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Some new Simpson's type inequalities for coordinated convex functions in quantum calculus

Muhammad Aamir Ali[1] | **Hüseyin Budak**[2] | **Zhiyue Zhang**[1] | **Hüseyin Yildirim**[3]

[1]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China
[2]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
[3]Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, Kahramanmaraş, 46000, Turkey

Correspondence
Zhiyue Zhang, Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China.
Email: 05298@njnu.edu.cn

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In this article, by using the notion of newly defined q_1q_2 derivatives and integrals, some new Simpson's type inequalities for coordinated convex functions are proved. The outcomes raised in this paper are extensions and generalizations of the comparable results in the literature on Simpson's inequalities for coordinated convex functions.

KEYWORDS
coordinated convexity, Simpson's inequalities, q_1q_2-integrals, quantum calculus, q_1q_2-derivatives

MSC CLASSIFICATION
26D10; 26D15; 26B25

1 | INTRODUCTION

Quantum calculus or q-calculus is sometimes referred to as calculus without limits. In this, we gain q-analogues of mathematical items that may be got back as $q \to 1$. There are two kinds of q-addition, the Nalli–Ward-Al-Salam q-addition (NWA) and the Jackson–Hahn–Cigler q-addition (JHC). The first one is commutative and associative, at the same time as the second one is neither. That is why from time to time, multiple q-analogues exist. These operators form the basis of the technique which unites hypergeometric collection and q-hypergeometric collection and which gives many formulations of q-calculus a natural shape. The past of quantum calculus may be traced back to Euler (1707–1783), who first added the q in the tracks of Newton's infinite series. In recent years, many researchers have shown an eager hobby in studying and investigating quantum calculus; accordingly, it emerges as an interdisciplinary subject. This is of course because of the fact that quantum analysis is very helpful in several fields and has huge applications in various areas of natural and carried-out sciences such as computer science and particle physics and additionally acts as an critical tool for researchers operating with analytic number theory or in theoretical physics. Quantum calculus can be considered as a bridge among mathematics and physics. Many scientists who use quantum calculus are physicists, as quantum calculus has many applications in quantum group theory. For some recent trends in quantum calculus, involved readers are referred to other studies.[1-6]

In recent decades, the idea of convex functions has been drastically studied because of its fantastic significance in numerous fields of pure and applied sciences. Theory of inequalities and concept of convex functions are closely related to each other; thus, diverse inequalities can be found within the literature which are proved for convex and differentiable convex functions of single and double variables, see other studies.[7-18]

Simpson's rules are well-known techniques for the numerical integration and the numerical estimations of definite integrals. This method is known to be developed by Thomas Simpson (1710–1761). However, Johannes Kepler used a similar approximation about 100 years ago, so this method is also known as Kepler's rule.

Simpson's quadrature formula (Simpson's 1/3 rule) is stated as

$$\frac{1}{b-a}\int_a^b F(x)\mathrm{d}x \approx \frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimations known as Simpson's inequality:

Theorem 1. *Suppose that* $F:[a,b]\to\mathbb{R}$ *is a four times continuously differentiable mapping on* (a,b), *and let* $\|F^{(4)}\|_\infty = \sup_{x\in(a,b)}|F^{(4)}(x)| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{F(a)+F(b)}{2}+2F\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b F(x)dx\right| \le \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (b-a)^4.$$

In recent years, many authors have focused on Simpson type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson and Newton type results for convex mappings, because convexity theory is effective and is a strong method for solving a great number of problems which arise different branches in pure and applied mathematics. For example, Dragomir et al.[19] presented new Simpson type results and their applications to quadrature formula in numerical integration in their study. What is more, some inequalities of Simpson type for s-convex functions are deduced by Alomari et al.[20] Afterwards, Sarikaya et al. observed the variants of Simpson type inequalities based on $s-$convexity in their study.[21] Kunt et al. also gave the Simpson's type inequalities for the functions of two variables. [22]

Inspired by this ongoing studies, we establish some new quantum analogues of Simpson's inequalities for q-differentiable coordinated convex functions. This is the primary motivation of this paper. The ideas and strategies of the paper may open new venues of further research in this field.

2 | PRELIMINARIES OF *Q*-CALCULUS AND SOME INEQUALITIES

In this section, we present some required definitions and related inequalities about q-calculus. For more information about q-calculus, one can refer to other studies.[2-4]

Definition 1 ([6]). For a continuous function $F:[a,b]\to\mathbb{R}$, the q_a-derivative of F at $x\in[a,b]$ is characterized by the expression

$$_a d_q F(x) = \frac{F(x)-F(qx+(1-q)a)}{(1-q)(x-a)},\ x\neq a. \tag{2.1}$$

Since $F:[a,b]\to\mathbb{R}$ is a continuous function, thus we can define $_a d_q F(a) = \lim_{x\to a} {_a d_q F(x)}$. The function F is said to be q_a-differentiable on $[a,b]$ if $_a d_q F(x)$ exists for all $x\in[a,b]$. If $a=0$ in (2.1), then $_0 d_q F(x) = d_q F(x)$, where $d_q F(x)$ is the familiar q-derivative of F at $x\in[a,b]$ defined by the expression (see Kac and Cheung[5])

$$d_q F(x) = \frac{F(x)-F(qx)}{(1-q)x},\ x\neq 0. \tag{2.2}$$

Definition 2 ([6]). Let $F : [a,b] \to \mathbb{R}$ be a continuous function. Then, the q_a-definite integral on $[a,b]$ is defined as

$$\int_a^x F(t)\ {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n F\left(q^n x + \left(1-q^n\right)a\right) \tag{2.3}$$

for $x \in [a,b]$.

We have to give the following notation which will be used many times in the next sections (see Kac and Cheung[5]):

$$[n]_q = \frac{q^n - 1}{q-1}.$$

Moreover, we will need the following Lemma in our main results.

Lemma 1 ([23]). *We have the equality*

$$\int_a^b (x-a)^{\alpha}\ {}_ad_qx = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q}$$

for $\alpha \in \mathbb{R} \setminus \{-1\}$.

On the other hand, Bermudo et al. gave the following new definitions and related Hermite–Hadamard type inequalities:

Definition 3 ([9]). For a continuous function $F : [a,b] \to \mathbb{R}$, the q^b-derivative of F at $x \in [a,b]$ is characterized by the expression

$$^b d_q F(x) = \frac{F(qx + (1-q)b) - F(x)}{(1-q)(b-x)},\ x \neq b.$$

Definition 4 ([9]). Let $F : [a,b] \to \mathbb{R}$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as

$$\int_x^b F(t)\ {}^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n F\left(q^n x + \left(1-q^n\right)b\right)$$

for $x \in [a,b]$.

Theorem 2 ([9]). *If* $F : [a,b] \to \mathbb{R}$ *is a convex differentiable function on* $[a,b]$ *and* $0<q<1$*, then we have the following q-Hermite–Hadamard inequalities*

$$F\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a}\int_a^b F(x)\ {}^bd_qx \leq \frac{F(a)+qF(b)}{[2]_q}. \tag{2.4}$$

In their study,[13] Latif et al. defined q_{ac}-integral and partial q-derivatives for two variables functions as follows:

Definition 5. Suppose that $F : [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Then, the definite q_{ac}-integral on $[a,b]\times[c,d]$ is defined by

$$\int_a^x\int_c^y F(t,s)\ {}_cd_{q_2}s\ {}_ad_{q_1}t = (1-q_1)(1-q_2)(x-a)(y-c) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + \left(1-q_1^n\right)a, q_2^m y + \left(1-q_2^m\right)c\right)$$

for $(x,y) \in [a,b]\times[c,d]$.

