorithm 3 are compared with those found by using the non-stabilized method and the usual grad-div stabilization (with parameter 1.0). The results can be seen in Table 3 and Table 4, and reveal that both velocity and magnetic errors of the proposed stabilization are much better with grad-div and modular grad-div even outperforms usual grad-div.

**TABLE 2.** Velocity and magnetic errors of non-stabilized, usual grad-div and modular grad-div methods with varying  $\nu, \nu_m$ .

$\nu = \nu_m$	$   \nabla(\mathbf{u} - \mathbf{u}_h)   _{2,0}$			$   \nabla(\mathbf{B} - \mathbf{B}_h)   _{2,0}$		
	No-stab.	Usual	Modular	No-stab.	Usual	Modular
1	1.91e-4	1.90e-4	1.92e-4	1.26e-3	1.26e-3	1.27e-3
1e-1	4.76e-3	4.76e-3	7.16e-3	7.11e-3	7.11e-3	7.16e-3
1e-2	2.23e-2	2.20e-2	2.17e-2	2.36e-2	2.33e-2	2.35e-2
1e-3	7.61e-2	6.38e-2	4.01e-2	7.27e-2	6.17e-2	4.43e-2
1e-4	3.04e-1	1.16e-1	5.35e-2	2.71e-1	9.20e-2	4.95e-2
1e-5	2.10	1.33e-1	5.73e-2	2.07	1.03e-1	5.10e-2
1e-6	3.00	1.35e-1	5.77e-2	3.00	1.05e-1	5.12e-2

When  $\beta > 0$ , the proposed algorithm, for example for the incompressible NSE, is consistent with the BE time discretization of the NSE by adding the penalization  $-\beta \nabla \nabla \cdot \mathbf{u}_t$ , and  $-\gamma \nabla \nabla \cdot \mathbf{u}$ , see [5] for details. To see the effect of  $\beta$  when it is non-zero, we rerun Algorithm 4.1 for the same test problem, but taking  $\gamma = 1.0, \beta = 1.0$ . The computed results of this experiment are presented in Table 5, reveals that the modular grad-div stabilization penalizes the divergence stronger then the usual grad-div stabilization.

**TABLE 3.** Velocity errors of the non-stabilized, usual grad-div and modular grad-div methods.

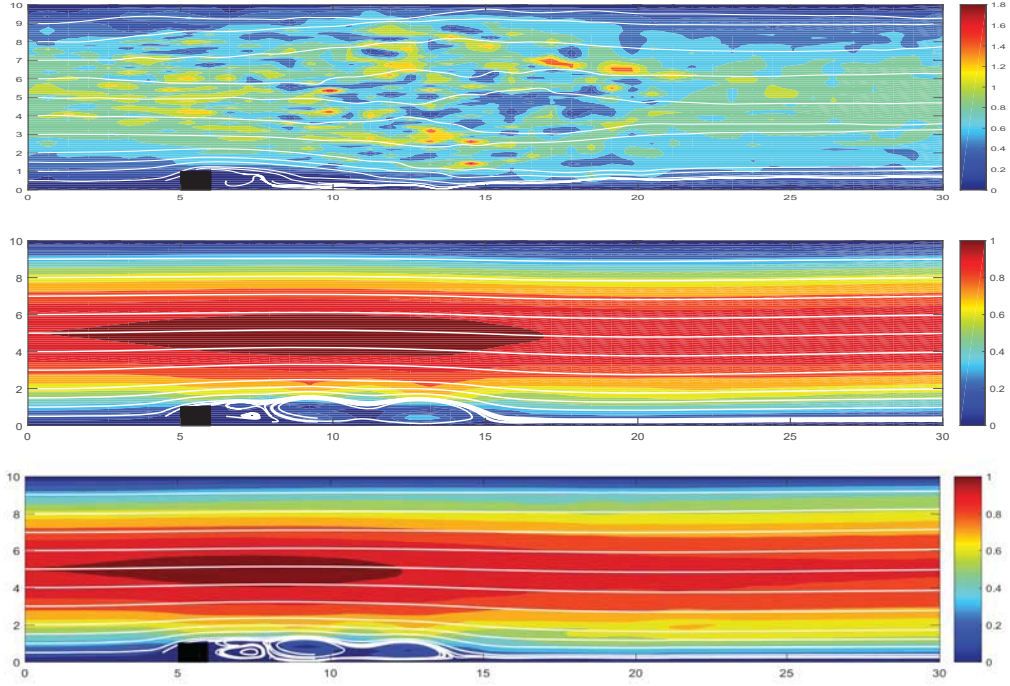
h	$   \nabla(\mathbf{u} - \mathbf{u}_h)   _{2,0}$			$   \nabla \cdot (\mathbf{u} - \mathbf{u}_h)   _{2,0}$		
	No-stab.	Usual	Modular	No-Stab.	Usual	Modular
1/2	444.89	126.11	9.152	317.38	60.03	4.42
1/4	318.53	83.60	4.85	232.57	32.46	2.15
1/8	176.27	25.31	1.56	148.55	5.01	2.84e-1
1/16	47.32	5.50	3.08e-1	46.22	6.63e-1	2.48e-2
1/32	7.01	9.40e-1	4.43e-2	6.93	7.99e-2	1.48e-3
1/64	8.92e-1	1.34e-1	9.02e-3	8.93e-1	9.42e-3	8.57e-5