Lemma 2 ([12]). *If the assumptions of Definition 5 hold, then*

$$\begin{aligned}\int_{y_1}^{y}\int_{x_1}^{x} F(t,s)_a d_{q_1} t_c d_{q_2} s &= \int_{y_1}^{y}\int_{a}^{x} F(t,s)_a d_{q_1} t_c d_{q_2} s - \int_{y_1}^{y}\int_{a}^{x_1} F(t,s)_a d_{q_1} t_c d_{q_2} s \\ &= \int_{c}^{y}\int_{a}^{x} F(t,s)_a d_{q_1} t_c d_{q_2} s - \int_{c}^{y_1}\int_{a}^{x} F(t,s)_a d_{q_1} t_c d_{q_2} s \\ &\quad - \int_{c}^{y}\int_{a}^{x_1} F(t,s)_a d_{q_1} t_c d_{q_2} s + \int_{c}^{y_1}\int_{a}^{x_1} F(t,s)_a d_{q_1} t_c d_{q_2} s.\end{aligned}$$

Definition 6 ([13]). Let $F : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then the partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(x,y) \in [a,b] \times [c,d]$ can be given as follows:

$$\begin{aligned}\frac{{}^{b}\partial_{q_1} F(x,y)}{{}^{b}\partial_{q_1} x} &= \frac{F(q_1x + (1-q_1)a, y) - F(x,y)}{(1-q_1)(x-a)}, x \neq b \\ \frac{{}^{d}\partial_{q_1} F(x,y)}{{}^{b}\partial_{q_2} y} &= \frac{F(x, q_2y + (1-q_2)c) - F(x,y)}{(1-q_2)(y-c)}, y \neq c \\ \frac{{}_{a,c}\partial^2_{q_1,q_2} F(x,y)}{{}_{a}\partial_{q_1} x_c \partial_{q_2} y} &= \frac{1}{(x-a)(y-c)(1-q_1)(1-q_2)}\left[F(q_1x + (1-q_1)a, q_2y + (1-q_2)c)\right. \\ &\quad \left. - F(q_1x + (1-q_1)a, y) - F(x, q_2y + (1-q_2)c) + F(x,y)\right], x \neq a, y \neq c.\end{aligned}$$

For more details related to q-integrals and derivatives for the functions of two variables, one can see Latif et al.[13]
On the other hand, Budak et al.[24] gave the following definitions of q_a^d, q_b^c, and q^{bd} integrals and related inequalities of Hermite–Hadamard type:

Definition 7 ([24]). Suppose that $F : [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous function. Then, the following q_a^d, q_c^b, and q^{bd} integrals on $[a,b] \times [c,d]$ are defined by

$$\int_a^x \int_y^d F(t,s)\, {}^{d}d_{q_2} s_a d_{q_1} t = (1-q_1)(1-q_2)(x-a)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + \left(1-q_1^n\right)a, q_2^m y + \left(1-q_2^m\right)d\right) \tag{2.5}$$

$$\int_x^b \int_c^y F(t,s)\, {}_{c}d_{q_2} s^b d_{q_1} t = (1-q_1)(1-q_2)(b-x)(y-c) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + \left(1-q_1^n\right)b, q_2^m y + \left(1-q_2^m\right)c\right) \tag{2.6}$$

and

$$\int_x^b \int_y^d F(t,s)\, {}^{d}d_{q_2} s^b d_{q_1} t = (1-q_1)(1-q_2)(b-x)(d-y) \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n x + \left(1-q_1^n\right)b, q_2^m y + \left(1-q_2^m\right)d\right) \tag{2.7}$$

respectively, for $(x,y) \in [a,b] \times [c,d]$.

Theorem 3 ([24]). *Let $F:\left[a,b\right]\times\left[c,d\right]\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinated convex function on $\left[a,b\right]\times\left[c,d\right]$. Then, we have the following inequalities:*

$$\begin{aligned}
&F\left(\frac{q_1a+b}{[2]_{q_1}},\frac{c+q_2d}{[2]_{q_2}}\right)\\
&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+q_2d}{[2]_{q_2}}\right)\,{}_ad_{q_1}x+\frac{1}{d-c}\int_c^d F\left(\frac{q_1a+b}{[2]_{q_1}},y\right)\,{}^dd_{q_2}y\right]\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}^dd_{q_2}y\,{}_ad_{q_1}x\\
&\leq\frac{q_1}{2[2]_{q_1}(d-c)}\int_c^d F(a,y)\,{}^dd_{q_2}y+\frac{1}{2[2]_{q_1}(d-c)}\int_c^d F(b,y)\,{}^dd_{q_2}y\\
&\quad+\frac{1}{2[2]_{q_2}(b-a)}\int_a^b F(x,c)\,{}_ad_{q_1}x+\frac{q_2}{2[2]_{q_2}(b-a)}\int_a^b F(x,d)\,{}_ad_{q_1}x\\
&\leq\frac{q_1F(a,c)+q_1q_2F(a,d)+F(b,c)+q_2F(b,d)}{[2]_{q_1}[2]_{q_2}}
\end{aligned}\tag{2.8}$$

for all $0<q_1,q_2<1$.

Theorem 4 ([24]). *Let $F:\left[a,b\right]\times\left[c,d\right]\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinated convex function on $\left[a,b\right]\times\left[c,d\right]$. Then, we have the following inequalities:*

$$\begin{aligned}
&F\left(\frac{a+q_1b}{[2]_{q_1}},\frac{q_2c+d}{[2]_{q_2}}\right)\\
&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{q_2c+d}{[2]_{q_2}}\right)\,{}^bd_{q_1}x+\frac{1}{d-c}\int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}},y\right)\,{}_cd_{q_2}y\right]\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}_cd_{q_2}y\,{}^bd_{q_1}x\\
&\leq\frac{1}{2[2]_{q_1}(d-c)}\int_c^d F(a,y)\,{}_cd_{q_2}y+\frac{q_1}{2[2]_{q_1}(d-c)}\int_c^d F(b,y)\,{}_cd_{q_2}y\\
&\quad+\frac{q_2}{2[2]_{q_2}(b-a)}\int_a^b F(x,c)\,{}^bd_{q_1}x+\frac{1}{2[2]_{q_2}(b-a)}\int_a^b F(x,d)\,{}^bd_{q_1}x\\
&\leq\frac{q_2F(a,c)+F(a,d)+q_1q_2F(b,c)+q_1F(b,d)}{[2]_{q_1}[2]_{q_2}}
\end{aligned}\tag{2.9}$$

for all $0<q_1,q_2<1$.

Theorem 5 ([24]). *Let $F:\left[a,b\right]\times\left[c,d\right]\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinated convex function on $\left[a,b\right]\times\left[c,d\right]$. Then, we have the following inequalities:*

$$\begin{aligned}
&F\left(\frac{a+q_1b}{[2]_{q_1}},\frac{c+q_2d}{[2]_{q_2}}\right)\\
&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+q_2d}{[2]_{q_2}}\right)\,{}^bd_{q_1}x+\frac{1}{d-c}\int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}}y,\right)\,{}^dd_{q_2}y\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\ {}^{d}d_{q_2}y\,{}^{b}d_{q_1}x \\
&\le \frac{1}{2[2]_{q_1}(d-c)}\int_c^d F(a,y)\ {}^{d}d_{q_2}y + \frac{q_1}{2[2]_{q_1}(d-c)}\int_c^d F(b,y)\ {}^{d}d_{q_2}y \\
&\quad + \frac{1}{2[2]_{q_2}(b-a)}\int_a^b F(x,c)\ {}^{d}d_{q_2}y + \frac{q_2}{2[2]_{q_2}(b-a)}\int_a^b F(x,d)\ {}^{b}d_{q_1}x \\
&\le \frac{F(a,c)+q_2F(a,d)+q_1F(b,c)+q_1q_2F(b,d)}{[2]_{q_1}[2]_{q_2}}
\end{aligned} \tag{2.10}
$$

for all $0<q_1, q_2<1$.

Theorem 6 (q_1q_2-Hölder's inequality for two variables functions,[13]). *Let* $x, y>0$, $0<q_1,\ q_2<1$, *and* $p_1>1$ *such that* $\frac{1}{p_1}+\frac{1}{r_1}=1$. *Then*

$$
\int_0^x\int_0^y |F(x,y)G(x,y)|\,d_{q_1}xd_{q_2}y \le \left(\int_0^x\int_0^y |F(x,y)|^{p_1}d_{q_1}xd_{q_2}y\right)^{\frac{1}{p_1}}\left(\int_0^x\int_0^y |G(x,y)|^{r_1}d_{q_1}xd_{q_2}y\right)^{\frac{1}{r_1}}.
$$

We can give the following new partial q-derivatives for functions of two variables.

Definition 8 ([25]). Let $F : [a,b]\times[c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables. Then the partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(x,y)\in[a,b]\times[c,d]$ can be given as follows:

$$
\begin{aligned}
\frac{{}^{b}\partial_{q_1}F(x,y)}{{}^{b}\partial_{q_1}x} &= \frac{F(q_1x+(1-q_1)b,y)-F(x,y)}{(1-q_1)(b-x)},\ x\neq b \\
\frac{{}^{d}\partial_{q_1}F(x,y)}{{}^{b}\partial_{q_2}y} &= \frac{F(x,q_2y+(1-q_2)d)-F(x,y)}{(1-q_2)(d-y)},\ d\neq y \\
\frac{{}_{a}^{d}\partial^2_{q_1,q_2}F(x,y)}{{}_{a}\partial_{q_1}x\,{}^{d}\partial_{q_2}y} &= \frac{1}{(x-a)(d-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)a,q_2y+(1-q_2)d)\right. \\
&\quad \left.-F(q_1x+(1-q_1)a,y)-F(x,q_2y+(1-q_2)d)+F(x,y)\right],\ x\neq a,\ y\neq d, \\
\frac{{}_{c}^{b}\partial^2_{q_1,q_2}F(x,y)}{{}^{b}\partial_{q_1}x\,{}_{c}\partial_{q_2}y} &= \frac{1}{(b-x)(y-c)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)b,q_2y+(1-q_2)c)\right. \\
&\quad \left.-F(q_1x+(1-q_1)b,y)-F(x,q_2y+(1-q_2)c)+F(x,y)\right],\ x\neq b,\ y\neq c, \\
\frac{{}^{b,d}\partial^2_{q_1,q_2}F(x,y)}{{}^{b}\partial_{q_1}x\,{}^{d}\partial_{q_2}y} &= \frac{1}{(b-x)(d-y)(1-q_1)(1-q_2)}\left[F(q_1x+(1-q_1)b,q_2y+(1-q_2)d)\right. \\
&\quad \left.-F(q_1x+(1-q_1)b,y)-F(x,q_2y+(1-q_2)d)+F(x,y)\right],\ x\neq b,\ y\neq d.
\end{aligned}
$$

3 | A CRUCIAL LEMMA

We deal with identity which is necessary to attain our main estimations in this section.

Let's start with the following useful Lemma.

Lemma 3. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$, then following identity holds for q_1q_2-integrals:*

$$ {}^{b,d}\mathcal{I}_{q_1,q_2}(F) = (b-a)(d-c)\int_0^1\int_0^1 \Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s, \tag{3.1} $$

where $0 < q_1, q_2 < 1$,

$$ \begin{aligned} {}^{b,d}\mathcal{I}_{q_1,q_2}(F) = {} & \frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9} \\ & + \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36} \\ & - \frac{1}{6(b-a)}\int_a^b \left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]^b d_{q_1}x \\ & - \frac{1}{6(d-c)}\int_c^d \left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]^d d_{q_2}y \\ & + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)^b d_{q_1}x^d d_{q_2}y, \end{aligned} $$

and

$$ \Lambda_{q_1}(t) = \begin{cases} q_1t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right), \\ q_1t-\frac{5}{6}, & t\in\left[\frac{1}{2},1\right], \end{cases} $$

$$ \Lambda_{q_2}(s) = \begin{cases} q_2s-\frac{1}{6}, & s\in\left[0,\frac{1}{2}\right), \\ q_2s-\frac{5}{6}, & s\in\left[\frac{1}{2},1\right]. \end{cases} $$

Proof. Because of the Lemma 2 and the definition of $\Lambda_{q_1}(t)$ and $\Lambda_{q_2}(s)$, it is easy to see that

$$ \begin{aligned} & \int_0^1\int_0^1 \Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s \\ & = \frac{4}{9}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s \\ & + \frac{2}{3}\int_0^{\frac{1}{2}}\int_0^1 \left(q_2s-\frac{5}{6}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s \\ & + \frac{2}{3}\int_0^1\int_0^{\frac{1}{2}} \left(q_1t-\frac{5}{6}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s \\ & + \int_0^1\int_0^1 \left(q_1t-\frac{5}{6}\right)\left(q_2s-\frac{5}{6}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s \\ & = I_1+I_2+I_3+I_4. \end{aligned} \tag{3.2} $$

From Definition 8, we have

$$\begin{aligned}
&\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\\
&=\frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)ts}\left[F(tq_1a+(1-tq_1)b,sq_2c+(1-sq_2)d)\right.\\
&-F(tq_1a+(1-tq_1)b,sc+(1-s)d)-F(ta+(1-t)b,sq_2c+(1-sq_2)d)\\
&\left.+F(ta+(1-t)b,sc+(1-s)d)\right].
\end{aligned}$$

It is needed to calculate the integrals in the right side of (3.2) in order to finish this proof. By using the definition of q_1q_2-integrals, we obtain that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}d_{q_1}t\,d_{q_2}s\\
&=\frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{1}{ts}\left[F(tq_1a+(1-tq_1)b,sq_2c+(1-sq_2)d)\right.\\
&-F(tq_1a+(1-tq_1)b,sc+(1-s)d)-F(ta+(1-t)b,sq_2c+(1-sq_2)d)\\
&\left.+F(ta+(1-t)b,sc+(1-s)d)\right]d_{q_1}t\,d_{q_2}s\\
&=\frac{1}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,\frac{q_2^{m+1}}{2}c+\left(1-\frac{q_2^{m+1}}{2}\right)d\right)\right.\\
&-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,\frac{q_2^{m}}{2}c+\left(1-\frac{q_2^{m}}{2}\right)d\right)\\
&-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,\frac{q_2^{m+1}}{2}c+\left(1-\frac{q_2^{m+1}}{2}\right)d\right)\\
&\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,\frac{q_2^{m}}{2}c+\left(1-\frac{q_2^{m}}{2}\right)d\right)\right.\\
&=\frac{1}{(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F\left(b,\frac{c+d}{2}\right)+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].
\end{aligned} \tag{3.3}$$