**TABLE 4.** Magnetic errors of the non-stabilized, usual grad-div and modular grad-div methods.

h	$   \nabla(\mathbf{B} - \mathbf{B}_h)   _{2,0}$			$   \nabla \cdot (\mathbf{B} - \mathbf{B}_h)   _{2,0}$		
	No-stab.	Usual	Modular	No-Stab.	Usual	Modular
1/2	352.20	19.33	5.47	246.94	5.03	2.10e-1
1/4	265.56	39.89	2.60	177.68	5.70	1.46e-1
1/8	137.24	6.16	6.40e-1	94.70	4.49e-1	5.33e-2
1/16	14.18	9.90e-1	6.64e-2	10.48	3.84e-2	1.57e-2
1/32	7.04e-1	7.02e-2	8.37e-3	5.15e-1	1.83e-3	3.85e-3
1/64	3.02e-2	8.20e-3	7.54e-3	2.06e-2	8.07e-5	9.26e-5

**TABLE 5.** Errors of the non-stabilized, usual grad-div and modular grad-div methods with  $\gamma = 1.0, \beta = 1.0$

h	$   \nabla(\mathbf{u} - \mathbf{u}_h)   _{2,0}$		$   \nabla \cdot (\mathbf{u} - \mathbf{u}_h)   _{2,0}$		$   \nabla(\mathbf{B} - \mathbf{B}_h)   _{2,0}$		$   \nabla \cdot (\mathbf{B} - \mathbf{B}_h)   _{2,0}$	
	Usual	Modular	Usual	Modular	Usual	Modular	Usual	Modular
1/2	64.98	4.33	30.26	2.02	14.00	2.02	2.35	1.42
1/4	48.02	2.73	14.57	1.02	21.99	1.46	1.56	1.01e-1
1/8	16.85	9.84e-1	2.37	1.38e-1	3.63	4.11e-1	1.75e-1	3.71e-2
1/16	4.37	2.13e-1	3.47e-1	1.28e-2	7.44e-1	5.03e-2	2.00e-2	1.13e-2
1/32	8.45e-1	3.24e-2	4.30e-2	8.41e-4	6.14e-2	8.22e-3	1.23e-3	2.90e-3
1/64	1.29e-1	8.21e-3	4.66e-3	5.35e-5	8.10e-3	7.56e-3	5.82e-5	7.32e-4



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**FIGURE 1.** Shown above are  $T=40$  velocity solutions (shown as streamlines over speed contours) for MHD Channel flow over a step with  $s = 0.01$ ,  $\Delta t = 0.01$ ,  $\gamma = 1.0$ ,  $\beta = 0.0$ .

### Channel Flow over a Step

This numerical experiment focuses on two dimensional channel flow over a forward/backward step, in the presence of a magnetic field through the channel from left to right [6]. We here emphasize that we replace the term  $(\nabla \mathbf{B}_h, \nabla \mathbf{w}_h)$  in Maxwell equation with  $(\nabla \times \mathbf{B}_h, \nabla \times \mathbf{w}_h)$  since the domain is not convex. The proposed scheme is tested and compared with an unstabilized method, and usual grad-div stabilization method by choosing the same flow parameters on a coarser mesh that provide 11,458 dof. The stabilization parameters for the modular grad-div are taken as  $\gamma = 1.0$ , and  $\beta = 0.0$ , and for the usual stabilization is  $\gamma = 1.0$ . All computations are run to  $T = 40$  by taking time step  $\Delta t = 0.01$ . The results from these computations are presented in Figure 1. Here the plots reveals that proposed stabilization method gives a much more accurate solution when compared to unstabilized method, and nearly identical to that found using usual grad-div stabilization.

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