So

$$I_1=\frac{4}{9(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F\left(b,\frac{c+d}{2}\right)+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].$$

Now from Definition 7, we obtain the following:

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^{1}s\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}d_{q_1}t\,d_{q_2}s\\
&=\frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)}\int_0^{\frac{1}{2}}\int_0^{1}\frac{1}{t}\left[F(tq_1a+(1-tq_1)b,sq_2c+(1-sq_2)d)\right.\\
&-F(tq_1a+(1-tq_1)b,sc+(1-s)d)-F(ta+(1-t)b,sq_2c+(1-sq_2)d)\\
&\left.+F(ta+(1-t)b,sc+(1-s)d)\right]d_{q_1}t\,d_{q_2}s\\
&=\frac{1}{(b-a)(d-c)}\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)\right)\right.
\end{aligned}$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,q_2^mc+\left(1-q_2^m\right)d\right)\right)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)\right)$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m\left(F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,q_2^mc+\left(1-q_2^m\right)d\right)\right)\Bigg\}$$
$$=\frac{1}{(b-a)(d-c)}\Bigg\{\sum_{m=0}^{\infty}q_2^m\Bigg[\sum_{n=0}^{\infty}F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)$$
$$-\sum_{n=0}^{\infty}\left(F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)\right)\Bigg]$$
$$+\sum_{m=0}^{\infty}q_2^m\Bigg[\sum_{n=0}^{\infty}F\left(\frac{q_1^{n}}{2}a+\left(1-\frac{q_1^{n}}{2}\right)b,q_2^mc+\left(1-q_2^m\right)d\right)$$
$$-\sum_{n=0}^{\infty}F\left(\frac{q_1^{n+1}}{2}a+\left(1-\frac{q_1^{n+1}}{2}\right)b,q_2^mc+\left(1-q_2^m\right)d\right)\Bigg]\Bigg\}$$
$$=\frac{1}{(b-a)(d-c)}\Bigg\{\sum_{m=0}^{\infty}q_2^mF\left(b,q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)-\sum_{m=0}^{\infty}q_2^mF\left(b,q_2^mc+\left(1-q_2^m\right)d\right)$$
$$+\sum_{m=0}^{\infty}q_2^mF\left(\frac{a+b}{2},q_2^mc+\left(1-q_2^m\right)d\right)-\sum_{m=0}^{\infty}q_2^mF\left(\frac{a+b}{2},q_2^{m+1}c+\left(1-q_2^{m+1}\right)d\right)\Bigg\}$$
$$=\frac{1}{(b-a)(d-c)}\Bigg\{\frac{1-q_2}{q_2}\sum_{m=0}^{\infty}q_2^mF\left(b,q_2^mc+\left(1-q_2^m\right)d\right)-\frac{1}{q_2}F(b,c)$$
$$-\frac{1-q_2}{q_2}\sum_{m-0}^{\infty}q_2^mF\left(\frac{a+b}{2},q_2^mc+\left(1-q_2^m\right)d\right)+\frac{1}{q_2}F\left(\frac{a+b}{2},c\right)$$
$$=\frac{1}{(b-a)(d-c)}\left[\frac{1}{d-c}\int_c^d F(b,y)\,{}^d d_{q_2}y-\frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right){}^d d_{q_2}y\right.$$
$$\left.-\frac{1}{q_2}F(b,c)+\frac{1}{q_2}F\left(\frac{a+b}{2},y\right)\right].\tag{3.4}$$

By using the similar operations used in (3.3), we have

$$\int_0^{\frac{1}{2}}\int_0^1\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}d_{q_1}t\,d_{q_2}s$$
$$=\frac{1}{(b-a)(d-c)}\left[F(b,d)-F\left(\frac{a+b}{2},d\right)-F(b,c)+F\left(\frac{a+b}{2},c\right)\right].\tag{3.5}$$

From (3.4) and (3.5), we obtain that

$$I_2=\frac{2}{3(b-a)(d-c)}\Bigg\{\frac{1}{d-c}\int_c^d F(b,y)\,{}^d d_{q_2}y-\frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right){}^d d_{q_2}y$$
$$-F(b,c)+F\left(\frac{a+b}{2},c\right)\Bigg\}-\frac{10}{18(b-a)(d-c)}\Bigg\{F(b,d)-F\left(\frac{a+b}{2},d\right)$$
$$-F(b,c)+F\left(\frac{a+b}{2},c\right)\Bigg\}.$$

Similarly, we have

$$I_3 = \frac{2}{3(b-a)(d-c)}\left\{\frac{1}{b-a}\int_a^b F(x,d)\,{}^b d_{q_1}x - \frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right){}^b d_{q_1}x - F(a,d) + F\left(a,\frac{c+d}{2}\right)\right\} - \frac{10}{18(b-a)(d-c)}\left\{F(b,d) - F\left(b,\frac{c+d}{2}\right) - F(a,d) + F\left(a,\frac{c+d}{2}\right)\right\}.$$

Also, we have

$$\int_0^1\int_0^1 \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s} d_{q_1}t\,d_{q_2}s = \frac{1}{(b-a)(d-c)}\left[F(b,d) - F(a,d) - F(b,c) + F(a,c)\right], \tag{3.6}$$

$$\int_0^1\int_0^1 s\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s} d_{q_1}t\,d_{q_2}s = \frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_2(d-c)}\int_c^d F(b,y)\,{}^d d_{q_2}y - \frac{1}{q_2(d-c)}\int_c^d F(a,y)\,{}^d d_{q_2}y - \frac{1}{q_2}F(b,c) + \frac{1}{q_2}F(a,c)\right\}, \tag{3.7}$$

$$\int_0^1\int_0^1 t\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s} d_{q_1}t\,d_{q_2}s = \frac{1}{(b-a)(d-c)}\left\{\frac{1}{q_1(b-a)}\int_a^b F(x,d)\,{}^b d_{q_1}x - \frac{1}{q_1(b-a)}\int_a^b F(x,c)\,{}^b d_{q_1}x - \frac{1}{q_1}F(a,d) + \frac{1}{q_1}F(a,c)\right\} \tag{3.8}$$

and

$$\int_0^1\int_0^1 ts\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s} d_{q_1}t\,d_{q_2}s = \frac{1}{(b-a)(d-c)}\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^{n+1}a + \left(1-q_1^{n+1}\right)b, q_2^{m+1}c + \left(1-q_2^{m+1}\right)d\right) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^{n+1}a + \left(1-q_1^{n+1}\right)b, q_2^{m}c + \left(1-q_2^{m}\right)d\right) - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^{n}a + \left(1-q_1^{n}\right)b, q_2^{m+1}c + \left(1-q_2^{m+1}\right)d\right) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^{n}a + \left(1-q_1^{n}\right)b, q_2^{m}c + \left(1-q_2^{m}\right)d\right)\right\}$$

$$= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_1 q_2} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right) b, q_2^m c + \left(1-q_2^m\right) d\right) \right.\right.$$
$$\left. - \sum_{m=0}^{\infty} q_2^m F\left(a, q_2^m c + \left(1-q_2^m\right) d\right) - \sum_{n=0}^{\infty} q_1^n F\left(q_1^n a + \left(1-q_1^n\right) b, c\right) + F(a,c) \right]$$
$$- \frac{1}{q_1} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right) b, q_2^m c + \left(1-q_2^m\right) d\right) \right.$$
$$\left. - \sum_{m=0}^{\infty} q_2^m F\left(a, q_2^m c + \left(1-q_2^m\right) d\right) \right] - \frac{1}{q_2} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right) b, q_2^m c + \left(1-q_2^m\right) d\right) \right.$$
$$\left.\left. - \sum_{n=0}^{\infty} q_1^n F\left(q_1^n a + \left(1-q_1^n\right) b, c\right) \right] + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right) b, q_2^m c + \left(1-q_2^m\right) d\right) \right\}$$
$$= \frac{1}{(b-a)(d-c)} \left\{ \frac{(1-q_1)(1-q_2)}{q_1 q_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right) b, q_2^m c + \left(1-q_2^m\right) d\right) \right.$$
$$\left. - \frac{1-q_2}{q_1 q_2} \sum_{m=0}^{\infty} q_2^m F\left(a, q_2^m c + \left(1-q_2^m\right) d\right) - \frac{1-q_1}{q_1 q_2} \sum_{n=0}^{\infty} q_1^n F\left(q_1^n a + \left(1-q_1^n\right) b, c\right) + \frac{1}{q_1 q_2} F(a,c) \right\}$$
$$= \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{q_1 q_2 (b-a)(d-c)} \int_a^b \int_c^d F(x,y)\, {}^b d_{q_1} x\, {}^d d_{q_2} y - \frac{1}{q_1 q_2 (b-a)} \int_a^b F(x,c)\, {}^b d_{q_1} x \right.$$
$$\left. - \frac{1}{q_1 q_2 (d-c)} \int_c^d F(a,x)\, {}^d d_{q_2} y + \frac{1}{q_1 q_2} F(a,c) \right\}. \tag{3.9}$$

From (3.6)–(3.9), we obtain that

$$I_4 = \frac{1}{(b-a)(d-c)} \left\{ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x,y)\, {}^b d_{q_1} x\, {}^d d_{q_2} y \right.$$
$$\left. - \frac{1}{b-a} \int_a^b F(x,c)\, {}^b d_{q_1} x - \frac{1}{d-c} \int_c^d F(a,y)\, {}^d d_{q_2} y + F(a,c) \right\}$$
$$- \frac{5}{6(b-a)(d-c)} \left\{ \frac{1}{b-a} \int_a^b F(x,d)\, {}^b d_{q_1} x - \frac{1}{b-a} \int_a^b F(x,c)\, {}^b d_{q_1} x \right.$$
$$\left. + F(a,c) - F(a,d) \right\} - \frac{5}{6(b-a)(d-c)} \left\{ \frac{1}{d-c} \int_c^d F(b,y)\, {}^d d_{q_2} y \right.$$
$$\left. - \frac{1}{d-c} \int_c^d F(a,y)\, {}^d d_{q_2} y + F(a,c) - F(b,c) \right\}$$
$$+ \frac{25}{36(b-a)(d-c)} \left[F(b,d) - F(a,d) - F(b,c) + F(a,c) \right].$$

Now using the calculated integrals (I_1)–(I_4) in (3.2) and multiplying the resulting one with $(b-a)(d-c)$, then we have desired equality (3.1) which accomplishes the proof. □

Remark 1. Under the given conditions of Lemma 3 with $q_1, q_2 \to 1^-$, we have the following identity:

$$\begin{aligned}
&\frac{F\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) + 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + F\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right)}{9} \\
&+ \frac{F(a,c) + F(a,d) + F(b,c) + F(b,d)}{36} \\
&- \frac{1}{6(b-a)} \int_a^b \left[F(x,c) + 4F\left(x, \frac{c+d}{2}\right) + F(x,d)\right] dx \\
&- \frac{1}{6(d-c)} \int_c^d \left[F(a,y) + 4F\left(\frac{a+b}{2}, y\right) + F(b,y)\right] dy \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(x,y)\, dxdy \\
&= (b-a)(d-c) \int_0^1 \int_0^1 \Lambda(t) \Lambda(s) \frac{\partial^2 F(ta + (1-t)b, sc + (1-s)d)}{\partial t \partial s} dtds,
\end{aligned} \tag{3.10}$$

where

$$\Lambda(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and

$$\Lambda(s) = \begin{cases} s - \frac{1}{6}, & s \in \left[0, \frac{1}{2}\right) \\ s - \frac{5}{6}, & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which is turned out by Özdemir et al.[22, Lemma 1]

4 | SOME NEW Q_1q_2-SIMPSON'S TYPE INEQUALITIES

For brevity, we give some calculated integrals before giving new estimates.

$$A_1(q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| t d_q t = \begin{cases} \frac{1-2q-2q^2}{24[2]_q[3]_q} & 0 < q < \frac{1}{3} \\ \frac{18q^2+18q-7}{216[2]_q[3]_q} & \frac{1}{3} \le q < 1, \end{cases} \tag{4.1}$$

$$A_2(q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| (1-t) d_q t = \begin{cases} \frac{1-4q^3}{24[2]_q[3]_q} & 0 < q < \frac{1}{3} \\ \frac{36q^3+12q^2+12q+1}{216[2]_q[3]_q} & \frac{1}{3} \le q < 1, \end{cases} \tag{4.2}$$

$$A_3(q) = \int_{\frac{1}{2}}^{1} \left|qt - \frac{5}{6}\right| t d_q t = \begin{cases} \frac{15-6q-6q^2}{24[2]_q[3]_q} & 0 < q < \frac{5}{6} \\ \frac{18q^2+18q+25}{216[2]_q[3]_q} & \frac{5}{6} \le q < 1 \end{cases} \tag{4.3}$$

$$A_4(q) = \int_{\frac{1}{2}}^{1} \left|qt - \frac{5}{6}\right| (1-t) d_q t = \begin{cases} \frac{-5+8q+8q^2-8q^3}{24[2]_q[3]_q} & 0 < q < \frac{5}{6} \\ \frac{12q^2+12q+5}{216[2]_q[3]_q} & \frac{5}{6} \le q < 1, \end{cases} \tag{4.4}$$

$$A_5(q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| d_q t = \begin{cases} \frac{1}{36[2]_q} & 0 < q < \frac{1}{3} \\ \frac{6q-1}{36[2]_q} & \frac{1}{3} \le q < 1, \end{cases}, \tag{4.5}$$

$$A_6(q)=\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|d_qt=\begin{cases}\frac{5-4q}{12[2]_q} & 0<q<\frac{5}{6}\\ \frac{5}{36[2]_q} & \frac{5}{6}\le q<1.\end{cases}. \tag{4.6}$$

Now we give some new quantum estimates by using the identity given in the last section.

Theorem 7. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|$ is convex on $[a,b]\times[c,d]$:*

$$\begin{aligned}\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right|\le(b-a)(d-c)\Bigg[&(A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\\&+(A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\\&+(A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\\&+(A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\Bigg],\end{aligned} \tag{4.7}$$

where $0<q_1,q_2<1$.

Proof. On taking modulus of the identity of Lemma 3.1, because of the properties of modulus, we find that

$$\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right|\le(b-a)(d-c)\int_0^1\int_0^1\left|\Lambda_{q_1}(t)\Lambda_{q_2}(s)\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|d_{q_1}t\,d_{q_2}s. \tag{4.8}$$

Now using the convexity of $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|$, then (4.8) becomes

$$\begin{aligned}&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right|\\&\le(b-a)(d-c)\int_0^1\Lambda_{q_2}(s)\left[\int_0^1\Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right.\right.\\&\quad\left.\left.+(1-t)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right\}d_{q_1}t\right]d_{q_2}s.\end{aligned} \tag{4.9}$$

Now we compute the integrals appear in right side of inequality (4.9)

$$\begin{aligned}
&\int_0^1 \Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right\}d_{q_1}t\\
&=\int_0^{\frac{1}{2}} t\left|q_1t-\frac{1}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|d_{q_1}t\\
&\quad+\int_0^{\frac{1}{2}}(1-t)\left|q_1t-\frac{1}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|d_{q_1}t\\
&\quad+\int_{\frac{1}{2}}^1 t\left|q_1t-\frac{5}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|d_{q_1}t\\
&\quad+\int_0^{\frac{1}{2}}(1-t)\left|q_1t-\frac{5}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|d_{q_1}t.
\end{aligned}$$

From (4.1)–(4.4), we obtain that

$$\begin{aligned}
&\int_0^1 \Lambda_{q_1}(t)\left\{t\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right\}d_{q_1}t\\
&=\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|(A_1(q_1)+A_3(q_1))\\
&\quad+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|(A_2(q_1)+A_4(q_1)).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right|\\
&\le(b-a)(d-c)\int_0^1\Lambda_{q_2}(s)\left[\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|(A_1(q_1)+A_3(q_1))\right.\\
&\quad\left.+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|(A_2(q_1)+A_4(q_1))\right]d_{q_2}s\\
&\le(b-a)(d-c)\int_0^1\Lambda_{q_2}(s)\left[\left\{s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right.\right.\\
&\quad\left.\times(A_1(q_1)+A_3(q_1))\right\}+\left\{s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right.\\
&\quad\left.\left.\times(A_2(q_1)+A_4(q_1))\right\}\right]d_{q_2}s\\
&=(b-a)(d-c)(A_1(q_1)+A_3(q_1))\left[\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_0^{\frac{1}{2}}s\left|q_2s-\frac{1}{6}\right|d_{q_2}s\right.\\
&\quad+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_0^{\frac{1}{2}}(1-s)\left|q_2s-\frac{1}{6}\right|d_{q_2}s+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_{\frac{1}{2}}^1 s\left|q_2s-\frac{5}{6}\right|d_{q_2}s\\
&\quad\left.+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_{\frac{1}{2}}^1(1-s)\left|q_2s-\frac{5}{6}\right|d_{q_2}s\right]\\
&\quad+(b-a)(d-c)(A_2(q_1)+A_4(q_1))\left[\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_0^{\frac{1}{2}}s\left|q_2s-\frac{1}{6}\right|d_{q_2}s\right.
\end{aligned}$$

$$+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_0^{\frac{1}{2}}(1-s)\left|q_2s-\frac{1}{6}\right|d_{q_2}s+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_{\frac{1}{2}}^{1}s\left|q_2s-\frac{5}{6}\right|d_{q_2}s$$
$$+\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\int_{\frac{1}{2}}^{1}(1-s)\left|q_2s-\frac{5}{6}\right|d_{q_2}s\Bigg].$$

From (4.1)–(4.4), we have

$$\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right|\leq(b-a)(d-c)\Bigg[(A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|$$
$$+(A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|$$
$$+(A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|$$
$$+(A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^{b,d}\partial_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\Bigg].$$

Hence, the proof is completed. □

Remark 2. Under the given conditions of Theorem 7 with $q_1, q_2 \to 1^-$, we attain the following inequality:

$$\Bigg|\frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9}$$
$$+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}$$
$$-\frac{1}{6(b-a)}\int_a^b\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]dx$$
$$-\frac{1}{6(d-c)}\int_c^d\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]dy$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dxdy\Bigg|$$
$$\leq\frac{25(b-a)(d-c)}{72}\left[\frac{\left|\frac{\partial^2F(a,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2F(a,d)}{\partial t\partial s}\right|+\left|\frac{\partial^2F(b,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2F(b,d)}{\partial t\partial s}\right|}{72}\right]\tag{4.10}$$

which is shown by Özdemir et al.[22, Theorem 3]

Theorem 8. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p>1$*

and $\frac{1}{r}+\frac{1}{p}=1$, *then we have following inequality:*

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\left(\int_0^1\int_0^1 \left|\Lambda_{q_1}(t)\Lambda_{q_2}(s)\right|^r d_{q_1}t d_{q_2}s\right)^{\frac{1}{r}} \\
&\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+\frac{q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\right. \\
&\left.+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\right]^{\frac{1}{p}},
\end{aligned} \tag{4.11}$$

where $0<q_1,q_2<1$.

Proof. Applying the well-known Hölder's inequality for q_1q_2-integrals to the integrals in right side of (4.8), it is found that

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\left[\left(\int_0^1\int_0^1 \left|\Lambda_{q_1}(t)\Lambda_{q_2}(s)\right|^r d_{q_1}t d_{q_2}s\right)^{\frac{1}{r}}\right. \\
&\left.\times\left(\int_0^1\int_0^1\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}\right].
\end{aligned} \tag{4.12}$$

By applying convexity of $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}$, then (4.12) becomes

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\left[\left(\int_0^1\int_0^1 \left|\Lambda_{q_1}(t)\Lambda_{q_2}(s)\right|^r d_{q_1}t d_{q_2}s\right)^{\frac{1}{r}}\right. \\
&\times\left(\int_0^1\int_0^1\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\right.\right. \\
&\left.\left.\left.+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\right]d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}\right].
\end{aligned} \tag{4.13}$$

Now, if we apply the concept of Lemma 1 for $a=0$ to the above quantum integrals, we attain

$$\begin{aligned}
\int_0^1\int_0^1 ts\,d_{q_1}t d_{q_2}s &= \left(\int_0^1 t\,d_{q_1}t\right)\left(\int_0^1 s\,d_{q_2}s\right) \\
&= \frac{1}{[2]_{q_1}[2]_{q_2}},
\end{aligned} \tag{4.14}$$

$$\int_0^1\int_0^1 t(1-s)\,d_{q_1}t d_{q_2}s = \frac{q_2}{[2]_{q_1}[2]_{q_2}}, \tag{4.15}$$

$$\int_0^1\int_0^1 (1-t)s\,d_{q_1}t d_{q_2}s = \frac{q_1}{[2]_{q_1}[2]_{q_2}}, \tag{4.16}$$

$$\int_0^1\int_0^1 (1-t)(1-s)\,d_{q_1}t d_{q_2}s = \frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}. \tag{4.17}$$

By substituting the calculated integrals (4.14)–(4.17) in (4.13), then we obtain the desired inequality (4.11) which finishes the proof. □

Theorem 9. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p\geq 1$, then we have following inequality:*

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\Bigg[A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_2(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_5^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_2(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_4(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_4(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}}\Bigg],
\end{aligned}\tag{4.18}$$

where $0<q_1,q_2<1$.

Proof. When applying well-known power mean inequality for q_1q_2-integrals to the integrals in right side of (4.8), it is found that

$$\begin{aligned}
&\left|{}^{b,d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\Bigg[\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{q_1}t\,d_{q_2}s\right)^{1-\frac{1}{p}} \\
&\times\Bigg(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right| \\
&\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_1}t\,d_{q_2}s\Bigg)^{\frac{1}{p}} \\
&+\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right)^{1-\frac{1}{p}}
\end{aligned}$$

$$\times\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right.$$

$$\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{q_1}t d_{q_2}s\right)^{1-\frac{1}{p}}$$

$$\times\left(\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\right.$$

$$\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{q_1}t d_{q_2}s\right)$$

$$\times\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{q_1}t d_{q_2}s\right.$$

$$\left.\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}\right]. \tag{4.19}$$

By applying convexity of $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p$, then we have

$$\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{q_1}t d_{q_2}s\right)^{1-\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{1}{6}\right|\right.$$

$$\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(ta+(1-t)b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_1}t d_{q_2}s\right)^{\frac{1}{p}}$$

$$\leq\left(\left(\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|d_{q_1}t\right)\left(\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|d_{q_2}s\right)\right)^{1-\frac{1}{p}}$$

$$\left[\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left\{\int_0^{\frac{1}{2}}\left|q_1t-\frac{1}{6}\right|\left(t\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(a,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right.\right.\right.$$

$$\left.\left.\left.+(1-t)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)d_{q_1}t\right\}d_{q_2}s\right]^{\frac{1}{p}}$$

$$=A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(a,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_2}s\right.$$

$$\left.+A_2(q_1)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F\left(b,sc+(1-s)d\right)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p d_{q_2}s\right]^{\frac{1}{p}}$$

$$\leq A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right|\left(s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right.\right.$$

$$
\begin{aligned}
&+(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\Bigg)d_{q_2}s+A_2(q_1)\int_0^{\frac{1}{2}}\left|q_2s-\frac{1}{6}\right| \\
&\times\left(s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right)d_{q_2}s\Bigg]^{\frac{1}{p}} \\
&=A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2)\left[A_1(q_1)\left\{A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right. \\
&\left.+A_2(q_1)\left\{A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right]^{\frac{1}{p}}. \qquad (4.20)
\end{aligned}
$$

By applying the similar operations, we obtain that

$$
\begin{aligned}
&\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right)^{1-\frac{1}{p}} \\
&\times\left(\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left|q_1t-\frac{1}{6}\right|\left|q_2s-\frac{5}{6}\right|\right. \\
&\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p d_{q_1}td_{q_2}s\right)^{\frac{1}{p}} \\
&\le A_5^{1-\frac{1}{p}}A_6^{1-\frac{1}{p}}\left[A_1(q_1)\left\{A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right. \\
&\left.+A_2(q_1)\left\{A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right]^{\frac{1}{p}}, \qquad (4.21)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|d_{q_1}td_{q_2}s\right)^{1-\frac{1}{p}} \\
&\times\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{1}{6}\right|\right. \\
&\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p d_{q_1}td_{q_2}s\right)^{\frac{1}{p}} \\
&\le A_6^{1-\frac{1}{p}}A_5^{1-\frac{1}{p}}\left[A_3(q_1)\left\{A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right. \\
&\left.+A_4(q_1)\left\{A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right]^{\frac{1}{p}}, \qquad (4.22)
\end{aligned}
$$

$$\left(\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{q_1}td_{q_2}s\right)$$
$$\times\left[\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left|q_1t-\frac{5}{6}\right|\left|q_2s-\frac{5}{6}\right|d_{q_1}td_{q_2}s\right.$$
$$\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p d_{q_1}td_{q_2}s\right]^{\frac{1}{p}}$$
$$\leq A_6^{1-\frac{1}{p}}A_6^{1-\frac{1}{p}}\left[A_3(q_1)\left\{A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right.$$
$$\left.+A_4(q_1)\left\{A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}\right|^p\right\}\right]^{\frac{1}{p}}. \tag{4.23}$$

From (4.19)–(4.23), we obtain desired inequality, and proof is ended. □

5 | ADDITIONAL RESULTS

In this section, we present some results without proof since the proofs are similar to the ones given in Section 3.

Lemma 4. *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$, then following identity holds for q_1q_2-integrals:*

$$\begin{aligned}&{}_a^d\mathcal{I}_{q_1,q_2}(F)\\&=(b-a)(d-c)\int_0^1\int_0^1\Lambda_{q_1}(t)\Lambda_{q_2}(s)\frac{{}_a^d\partial^2_{q_1,q_2}F(tb+(1-t)a,sc+(1-s)d)}{{}_a\partial_{q_1}t^{d}\partial_{q_2}s}d_{q_1}td_{q_2}s,\end{aligned} \tag{5.1}$$

where $0<q_1,q_2<1$, Λ_{q_1}, and Λ_{q_2} are defined as in Lemma 3 and

$$\begin{aligned}{}_a^d\mathcal{I}_{q_1,q_2}(F)=&\frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9}\\&+\frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36}\\&-\frac{1}{6(b-a)}\int_a^b\left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]{}_ad_{q_1}x\\&-\frac{1}{6(d-c)}\int_c^d\left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]{}^dd_{q_2}y\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}_ad_{q_1}x\,{}^dd_{q_2}y.\end{aligned}$$

and $0<q_1$ and $q_2<1$.

Proof. If the strategy which was used in the proof of Lemma 3 is applied by taking into account the definition of $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t^{d}\partial_{q_2}s}$, the desired inequality (5.1) can be attained. □

Remark 3. If we choose $q_1,q_2\to1^-$ in Lemma 4, then the identity (5.1) reduces to identity (3.10).

Now we use Lemma 4 to find some new quantum estimates. We first examine a new result for functions whose partially q_1q_2-derivatives in modulus are convex in the following theorem.

Theorem 10. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^d_a\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}^d_a\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|$ is convex on $[a,b]\times[c,d]$:*

$$\begin{aligned}\left|{}^d_a\mathcal{I}_{q_1,q_2}(F)\right| \le (b-a)(d-c)\Bigg[&(A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^d_a\partial_{q_1,q_2}F(a,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\\ &+(A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^d_a\partial_{q_1,q_2}F(a,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\\ &+(A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2))\left|\frac{{}^d_a\partial_{q_1,q_2}F(b,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\\ &+(A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2))\left|\frac{{}^d_a\partial_{q_1,q_2}F(b,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\Bigg],\end{aligned} \tag{5.2}$$

where $0<q_1, q_2<1$.

Theorem 11. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p>1$ and $\frac{1}{r}+\frac{1}{p}=1$, then we have following inequality:*

$$\begin{aligned}&\left|{}^d_a\mathcal{I}_{q_1,q_2}(F)\right|\\ &\le(b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda_{q_1}(t)\Lambda_{q_2}(s)\right|^r d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{r}}\\ &\quad\Bigg[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^d_a\partial_{q_1,q_2}F(a,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+\frac{q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^d_a\partial_{q_1,q_2}F(a,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\\ &\quad+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^d_a\partial_{q_1,q_2}F(b,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p+\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^d_a\partial_{q_1,q_2}F(b,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p\Bigg]^{\frac{1}{p}},\end{aligned} \tag{5.3}$$

where $0<q_1, q_2<1$.

Theorem 12. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}^d_a\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $\left|\frac{{}^d_a\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$*

for some $p \geq 1$, then we have following inequality:

$$
\begin{aligned}
&\left|{}_{a}^{d}\mathcal{I}_{q_1,q_2}(F)\right| \\
&\leq (b-a)(d-c)\Bigg[A_5^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_2(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_5^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_1(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_2(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_5^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_4(q_1)\left(A_1(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_2(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}} \\
&+A_6^{1-\frac{1}{p}}(q_1)A_6^{1-\frac{1}{p}}(q_2) \\
&\times\Bigg\{A_3(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right) \\
&+A_4(q_1)\left(A_3(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p+A_4(q_2)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^p\right)\Bigg\}^{\frac{1}{p}}\Bigg],
\end{aligned}
\tag{5.4}
$$

where $0 < q_1, q_2 < 1$.

Lemma 5. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$, then following identity holds for q_1q_2-integrals:*

$$
\begin{aligned}
&{}_{c}^{b}\mathcal{I}_{q_1,q_2}(F) \\
&=(b-a)(d-c)\int_0^1\int_0^1\Lambda_{q_1}(t)\,\Lambda_{q_2}(s)\,\frac{{}_{c}^{b}\partial^2_{q_1,q_2}F(ta+(1-t)b,sd+(1-s)c)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s}d_{q_1}td_{q_2}s,
\end{aligned}
\tag{5.5}
$$

where $0<q_1, q_2<1$, Λ_{q_1}, and Λ_{q_2} are defined as in Lemma 3 and

$$
\begin{aligned}
{}_c^b\mathcal{I}_{q_1,q_2}(F) = & \frac{F\left(\frac{a+b}{2},c\right)+F\left(\frac{a+b}{2},d\right)+4F\left(\frac{a+b}{2},\frac{c+d}{2}\right)+F\left(a,\frac{c+d}{2}\right)+F\left(b,\frac{c+d}{2}\right)}{9} \\
& + \frac{F(a,c)+F(a,d)+F(b,c)+F(b,d)}{36} \\
& - \frac{1}{6(b-a)}\int_a^b \left[F(x,c)+4F\left(x,\frac{c+d}{2}\right)+F(x,d)\right]^b d_{q_1}x \\
& - \frac{1}{6(d-c)}\int_c^d \left[F(a,y)+4F\left(\frac{a+b}{2},y\right)+F(b,y)\right]_c d_{q_2}y \\
& + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)^b d_{q_1}x_c d_{q_2}y.
\end{aligned}
$$

Proof. If the strategy which was used in the proof of Lemma 3 is applied by taking into account the definition of $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}$, the desired inequality (5.5) can be attained. □

Remark 4. If we choose $q_1, q_2 \to 1^-$ in Lemma 5, then the identity (5.5) reduces to identity (3.10).

Now we use Lemma 5 to find some new quantum estimates. We first examine a new result for functions whose partially q_1q_2-derivatives in modulus are convex in the following theorem.

Theorem 13. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d] \subseteq \Delta^\circ$. Then we have following inequality provided that $\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right|$ is convex on $[a,b]\times[c,d]$:*

$$
\begin{aligned}
\left|{}_c^b\mathcal{I}_{q_1,q_2}(F)\right| \le (b-a)(d-c) \Bigg[& (A_1(q_1)+A_3(q_1))(A_1(q_2)+A_3(q_2)) \left|\frac{{}_c^b\partial_{q_1,q_2}F(a,c)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right| \\
& + (A_1(q_1)+A_3(q_1))(A_2(q_2)+A_4(q_2)) \left|\frac{{}_c^b\partial_{q_1,q_2}F(a,d)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right| \\
& + (A_2(q_1)+A_4(q_1))(A_1(q_2)+A_3(q_2)) \left|\frac{{}_c^b\partial_{q_1,q_2}F(b,c)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right| \\
& + (A_2(q_1)+A_4(q_1))(A_2(q_2)+A_4(q_2)) \left|\frac{{}_c^b\partial_{q_1,q_2}F(b,d)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right| \Bigg],
\end{aligned}
\tag{5.6}
$$

where $0<q_1, q_2<1$.

Theorem 14. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d] \subseteq \Delta^\circ$. If $\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t_c\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$*

for some $p > 1$ and $\frac{1}{r} + \frac{1}{p} = 1$, then we have following inequality:

$$\begin{aligned}
&\left| {}_{c}^{b}\mathcal{I}_{q_1,q_2}(F) \right| \\
&\le (b-a)(d-c)\left(\int_0^1 \int_0^1 \left| \Lambda_{q_1}(t)\Lambda_{q_2}(s) \right|^r d_{q_1}t d_{q_2}s \right)^{\frac{1}{r}} \\
&\left[\frac{1}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}_{c}^{b}\partial_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s} \right|^p + \frac{q_2}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}_{c}^{b}\partial_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s} \right|^p \right. \\
&\quad \left. + \frac{q_1}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}_{c}^{b}\partial_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s} \right|^p + \frac{q_1q_2}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}_{c}^{b}\partial_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s} \right|^p \right]^{\frac{1}{p}},
\end{aligned} \tag{5.7}$$

where $0 < q_1, q_2 < 1$.

Theorem 15. *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that partial q_1q_2-derivative $\frac{{}_{c}^{b}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d] \subseteq \Delta^\circ$. If $\left|\frac{{}_{c}^{b}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t_c\partial_{q_2}s}\right|^p$ is convex on $[a,b]\times[c,d]$ for some $p \ge 1$, then we have following inequality:*

$$\begin{aligned}
&\left| {}_{c}^{b}\mathcal{I}_{q_1,q_2}(F) \right| \\
&\le (b-a)(d-c)\left[A_5^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \right. \\
&\times \left\{ A_1(q_1)\left(A_1(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_2(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right. \\
&\left. + A_2(q_1)\left(A_1(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_2(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right\}^{\frac{1}{p}} \\
&+ A_5^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
&\times \left\{ A_1(q_1)\left(A_3(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_4(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right. \\
&\left. + A_2(q_1)\left(A_3(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_4(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right\}^{\frac{1}{p}} \\
&+ A_6^{1-\frac{1}{p}}(q_1) A_5^{1-\frac{1}{p}}(q_2) \\
&\times \left\{ A_3(q_1)\left(A_1(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_2(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right. \\
&\left. + A_4(q_1)\left(A_1(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_2(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right\}^{\frac{1}{p}} \\
&+ A_6^{1-\frac{1}{p}}(q_1) A_6^{1-\frac{1}{p}}(q_2) \\
&\times \left\{ A_3(q_1)\left(A_3(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_4(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right. \\
&\left.\left. + A_4(q_1)\left(A_3(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p + A_4(q_2)\left| \frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t^d\partial_{q_2}s} \right|^p \right) \right\}^{\frac{1}{p}} \right],
\end{aligned} \tag{5.8}$$

where $0 < q_1, q_2 < 1$.

6 | CONCLUSION

In this paper, Simpson's type inequalities for coordinated convex functions by applying the q_1q_2-integrals are obtained. It is also shown that the results proved in this paper are a potential generalization of the existing comparable results in the literature. As future directions, one may find similar inequalities through different types of convexities.

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CONFLICT OF INTEREST

This work does not have a conflict of interest.

ORCID

Muhammad Aamir Ali https://orcid.org/0000-0001-5341-4926
Hüseyin Budak https://orcid.org/0000-0001-8843-955X
Zhiyue Zhang https://orcid.org/0000-0001-7070-2532
Hüseyin Yildirim https://orcid.org/0000-0001-8855-9260

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